WSGP 24

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In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 24th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2005. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 75. pp. [325]–332.

Persistent URL: http://dml.cz/dmlcz/701758

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 75 (2005), pp. 325-332

RESTRICTIONS OF 3–FORMS IN DIMENSION 7 TO SUBSPACES OF CODIMENSION 1

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ABSTRACT. On a 6-dimensional real vector space there are six types of 3-forms. We take all types of 3-forms on a 7-dimensional space and determine types of restrictions to all subspaces of codimension 1.

Let V be a finite dimensional vector space. A k-form $\omega \in \Lambda^k V^*$ is called *multisymplectic* or *regular* if the homomorphism

$$V \to \Lambda^{k-1} V^*, \quad v \mapsto \iota_v \omega = \omega(v, \dots)$$

is a monomorphism. If a k-form ω is not regular, we shall call it singular. We denote by $\Lambda^k_r V^* \subset \Lambda^k V^*$ ($\Lambda^k_s V^* \subset \Lambda^k V^*$) the subset consisting of all regular (singular) forms. The general linear group $GL(V)$ operates in a natural way on $\Lambda^k V^*$, and it is easy to see that this action preserves $\Lambda^k_r V^*$ ($\Lambda^k_s V^*$). Consequently, $\Lambda^k_r V^*$ ($\Lambda^k_s V^*$) decomposes into orbits of this action. In this paper we take $k = 3$, i.e. we consider 3-forms. It is known, that the number of orbits of 3-forms is finite if and only if $\dim V \leq 8$.

Let us treat first a 6-dimensional real vector space W. We choose its basis $f_1, \dots, f_6$, and we denote $\beta_1, \dots, \beta_6$ the corresponding dual basis. There are three orbits consisting of singular forms represented by the forms

(S1) $\sigma_1 = 0$,
(S2) $\sigma_2 = \beta_1 \wedge \beta_2 \wedge \beta_3$,
(S3) $\sigma_3 = \beta_1 \wedge (\beta_2 \wedge \beta_3 + \beta_4 \wedge \beta_5)$.

2000 *Mathematics Subject Classification*: 15A75.

Key words and phrases: 3-form, 7-dimensional vector space, hyperplane.

Supported by the Grant Agency of the Academy of Sciences of the Czech Republic, grant no. A1019204.

The paper is in final form and no version of it will be submitted elsewhere.

There are also three orbits consisting of regular forms. They are represented by the forms

(R1) $\rho_1 = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_4 \wedge \beta_5 \wedge \beta_6$,

(R2) $\rho_2 = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_4 \wedge \beta_6 - \beta_3 \wedge \beta_5 \wedge \beta_6$,

(R3) $\rho_3 = \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_4 \wedge \beta_6 + \beta_3 \wedge \beta_5 \wedge \beta_6$.

Now, let us pass to a 7-dimensional real vector space V. We choose a basis $e_1, \dots, e_7$ of V, and we denote by $\alpha_1, \dots, \alpha_7$ the corresponding dual basis. Here the subset $\Lambda^3_r V^*$ decomposes into eight orbits. They are represented by the following forms.

$$\omega_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6 \,, \tag{1}$$

$$\begin{aligned} \omega_2 = {} & \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 \\ & - \alpha_2 \wedge \alpha_3 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 \,, \end{aligned} \tag{2}$$

$$\omega_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) \,, \tag{3}$$

$$\omega_4 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 \,, \tag{4}$$

$$\begin{aligned} \omega_5 = {} & \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ & + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6 \,, \end{aligned} \tag{5}$$

$$\begin{aligned} \omega_6 = {} & \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 \\ & + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 \,, \end{aligned} \tag{6}$$

$$\begin{aligned} \omega_7 = {} & \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 \\ & + \alpha_2 \wedge \alpha_3 \wedge \alpha_7 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \,, \end{aligned} \tag{7}$$

$$\begin{aligned} \omega_8 = {} & \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ & + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 \\ & - \alpha_3 \wedge \alpha_5 \wedge \alpha_6 \,. \end{aligned} \tag{8}$$

The subset $\Lambda^3_s V^*$ decomposes into six orbits. They are represented by the following forms

$$\omega_9 = 0 \,, \tag{9}$$

$$\omega_{10} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \,, \tag{10}$$

$$\omega_{11} = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5) \,, \tag{11}$$

(12) $\omega_{12} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$,

(13) $\omega_{13} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6$,

(14) $\omega_{14} = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$.

Let us recall that with a 3-form τ on a 6-dimensional space W we can associate an endomorphism $Q(\tau)$ in the following way. We choose a nonzero 6-form θ on W, and for $w \in W$ we define $Q(\tau)w$ by the formula

$$(\iota_w \tau) \wedge \tau = \iota_{Q(\tau)w} \theta .$$

We have

$$Q(\sigma_1) = 0 , \quad Q(\sigma_2) = 0 , \quad Q(\sigma_3)^2 = 0 , \quad \dim \operatorname{im} Q(\sigma_3) = 1 .$$

Replacing θ by $a\theta$ if necessary, we get moreover

$$Q(\rho_1)^2 = I , \quad \dim \operatorname{im}(Q(\rho_1) + I) = \dim \operatorname{im}(Q(\rho_1) - I) = 3 ,$$
$$Q(\rho_2)^2 = -I , \quad Q(\rho_3)^2 = 0, \dim \operatorname{im} Q(\rho_3) = 3 .$$

More information about the endomorphism Q you can find in [BV1].

Further, let ω be a 3-form on a 7-dimensional space V. We choose again a 7-form θ on V. Then we can define a symmetric bilinear form q on V by the formula

$$(\iota_v \omega) \wedge (\iota_{v'} \omega) \wedge \omega = q(v, v')\theta .$$

It is obvious that the definition of the symmetric bilinear form q depends on the choice of the 7-form θ. In other words the form q is determined up to a nonzero scalar multiple. More information about 3-forms on a 7-dimensional space you can find in [BV2].

Finally, for any 3-form ζ on a vector space Z we define

$$\Delta^2(\zeta) = \{z \in Z; (\iota_z \zeta)^{\wedge 2} = 0\} , \quad \Delta^3(\zeta) = \{z \in Z; (\iota_z \zeta)^{\wedge 3} = 0\} .$$

In the sequel we take the 3-forms $\omega_1, \ldots, \omega_{14}$ on the 7-dimensional space V, and consider their restrictions on all 6-dimensional subspaces $W \subset V$. I present the results without proofs. The proofs have computational character. For every restriction $\omega_i | W$ I have computed the corresponding endomorphism $Q(\omega_i | W)$, which (with the exceptions of the types (S1) and (S2)) enables to recognize type of the restriction $\omega_i | W$.

Type 1

For this form we have

$$\Delta^2(\omega_1) = V_3^a \cup V_3^b, \quad \text{where} \quad V_3^a = [e_3, e_4, e_7], \quad V_3^b = [e_5, e_6, e_7], \quad V_1 = V_3^a \cap V_3^b$$
$$\Delta^3(\omega_1) = V_6^a \cup V_6^b, \quad \text{where} \quad V_6^a = [e_1, e_3, e_4, e_5, e_6, e_7], \quad V_6^b = [e_2, e_3, e_4, e_5, e_6, e_7].$$

1. Proposition.

(S1) *There is no W such that $\omega_1|W$ is of type* (S1).
(S2) *$\omega_1|W$ is of type* (S2) *if and only if $W = V_6^a$ or $W = V_6^b$.*
(S3) *$\omega_1|W$ is of type* (S3) *if and only if $W \supset V_3^a$ or $W \supset V_3^b$ and $W \neq V_6^a, V_6^b$.*
(R1) *$\omega_1|W$ is of type* (R1) *if and only if $W \not\supset V_1$.*
(R2) *There is no W such that $\omega_1|W$ is of type* (R2).
(R3) *$\omega_1|W$ is of type* (R3) *if and only if $W \supset V_1$, $W \not\supset V_3^a$, and $W \not\supset V_3^b$.*

Type 2

Let us write $v = c_1e_1 + \cdots + c_7e_7$ and $v' = c_1'e_1 + \cdots + c_7'e_7$. For this form we have

$$\Delta^2(\omega_2) = \{v \in V; c_1 = c_2 = c_3 = c_4 = 0, c_5c_6 + c_6c_7 + c_7c_5 = 0\},$$
$$\Delta^3(\omega_2) = \{v \in V; c_1c_4 - c_2c_3 = 0\}.$$

Obviously, $\Delta^2(\omega_2)$ determines a subspace $V_3 \subset V$, $V_3 = [e_5, e_6, e_7]$. Moreover, on V we have a a symmetric bilinear form q (determined up to a nonzero multiple) defined by the formula

$$q(v, v') = c_1c_4' - c_2c_3' - c_3c_2' + c_4c_1'.$$

We can immediately see that $\ker q = V_3$. Consequently, q determines a regular symmetric bilinear form on V/V_3, and this one in turn determines a quadric $\mathcal{Q}$ in the projective space $P(V/V_3)$ associated with the vector space V/V_3. If $W \subset V$ is a subspace of codimension 1 such that $W \supset V_3$, then W determines a subspace of codimension 1 in V/V_3, and this one in turn determines a hyperplane $\mathcal{W}$ in the projective space $P(V/V_3)$. Finally, on V_3 we have a regular symmetric bilinear form q_3 (determined up to a nonzero multiple) defined by the formula

$$q_3(v, v') = c_5c_6' + c_5c_7' + c_6c_5' + c_6c_7' + c_7c_5' + c_7c_6'.$$

Let us remark that for each 2-dimensional subspace $Z \subset V_3$ the restriction $q_3|Z$ is a regular bilinear form.

2. Proposition.

(S1) *There is no W such that $\omega_2|W$ is of type* (S1).
(S2) *There is no W such that $\omega_2|W$ is of type* (S2).
(S3) *$\omega_2|W$ is of type* (S3) *if and only if $W \supset V_3$ and the hyperplane $\mathcal{W}$ is tangent to the quadric $\mathcal{Q}$.*

(R1) *$\omega_2|W$ is of type* (R1) *if and only if $W \not\supset V_3$ and the restriction $q_3|(W \cap V_3)$ is indefinite.*

(R2) *$\omega_2|W$ is of type* (R2) *if and only if $W \not\supset V_3$ and the restriction $q_3|(W \cap V_3)$ is definite.*

(R3) *$\omega_2|W$ is of type* (R3) *if and only if $W \supset V_3$ and the hyperplane $\mathcal{W}$ is not tangent to the quadric $\mathcal{Q}$.*

TYPE 3

For this form we have

$$\Delta^2(\omega_3) = \Delta^3(\omega_3) = V_6 = [e_2, e_3, e_4, e_5, e_6, e_7].$$

3. Proposition.

(S1) *$\omega_3|W$ is of type* (S1) *if and only if $W = V_6$.*

(S2) *There is no W such that $\omega_3|W$ is of type* (S2).

(S3) *$\omega_3|W$ is of type* (S3) *if and only if $W \neq V_6$.*

(R1) *There is no W such that $\omega_3|W$ is of type* (R1).

(R2) *There is no W such that $\omega_3|W$ is of type* (R2).

(R3) *There is no W such that $\omega_3|W$ is of type* (R2).

TYPE 4

For this form we have

$$\Delta^2(\omega_4) = V_3 = [e_3, e_5, e_7], \quad \Delta^3(\omega_4) = V_6 = [e_2, e_3, e_4, e_5, e_6, e_7].$$

4. Proposition.

(S1) *There is no W such that $\omega_4|W$ is of type* (S1).

(S2) *$\omega_4|W$ is of type* (S2) *if and only if $W = V_6$.*

(S3) *$\omega_4|W$ is of type* (S3) *if and only if $W \supset V_3$ and $W \neq V_6$.*

(R1) *There is no W such that $\omega_4|W$ is of type* (R1).

(R2) *There is no W such that $\omega_4|W$ is of type* (R2).

(R3) *$\omega_4|W$ is of type* (R3) *if and only if $W \not\supset V_3$.*

TYPE 5

Let us write again $v = c_1e_1 + \cdots + c_7e_7$ and $v' = c_1'e_1 + \cdots + c_7'e_7$. For this form we have

$$\Delta^2(\omega_5) = \{0\}, \quad \Delta^3(\omega_5) = \{v \in V; -c_1^2 - c_2^2 - c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2 = 0\}.$$

This time again, on V we have a a symmetric bilinear form q (determined up to a nonzero multiple) defined by the formula

$$q(v, v') = -c_1c_1' - c_2c_2' - c_3c_3' + c_4c_4' + c_5c_5' + c_6c_6' + c_7c_7'.$$

This form has obviously signature $\{3,4\}$. (We use this notation in order to underline that the bilinear form is determined up to a nonzero multiple. Depending on our choice it can have signature $(4,3)$ or $(3,4)$.)

5. Proposition.

(S1) *There is no W such that $\omega_5|W$ is of type* (S1).
(S2) *There is no W such that $\omega_5|W$ is of type* (S2).
(S3) *There is no W such that $\omega_5|W$ is of type* (S3).
(R1) *$\omega_5|W$ is of type* (R1) *if and only if the restriction $q|W$ is a regular form of signature* $\{3,3\}$.
(R2) *$\omega_5|W$ is of type* (R2) *if and only if the restriction $q|W$ is a regular form of signature* $\{2,4\}$.
(R3) *$\omega_5|W$ is of type* (R3) *if and only if the restriction $q|W$ is a singular form.*

TYPE 6

For this form we have

$$\Delta^2(\omega_6) = V_1 = [e_7], \quad \Delta^3(\omega_6) = V_5 = [e_3, e_4, e_5, e_6, e_7].$$

6. Proposition.

(S1) *There is no W such that $\omega_6|W$ is of type* (S1).
(S2) *There is no W such that $\omega_6|W$ is of type* (S2).
(S3) *$\omega_6|W$ is of type* (S3) *if and only if $W \supset V_5$.*
(R1) *There is no W such that $\omega_6|W$ is of type* (R1).
(R2) *$\omega_6|W$ is of type (R2) if and only if $W \not\supset V_1$.*
(R3) *$\omega_6|W$ is of type* (R3) *if and only if $W \supset V_1$ and $W \not\supset V_5$.*

TYPE 7

For this form we have

$$\Delta^2(\omega_7) = \{0\}, \quad \Delta^3(\omega_7) = V_3 = [e_5, e_6, e_7].$$

7. Proposition.

(S1) *There is no W such that $\omega_7|W$ is of type* (S1).
(S2) *There is no W such that $\omega_7|W$ is of type* (S2).
(S3) *There is no W such that $\omega_7|W$ is of type* (S3).
(R1) *There is no W such that $\omega_7|W$ is of type* (R1).
(R2) *$\omega_7|W$ is of type* (R2) *if and only if $W \not\supset V_3$.*
(R3) *$\omega_7|W$ is of type* (R3) *if and only if $W \supset V_3$.*

TYPE 8

For this form we have

$$\Delta^2(\omega_8) = \Delta^3(\omega_8) = \{0\}.$$

8. Proposition. *The restriction $\omega_8|W$ is always of type* (R2).

TYPE 9

9. Proposition. *The restriction $\omega_9|W$ is always of type* (S1).

TYPE 10

For this form we have $\ker\omega_{10} = V_4 = [e_4, e_5, e_6, e_7]$.

10. Proposition.

(S1) $\omega_{10}|W$ *is of type* (S1) *if and only if* $W \supset V_4$.
(S2) $\omega_{10}|W$ *is of type* (S2) *if and only if* $W \not\supset V_4$.
(S3) *There is no* W *such that* $\omega_{10}|W$ *is of type* (S3).
(R1) *There is no* W *such that* $\omega_{10}|W$ *is of type* (R1).
(R2) *There is no* W *such that* $\omega_{10}|W$ *is of type* (R2).
(R3) *There is no* W *such that* $\omega_{10}|W$ *is of type* (R3).

TYPE 11

For this form we have

$$\ker\omega_{11} = V_2 = [e_6, e_7] \quad \text{and} \quad \Delta^2(\omega_{11}) = V_6 = [e_2, e_3, e_4, e_5, e_6, e_7].$$

11. Proposition.

(S1) $\omega_{11}|W$ *is of type* (S1) *if and only if* $W = V_6$.
(S2) $\omega_{11}|W$ *is of type* (S2) *if and only if* $W \supset V_2$ *and* $W \neq V_6$.
(S3) $\omega_{11}|W$ *is of type* (S3) *if and only if* $W \not\supset V_2$.
(R1) *There is no* W *such that* $\omega_{11}|W$ *is of type* (R1).
(R2) *There is no* W *such that* $\omega_{11}|W$ *is of type* (R2).
(R3) *There is no* W *such that* $\omega_{11}|W$ *is of type* (R3).

TYPE 12

For this form we have

$$\ker\omega_{12} = V_1 = [e_7] \quad \text{and} \quad \Delta^2(\omega_{12}) = V_4^a \cup V_4^b,$$

where $V_4^a = [e_1, e_2, e_3, e_7]$ and $V_4^b = [e_4, e_5, e_6, e_7]$.

12. Proposition.

(S1) *There is no* W *such that* $\omega_{12}|W$ *is of type* (S1).
(S2) $\omega_{12}|W$ *is of type* (S2) *if and only if* $W \supset V_4^a$ *or* $W \supset V_4^b$.
(S3) $\omega_{12}|W$ *is of type* (S3) *if and only if* $W \supset V_1$, $W \not\supset V_4^a$, *and* $W \not\supset V_4^b$.
(R1) $\omega_{12}|W$ *is of type* (R1) *if and only if* $W \not\supset V_1$.
(R2) *There is no* W *such that* $\omega_{12}|W$ *is of type* (R2).
(R3) *There is no* W *such that* $\omega_{12}|W$ *is of type* (R3).

Type 13

For this form we have

$$\ker\omega_{13} = \Delta^2(\omega_{13}) = V_1 = [e_7]\,.$$

13. Proposition.

(S1) *There is no W such that $\omega_{13}|W$ is of type* (S1).
(S2) *There is no W such that $\omega_{13}|W$ is of type* (S2).
(S3) *$\omega_{13}|W$ is of type* (S3) *if and only if $W \supset V_1$.*
(R1) *There is no W such that $\omega_{13}|W$ is of type* (R1).
(R2) *$\omega_{13}|W$ is of type* (R2) *if and only if $W \not\supset V_1$.*
(R3) *There is no W such that $\omega_{13}|W$ is of type* (R3).

Type 14

For this form we have

$$\ker\omega_{14} = V_1 = [e_7] \quad \text{and} \quad \Delta^2(\omega_{14}) = V_4 = [e_1, e_2, e_3, e_7]\,.$$

14. Proposition.

(S1) *There is no W such that $\omega_{14}|W$ is of type* (S1).
(S2) *$\omega_{14}|W$ is of type* (S2) *if and only if $W \supset V_4$.*
(S3) *$\omega_{14}|W$ is of type* (S3) *if and only if $W \not\supset V_4$ and $W \supset V_1$.*
(R1) *There is no W such that $\omega_{14}|W$ is of type* (R1).
(R2) *There is no W such that $\omega_{14}|W$ is of type* (R2).
(R3) *$\omega_{14}|W$ is of type* (R3) *if and only if $W \not\supset V_1$.*

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