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TWO CONSTRUCTIONS WITH PARABOLIC GEOMETRIES

ANDREAS ČAP

ABSTRACT. This is an expanded version of a series of lectures delivered at the 25th Winter School "Geometry and Physics" in Srni.

After a short introduction to Cartan geometries and parabolic geometries, we give a detailed description of the equivalence between parabolic geometries and underlying geometric structures.

The second part of the paper is devoted to constructions which relate parabolic geometries of different type. First we discuss the construction of correspondence spaces and twistor spaces, which is related to nested parabolic subgroups in the same semisimple Lie group. An example related to twistor theory for Grassmannian structures and the geometry of second order ODE's is discussed in detail.

In the last part, we discuss analogs of the Fefferman construction, which relate geometries corresponding different semisimple Lie groups.

1. INTRODUCTION

This is an expanded version of a series of plenary lectures at the 25th Winter School "Geometry and Physics" in Srni. I would like to thank the organizers for giving me the opportunity to present this series.

The concept which is nowadays known as a Cartan geometry was introduced by E. Cartan under the name "generalized space" in order to build a bridge between geometry in the sense of F. Klein's Erlangen program and differential geometry. This concept associates to an arbitrary homogeneous space G/H the notion of a *Cartan geometry of type* (G, H), which is a differential geometric structure on smooth manifolds whose dimension equals the dimension of G/H. A manifold endowed with such a geometry can be considered as a "curved analog" of the homogeneous spaces G/H. Although Cartan geometries are an extremely general concept, there are several remarkable results which hold for all of them, see 2.2.

The most interesting examples of Cartan geometries are those, in which the Cartan geometry is equivalent to some simpler underlying structure. Obtaining the Cartan geometry from the underlying structure usually is a highly nontrivial process which often involves prolongation. Cartan himself found many examples of this situation,

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ranging from conformal and projective structures via 3-dimensional CR structures to generic rank two distributions in manifolds of dimension five.

Parabolic geometries are Cartan geometries of type (G, P), where G is a semisimple Lie group and $P \subset G$ is a parabolic subgroup. The corresponding homogeneous spaces G/P are the so-called generalized flag manifolds which are among the most important examples of homogeneous spaces. Under the conditions of regularity and normality, parabolic geometries always are equivalent to underlying structures. This basically goes back to the pioneering works of N. Tanaka, see e.g. [29].

In section 2 of this article we give a precise description of the underlying structures which are equivalent to regular normal parabolic geometries. In this underlying picture, the structures are very diverse, including in particular the four examples of structures listed above. From that point of view, parabolic geometries offer a unified approach to a broad variety of geometric structures.

Some of the advantages of this unified approach will be discussed in the remaining two sections. They are devoted to constructions which relate parabolic geometries of different types. The common feature of these constructions is that they are quite transparent in the picture of Cartan geometries, while from the point of view of the underlying structures they are often surprising.

Section 3 is devoted to the construction of correspondence spaces, which is associated to nested parabolic subgroups in one semisimple Lie group. Trying to characterize the geometries obtained in that way, one is lead to the notion of a twistor space and obtains several classical examples of twistor theory. In the end one arrives at a complete local characterization of correspondence spaces in terms of the harmonic curvature. We give a detailed discussion of one example of this situation related to the geometry of systems of second order ODE's.

The last section is devoted to Fefferman's construction of a conformal structure on the total space of a circle bundle over a CR manifold and analogs of this construction. From the point of view of Cartan geometries, the basic input for these constructions is an inclusion $i : G \to \tilde{G}$ between semisimple groups together with appropriately chosen parabolic subgroups $P \subset G$ and $\tilde{P} \subset \tilde{G}$. Then the construction relates geometries of type (G, P) to geometries of type $(\tilde{G}, \tilde{P})$.

2. Cartan geometries and parabolic geometries

We start with some general background on Cartan geometries.

2.1. Homogeneous spaces and the Maurer Cartan form. Let G be any Lie group and let $H \subset G$ be a closed subgroup. The basic idea behind Cartan geometries is to endow the homogeneous space G/H with a geometric structure, whose automorphisms are exactly the left actions of the elements of G. The natural projection $G \to G/H$ is well known to be a principal bundle with structure group H. Left multiplication by $g \in G$ lifts the action of g on G/H to an automorphism of this principal bundle. Of course, the group of principal bundle automorphisms of $G \to G/H$ is much bigger than just the left translations, so an additional ingredient is needed to recognize left translations.

It turns out that the right ingredient is the (left) *Maurer Cartan form* $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$. Recall that this is just a different way to encode the trivialization of the tangent bundle TG by left translations. By definition, for $\xi \in T_gG$ we have

$$\omega^{MC}(\xi) = T\lambda_{g^{-1}} \cdot \xi \in T_eG = \mathfrak{g},$$

where $\lambda_{g^{-1}}$ denotes left translation by g^{-1}.

Proposition. *Let G be a Lie group and let $H \subset G$ be a closed subgroup such that the homogeneous space G/H is connected. Then the left translations λ_g are exactly the principal bundle automorphisms of $G \to G/H$ which pull back ω^{MC} to itself.*

For later use, we note some further properties of ω^{MC}. Denoting by L_X the left invariant vector field generated by $X \in \mathfrak{g}$, we by definition have $\omega^{MC}(L_X) = X$.

Note that for $X \in \mathfrak{h}$, the vector field L_X coincides with the fundamental vector field ζ_X on the principal bundle $G \to G/H$ generated by X. For $g \in H$, consider the right translation r^g by g. Using that the adjoint action of g is the derivative of the conjugation by g, one immediately verifies that $(r^g)^*\omega^{MC} = \mathrm{Ad}(g^{-1}) \circ \omega^{MC}$. Note that for $g \in H$, the map r^g is the principal right action on the bundle $G \to G/H$. Finally, there is the Maurer–Cartan equation: The fact that $[L_X, L_X] = L_{[X,Y]}$ for all $X, Y \in \mathfrak{g}$ implies that $d\omega^{MC}(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$ for all vector fields ξ and η on G.

2.2. **Cartan geometries.** The definition of a Cartan geometry is now obtained by replacing $G \to G/H$ by an arbitrary principal H-bundle and ω^{MC} by a form which has all the properties of ω^{MC} that make sense in the more general setting.

Definition. (1) A *Cartan geometry* of type (G, H) on a smooth manifold M is a principal H-bundle $p : \mathcal{G} \to M$ together with a one form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ (the *Cartan connection*) such that

- $(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega$ for all $h \in H$.
- $\omega(\zeta_A) = A$ for all $A \in \mathfrak{h}$.
- $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

(2) A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ is a principal bundle homomorphism $\Phi : \mathcal{G} \to \tilde{\mathcal{G}}$ such that $\Phi^*\tilde{\omega} = \omega$.
(3) The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, H) is defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$.

Notice that a Cartan geometry is a local structure, i.e. it can be restricted to open subsets: For $(p : \mathcal{G} \to M, \omega)$ and an open subset $U \subset M$, we simply have the restriction $(p : p^{-1}(U) \to U, \omega|_{p^{-1}(U)})$. The curvature evidently is a local invariant, i.e. the curvature of this restricted geometry is the restriction of the original curvature.

Any morphism Φ between two Cartan geometries as in (2) has an underlying smooth map $\varphi : M \to \tilde{M}$. It turns out (see [27, chapter 5]) that Φ is determined by φ up to a smooth function from M to the maximal normal subgroup of G which is contained in H. In all cases of interest, this subgroup is trivial or at least discrete, whence this map has to be locally constant. In fact, it is necessary to included the possibility of having various morphisms covering the same base map to deal with structures analogous to Spin structures.

By definition $(G \to G/H, \omega^{MC})$ is a Cartan geometry of type (G, H), and Proposition 2.1 exactly tells us that the automorphisms of this geometry are exactly the left translations by elements of G. This geometry is called the *homogeneous model* of Cartan geometries of type (G, H).

The Maurer–Cartan equation noted in the end of 2.1 exactly says that the curvature of the homogeneous model vanishes identically. Indeed, the curvature exactly measures to what extent the Maurer–Cartan equation fails to hold. One of the nice features of Cartan geometries is that vanishing of the curvature characterizes the homogeneous model locally, i.e. any Cartan geometry of type (G, H) with vanishing curvature is locally isomorphic to $(G \to G/H, \omega^{MC})$, see [27, chapter 5]. More generally, the curvature (at least in principle) provides a solution to the equivalence problem. This is one of the reasons why already associating to some geometric structure a canonical Cartan connection is a powerful result. For the main part of the theory of parabolic geometries however, the existence of a canonical Cartan connection is only the starting point.

There are other interesting features of general Cartan geometries, for example:

- For any Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, H), the automorphism group $\mathrm{Aut}(\mathcal{G}, \omega)$ is a Lie group of dimension $\leq \dim(G)$. The Lie algebra $\mathrm{aut}(\mathcal{G}, \omega)$ can be described completely, and analyzing its algebraic structure leads to interesting results, see [9].
- The homogeneous model $(G \to G/H, \omega^{MC})$ satisfies a Liouville type theorem. If U and V are connected open subsets of G/H then any isomorphism between the restrictions of the geometry to these open subsets uniquely extends to an automorphism of the homogeneous model.
- There are various general tools available for Cartan geometries, for example the notions of distinguished curves and of normal coordinates.

2.3. **Cartan geometries determined by underlying structures.** The results listed above become particularly powerful if a Cartan geometry is obtained as an equivalent description of some underlying geometric structure. A very simple example is provided by Riemannian geometries, which correspond to the case that G is the Euclidean group $\mathrm{Euc}(n)$ and $H = O(n)$. The Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h} \oplus \mathbb{R}^n$ as an H-module. Therefore, a Cartan connection of type (G, H) on a principal $O(n)$-bundle $\mathcal{G} \to M$ decomposes into an $\mathbb{R}^n$-valued form θ and a $\mathfrak{h}$-valued form γ which both are H-equivariant. Then θ defines a reduction of the linear frame bundle of M to the structure group $O(n)$, which is equivalent to a Riemannian metric on M. The form γ defines a principal connection on $\mathcal{G}$ which is equivalent to a metric connection on M. If γ is torsion free, then it must be the Levi–Civita connection. Conversely, starting from a Riemannian manifold, one obtains a torsion free Cartan geometry by using the orthonormal frame bundle endowed with the soldering form and the Levi–Civita connection. In that way, one obtains an equivalence between torsion free Cartan geometries of type (G, H) and n-dimensional Riemannian manifolds.

The results discussed above then imply

- The isometry group of any Riemannian manifold is a Lie group of dimension $\leq \frac{1}{2}\dim(M)(\dim(M)+1)$.

- Any isometry between two connected open subsets of Euclidean space is the restriction of a uniquely determined Euclidean motion.
- The concepts of geodesics and Riemann normal coordinates for Riemannian manifolds.

The case of Riemannian metrics is rather easy, since the bundle $\mathcal{G}$ can be directly obtained from the underlying structure. In other cases, one also has to construct this principal bundle, a process which is usually called *prolongation*. This also leads to additional features. Let us discuss this in the case of conformal structures, which is a model case for parabolic geometries:

A conformal structure on a smooth manifold M is given by an equivalence class $[g]$ of Riemannian metrics on M. Here two metrics g and $\hat{g}$ are considered as equivalent if and only if $\hat{g} = e^{2f}g$ for some smooth function f on M. Equivalently, a conformal structure can be defined as a reduction of structure group of the frame bundle $\mathcal{P}M$ to the group $CO(n)$ of linear conformal isometries of $\mathbb{R}^n$.

It is a classical result of E. Cartan, see [16] that for $n = \dim(M) \geq 3$ conformal structures admit a canonical normal Cartan connection. Consider the semisimple Lie group $G := SO(n+1, 1)$. This naturally acts on $\mathbb{R}^{n+2}$ and the action preserves the null cone. Fix a nonzero null vector v and let $P \subset G$ be the stabilizer of the line $\mathbb{R}v$. Then P is an example of a *parabolic subgroup* of the semisimple Lie group G. It turns out that P contains an Abelian normal subgroup $P_+ \cong \mathbb{R}^n$ such that $P/P_+ =: G_0 \cong CO(n)$.

The relation of these groups to conformal geometry is the following: The group G acts transitively on the space of null lines in $\mathbb{R}^{n+2}$, which is easily seen to be isomorphic to S^n. Since by definition P is the stabilizer of one null line we get $G/P \cong S^n$ and this identifies G with the group of conformal isometries of S^n and P with the group of conformal isometries fixing a point $x_0 \in S^n$. It turns out that the projection from P to $G_0 \cong CO(n)$ is given by passing from a conformal isometry fixing x_0 to its tangent map in x_0, see [17] for more details.

Now consider a manifold M of dimension n endowed with a conformal structure $[g]$. The corresponding reduction of structure group is a G_0-principal bundle $p_0 : \mathcal{G}_0 \to M$ endowed with a canonical differential form θ called the *soldering form*. Cartan's result states that this bundle can be canonically extended to a principal bundle $p : \mathcal{G} \to M$ with structure group P and the soldering form θ can be extended to a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. If one requires the Cartan connection ω to satisfy a normalization condition, then it is uniquely determined.

Conversely, given a principal P-bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, one obtains a G_0-principal bundle $\mathcal{G}_0 := \mathcal{G}/P_+ \to M$ and ω induces a soldering form on that bundle, thus giving rise to a conformal structure on M. In the end one obtains an equivalence of categories between conformal structures and Cartan geometries of type (G, P).

The additional feature provided by this is that one obtains new geometric objects. Viewing a conformal structure as a reduction to the structure group $CO(n)$ of the linear frame bundle, one obtains a natural vector bundle associated to each representation of $CO(n)$. Since $CO(n)$ is a quotient of P, this gives rise to a representation of P and the resulting vector bundles can also be viewed as associated to the Cartan bundle $\mathcal{G}$. But the group P admits more general representations than those coming from G_0, and these give rise to new natural vector bundles and thus new geometric objects.

A particularly interesting case is to consider restrictions to P of representations of G. This leads to the so-called tractor bundles, see [1, 10].

In a series of pioneering papers in the 1960's and 70's culminating in [29], N. Tanaka showed that for all semisimple Lie groups and parabolic subgroups normal Cartan geometries are determined by underlying structures. These results have been put into the more general context of filtered manifolds in the work of T. Morimoto (see e.g. [23]) and a new version of the result tailored to the parabolic case was given in [12]. Otherwise put, these results show that these underlying structures (which seemingly are very diverse) admit canonical Cartan connections. Our next aim is to give a uniform description of the underlying structures.

2.4. **Generalized flag manifolds.** We first collect some background on the homogeneous models of parabolic geometries. We will use elementary definitions, which avoid structure theory of Lie algebras.

Definition. Let $\mathfrak{g}$ be a semisimple Lie algebra. A $|k|$-*grading* on $\mathfrak{g}$ is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$.

For given $\mathfrak{g}$ there is a simple complete description of such gradings (up to isomorphism) in terms of structure theory. For complex $\mathfrak{g}$, they are in bijective correspondence with sets of simple roots of $\mathfrak{g}$ and hence are conveniently denoted by Dynkin diagrams with crosses. For real $\mathfrak{g}$ there is a similar description in terms of the Satake diagram.

Let us make this more explicit for the case $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Up to isomorphism, each $|k|$-grading is determined by a block decomposition of matrices: One decomposes $\mathbb{R}^{n+1}$ into $k+1$ blocks of sizes $i_0, \dots, i_k$. The $\mathfrak{g}_0$ consists of all block diagonal matrices, and for $i > 0$, the component $\mathfrak{g}_i$ (respectively $\mathfrak{g}_{-i}$) consists of those matrices, which only have nonzero entries in the ith blocks above (respectively below) the main diagonal. The corresponding Dynkin diagram is obtained as follows: Look at the matrix entries in the first diagonal above the main diagonal. The block in which they are contained either lies in $\mathfrak{g}_0$ or in $\mathfrak{g}_1$. Use a dot in the first and a cross in the second case and connect each entry with a line to its (one or two) neighbors. More explicitly, consider $\mathfrak{sl}(4, \mathbb{K})$ with blocks of sizes 1, 1, and 2. Then one obtains a $|2|$-grading of the form

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 \end{pmatrix}$$

and the corresponding Dynkin (respectively Satake) diagram with crosses is ×—×—○.

Putting $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ we obtain a filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^k$ of $\mathfrak{g}$ such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. In particular, $\mathfrak{p} := \mathfrak{g}^0$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_+ := \mathfrak{g}^1$ is a nilpotent ideal in $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is a semidirect sum. The subalgebras $\mathfrak{p}$ obtained in that way are exactly the *parabolic* subalgebras of $\mathfrak{g}$ used in representation theory. In the complex case, a subalgebra of $\mathfrak{g}$ is parabolic if and only if it contains a maximal solvable subalgebra (i.e. a Borel subalgebra) of $\mathfrak{g}$. In the real case, parabolic subalgebras are defined via complexification.

Given a (not necessarily connected) Lie group G with Lie algebra $\mathfrak{g}$, it turns out that the normalizer $P := N_G(\mathfrak{p})$ of $\mathfrak{p}$ in G has Lie algebra $\mathfrak{p}$. This is the *parabolic subgroup* of G associated to the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. It turns out that for $g \in P$, the adjoint action $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ not only preserves the filtration component $\mathfrak{g}^0$ but all filtration components $\mathfrak{g}^i$. Indeed, the whole filtration can be reconstructed algebraically from $\mathfrak{g}^0 = \mathfrak{p}$. Further, one defines a closed subgroup $G_0 \subset P$ as the set of those $g \in P$, whose adjoint action even preserves the grading of $\mathfrak{g}$. Then G_0 is reductive and has Lie algebra $\mathfrak{g}_0$. One shows that exp defines a diffeomorphism from $\mathfrak{p}_+$ onto a closed subgroup $P_+ \subset P$ and P is the semidirect product of G_0 and P_+.

A *generalized flag variety* is a homogeneous space G/P for a semisimple Lie group G and a parabolic subgroup $P \subset G$. These homogeneous spaces are always compact and for complex G they are the only compact homogeneous spaces of G. In the complex case, G/P carries a Kähler metric. Generalized flag manifolds are among the most important examples of homogeneous spaces. They show up in many areas of mathematics.

Parabolic geometries are Cartan geometries of type (G, P) for G and P as above. In 2.3 we have seen that for an appropriate choice of G and P, such a structure (satisfying an additional normalization condition) is equivalent to a conformal Riemannian structure. Under the conditions of regularity and normality, general parabolic geometries are equivalent to a certain underlying structure. We will next describe how this underlying structure is obtained.

2.5. **The filtration of the tangent bundle.** We first show how a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) gives rise to a filtration of the tangent bundle TM. Define the *adjoint tractor bundle* $\mathcal{A}M$ of M as $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$. (This is an important example of the concept of tractor bundles discussed in 2.3.) Then we have the P-invariant filtration $\{\mathfrak{g}^i\}$ of $\mathfrak{g}$, which gives rise to a filtration

$$\mathcal{A}M = \mathcal{A}^{-k}M \supset \mathcal{A}^{-k+1}M \supset \cdots \supset \mathcal{A}^{k}M$$

of the adjoint tractor bundle by smooth subbundles. The Lie bracket on $\mathfrak{g}$ induces a tensorial map $\{\ ,\ \} : \mathcal{A}M \times \mathcal{A}M \to \mathcal{A}M$. In particular, each fiber of $\mathcal{A}M$ is a filtered Lie algebra isomorphic to $\mathfrak{g}$.

The Cartan connection ω leads to an identification $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, with the action coming from the adjoint action. The Killing form of $\mathfrak{g}$ induces a duality between this P-module and $\mathfrak{p}_+ = \mathfrak{g}^1$, so $T^*M \cong \mathcal{G} \times_P \mathfrak{p}_+ = \mathcal{A}^1M$. Hence T^*M is a bundle of nilpotent filtered Lie algebras. On the tangent bundle, there are similar but more subtle structures: From above, we see that $TM \cong \mathcal{A}M/\mathcal{A}^0M$, and we obtain an induced filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of the tangent bundle by putting $T^iM := \mathcal{A}^iM/\mathcal{A}^0M$. The associated graded bundle is

$$\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus \cdots \oplus \mathrm{gr}_{-1}(TM),$$

where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$. By construction, this implies that $\mathrm{gr}_i(TM) \cong \mathcal{G} \times_P \mathfrak{g}^i/\mathfrak{g}^{i+1}$. By definition, the subgroup $P_+ \subset P$ acts trivially on this quotient. Hence the P-action factorizes over $P/P_+ \cong G_0$ and as a G_0-module we have $\mathfrak{g}^i/\mathfrak{g}^{i+1} \cong \mathfrak{g}_i$.

On the level of principal bundles, we observe that the subgroup $P_+ \subset P$ acts freely on $\mathcal{G}$. Hence the quotient $\mathcal{G}_0 := \mathcal{G}/P_+$ is a principal bundle over M with structure group $P/P_+ = G_0$. The Cartan connection ω induces a bundle map from $\mathcal{G}_0$ to the frame bundle of $\mathrm{gr}(TM)$ which defines a reduction of structure group. In particular, $\mathrm{gr}_i(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$, which is a refined version of the identification of the representation spaces above. Putting the components together, we see that $\mathrm{gr}(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$. The Lie bracket on $\mathfrak{g}_-$ is G_0-invariant and hence gives rise to a tensorial map $\{\ ,\ \}$ on $\mathrm{gr}(TM)$. Hence for each $x \in M$, the space $\mathrm{gr}(T_xM)$ is a nilpotent graded Lie algebra isomorphic to $\mathfrak{g}_-$.

2.6. **Filtered manifolds and their symbol algebras.** A crucial observation for the sequel is that under a weak condition, a filtration of the tangent bundle of a manifold gives rise to the structure of a nilpotent graded Lie algebra on the associated graded of each tangent space.

A *filtered manifold* is a smooth manifold M together with a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of the tangent bundle by smooth subbundles, which is compatible with the Lie bracket of vector fields, i.e. such that for $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ one always has $[\xi, \eta] \in \Gamma(T^{i+j}M)$.

Let $q : T^{i+j}M \to T^{i+j}M/T^{i+j+1}M = \mathrm{gr}_{i+j}(TM)$ be the natural map, and consider the operator $\Gamma(T^iM) \times \Gamma(T^jM) \to \Gamma(\mathrm{gr}_{i+j}(TM))$ defined by $(\xi, \eta) \mapsto q([\xi, \eta])$. Since the indices of the filtration components are always negative, the bundles T^iM and T^jM are contained in $T^{i+j+1}M$, which implies that this operator is bilinear over smooth functions. Therefore, it is induced by a tensor $T^iM \times T^jM \to \mathrm{gr}_{i+j}(TM)$. If $\xi \in \Gamma(T^{i+1}M)$, then $[\xi, \eta] \in \Gamma(T^{i+j+1}M)$ so the result of this tensor depends only on the classes of ξ in $\mathrm{gr}_i(TM)$ and $\eta \in \mathrm{gr}_j(TM)$. Taking together the various components, we obtain a tensor $\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$ which is called the *Levi bracket*. By construction, this makes each of the spaces $\mathrm{gr}(T_xM)$ into a nilpotent graded Lie algebra, called the *symbol algebra* of the filtered manifold at the point x. Consider a local isomorphism between filtered manifolds, i.e. a local diffeomorphism f such that each of the maps T_xf is an isomorphism of filtered vector spaces. Then each T_xf induces and isomorphism between the associated graded spaces to the tangent spaces, which is easily seen to be an isomorphism of the symbol algebras.

Therefore, the symbol algebra should be considered as the first order approximation of a filtered manifold in a point, similarly to the tangent space at a point of an ordinary manifold. The usual tangent space (viewed as an Abelian Lie algebra) is recovered in the case of the trivial filtration $T^{-1}M = TM$.

A priory, the isomorphism class of the symbol algebra may change from point to point, but the case that all symbol algebras are isomorphic to a fixed nilpotent graded Lie algebra $\mathfrak{a}$ is of particular interest. In this case, there is a natural frame bundle for the vector bundle $\mathrm{gr}(TM)$ with structure group the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{a})$ of all automorphisms of the Lie algebra $\mathfrak{a}$, which in addition preserve the grading. This is the replacement for the usual frame bundle of a smooth manifold, which is again recovered in the special case of the trivial filtration.

2.7. **Regularity and normality.** Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P). Then we have the curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of ω as introduced in 2.2. The defining properties of K easily imply that it is horizontal and P-equivariant, so it defines a two-form κ on M with values in the bundle $\mathcal{G} \times_P \mathfrak{g} = \mathcal{A}M$. Hence the Cartan curvature can be viewed as a two form on M with values in the adjoint tractor bundle.

The geometry $(p : \mathcal{G} \to M, \omega)$ is called *regular* if and only if the curvature κ has the property that $\kappa(T^iM, T^jM) \subset \mathcal{A}^{i+j+1}M$ for all $i, j < 0$. Otherwise put, regularity means that the curvature is concentrated in positive homogeneities.

Recall that a Cartan geometry of type (G, P) is called *torsion free*, if its curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ actually has values in $\mathfrak{p} \subset \mathfrak{g}$. In the parabolic case, this can be nicely reformulated as κ lying in the subspace $\Omega^2(M, \mathcal{A}^0M) \subset \Omega^2(M, \mathcal{A}M)$. From this description, it is evident that torsion free parabolic geometries are automatically regular, so regularity can be viewed as a condition avoiding particularly bad types of torsion. Note that the condition is vacuous for $|1|$-gradings.

The geometric meaning of the regularity condition is easy to describe (and also easy to prove):

Proposition. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P), let T^iM be the induced filtration components in the tangent bundle, and let $\{\ ,\ \}$ be the tensorial Lie bracket on $\mathrm{gr}(TM)$ introduced in 2.5.*

Then the geometry is regular if and only if the T^iM make M into a filtered manifold such that the natural bracket on each symbol algebra coincides with $\{\ ,\ \}$. In particular, each symbol algebra is isomorphic to $\mathfrak{g}_-$.

For regular geometries, the bundle $\mathcal{G}_0 \to M$ from 2.5 nicely ties into the concepts for filtered manifolds. The adjoint action of G_0 on $\mathfrak{g}_-$ is by Lie algebra automorphisms which preserve the grading (by definition of G_0), so it defines a homomorphism $G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. This homomorphism turns out to be infinitesimally injective provided that none of the simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$. This condition is very harmless, since simple ideals contained in $\mathfrak{g}_0$ can be left out without problems, so we will assume throughout that it is satisfied. As we have noted in 2.6, the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ is the natural structure group for the vector bundle $\mathrm{gr}(TM)$ since each symbol algebra is isomorphic to $\mathfrak{g}_-$. The bundle $\mathcal{G}_0$ can thus be interpreted as the filtered manifold version of a first order G_0-structure.

Now we have collected the two structures underlying a regular parabolic geometry of type (G, P) that we will need in the sequel:

- A filtration $\{T^iM\}$ of the tangent bundle such that each symbol algebra is isomorphic to $\mathfrak{g}_-$.
- A reduction of structure group of the associated graded $\mathrm{gr}(TM)$ to the structure group $G_0 \subset \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$.

Similarly to the soldering form used for classical first order structures, this reduction of structure group can be expressed by certain partially defined differential forms on the bundle $\mathcal{G}_0$. This leads to the description of underlying structures used in [12]. The collection of these two underlying structures is called a *regular infinitesimal flag structure*, see [13].

Fixing the underlying regular infinitesimal flag structure still leaves a lot of freedom for the Cartan connection ω, so we need an additional *normalization condition*: Recall the the cotangent bundle T^*M can be naturally viewed as $\mathcal{G} \times_P \mathfrak{p}_+ = \mathcal{A}^1M$. Hence it naturally is a bundle of nilpotent Lie algebras with the restriction of the algebraic bracket $\{\ ,\ \}$ of $\mathcal{A}M$. Now for $\ell > 0$ we define a tensorial operator $\partial^* : \Lambda^\ell T^*M \otimes \mathcal{A}M \to \Lambda^{\ell-1}T^*M \otimes \mathcal{A}M$ by

$$\partial^*(\alpha_1 \wedge \cdots \wedge \alpha_\ell \otimes s) := \sum_{i=1}^{\ell} (-1)^i \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_\ell \otimes \{\alpha_i, s\}$$
$$+ \sum_{i<j} (-1)^{i+j} \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \widehat{\alpha_j} \wedge \cdots \wedge \alpha_\ell \otimes s$$

for $\alpha_r \in T^*M$ and $s \in \mathcal{A}M$, where as usual the hats denote omission. This is the differential in the standard complex computing Lie algebra homology. In particular, $\partial^* \circ \partial^* = 0$, and the quotients $\ker(\partial^*)/\operatorname{im}(\partial^*)$ are the pointwise Lie algebra homologies of the Lie algebras T^*_xM with coefficients in the modules $\mathcal{A}_xM$.

The homology groups $H_*(\mathfrak{p}_+, \mathfrak{g})$ are naturally P-modules and it is easy to see that P_+ acts trivially, so they are obtained by trivially extending the action of G_0. Hence the above bundles $\ker(\partial^*)/\operatorname{im}(\partial^*)$ can be naturally viewed as either $\mathcal{G} \times_P H_\ell(\mathfrak{p}_+, \mathfrak{g})$ or $\mathcal{G}_0 \times_{G_0} H_\ell(\mathfrak{p}_+, \mathfrak{g})$. The latter interpretation shows that they can be directly interpreted in terms of the underlying structure. It is a crucial point in the theory that the G_0-representations $H_\ell(\mathfrak{p}_+, \mathfrak{g})$ can be computed explicitly and algorithmically using Kostant's version of the Bott–Borel–Weil theorem, see [30]. (In that reference, as well as in large parts of the literature, cohomology groups rather than homology groups are used, but switching between the two points of view is easy.)

A parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is called *normal* if and only if its curvature κ has the property that $\partial^*(\kappa) = 0$.

Theorem. *Let $(M, \{T^iM\})$ be a filtered manifold such that each symbol algebra is isomorphic to $\mathfrak{g}_-$, and let $\mathcal{G}_0 \to M$ be a reduction of $\operatorname{gr}(TM)$ to the structure group $G_0 \subset \operatorname{Aut}_{\operatorname{gr}}(\mathfrak{g}_-)$. Then there is a regular normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$ inducing the given data. If $H_1(\mathfrak{p}_+, \mathfrak{g})$ is concentrated in non-positive homogeneous degrees, then the pair $(\mathcal{G}, \omega)$ is unique up to isomorphism.*

Remark. (1) The condition on $H_1(\mathfrak{p}_+, \mathfrak{g})$ can be easily turned into something much more concrete, see [30, 12]. If $\mathfrak{g}$ is simple, then it excludes exactly two series of examples corresponding to the crossed Dynkin diagrams ×—o—· · ·—o—o and ×—o—· · ·—o⇐o . Except for the very degenerate case of the Dynkin diagram × (i.e. the Borel subalgebra in $\mathfrak{sl}(2, \mathbb{K})$), the corresponding regular normal parabolic geometries are still determined by some underlying structure. Geometrically, these give rise to classical projective structures and a contact version of projective structures.
(2) One actually obtains an equivalence of categories between regular normal parabolic geometries and regular infinitesimal flag structures.

2.8. **Examples.** By Theorem 2.7, a regular normal parabolic geometry on M of type (G, P) is for almost all choices of G and P equivalent to a filtration $\{T^iM\}$ of the tangent bundle such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ plus a reduction of the structure group of $\operatorname{gr}(TM)$ to the group G_0. In many situation, this simplifies further, and we will discuss this next.

(1) **|1|-gradings.** Here we are in the situation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The classification of such gradings is equivalent to the classification of Hermitian and pseudo-Hermitian symmetric spaces and therefore well known. Geometrically, the

main point is that the filtration $\{T^iM\}$ by definition consists of just one bundle. Moreover, the regularity condition is easily seen to be vacuous in this case.

Hence if (G, P) corresponds to a $|1|$-grading, then Theorem 2.7 says that normal parabolic geometries of type (G, P) are equivalent to classical first order G_0-structures on M. Here G_0 is considered as a (covering of a) subgroup of $GL(\dim(M), \mathbb{R})$ via $\mathrm{Ad} : G_0 \to GL(\mathfrak{g}_{-1})$.

The most important examples of these structures are conformal, almost quaternionic, and almost Grassmannian structures. The exceptional case corresponding to the Dynkin diagram $\times\!\!-\!\!\circ\!\!-\cdots-\!\!\circ\!\!-\!\!\circ$ corresponds to a $|1|$-grading. Here $G_0 = GL(\mathfrak{g}_{-1})$ so the underlying infinitesimal flag structure contains no information at all. Normal parabolic geometries of this type are equivalent to classical projective structures, which will be discussed in more detail in 3.2 below.

(2) **Structures determined by the filtration.** We have seen in 2.7 that the adjoint action defines a homomorphism $G_0 \to \mathrm{Aut}_{gr}(\mathfrak{g}_-)$. If this is an isomorphism, then $\mathcal{G}_0$ is the full natural frame bundle of $\mathrm{gr}(TM)$ and there is no additional reduction of structure group. Hence in this case Theorem 2.7 shows that a regular normal parabolic geometry on M is equivalent to a filtration $\{T^iM\}$ such that each symbol algebra is isomorphic to $\mathfrak{g}_-$.

There is a simple way to obtain structures of this type: For any semisimple $\mathfrak{g}$, the group $G := \mathrm{Aut}(\mathfrak{g})$ has Lie algebra $\mathfrak{g}$. It turns out (see [25]) that for this choice of G we obtain $G_0 \cong \mathrm{Aut}_{gr}(\mathfrak{g}_-)$ provided that $H_1(\mathfrak{p}_+, \mathfrak{g})$ is concentrated in negative homogeneous degrees. Again this homological condition is easy to verify, and it turns out that it is often satisfied. The paper [30] contains a complete list of pairs $(\mathfrak{g}, \mathfrak{p})$ such that the condition is not satisfied.

This class of examples contains the quaternionic contact structures introduced by O. Biquard, see [3, 4], generic distributions of rank 2 in dimension 5 (which were studied in Cartan's classic [15]), rank 3 in dimension 6, and rank 4 in dimension 7.

(3) **Parabolic contact structures.** These correspond to $|2|$-gradings such that $\mathfrak{g}_{-2}$ is one-dimensional and such that the bilinear form $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ defined by the bracket is non degenerate. The classification of such gradings is equivalent to the classification of quaternionic symmetric spaces and therefore well know. Gradings of this type exist only on simple Lie algebras and are unique up to isomorphism. With a few exceptions, they exist on all non-compact, non-complex simple Lie algebras.

Since $\mathfrak{g}_-$ by definition is a real Heisenberg algebra, a filtration $TM = T^{-2}M \supset T^{-1}M$ of TM such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ is exactly a contact structure $T^{-1}M \subset TM$. Hence the filtration cannot be enough to determine the geometry and one needs the additional reduction to the structure group G_0, which can be expressed as an additional structure on $T^{-1}M$.

This class contains non-degenerate partially integrable almost CR structures of hypersurface type, for which the additional structure on $T^{-1}M$ is an almost complex structure, as well as Lagrangean contact structures, where the additional structure is a decomposition of $T^{-1}M$ into the direct sum of two isotropic subbundles. Next, there is the example of Lie contact structures (see [26]), in which the additional structure is a decomposition of $T^{-1}M$ as the tensor product of two auxiliary bundles, one of which has rank 2 while the other one is endowed with a pseudo-euclidean metric of some fixed signature. Finally, this class also contains the second exceptional structure mentioned in Remark 2.7 (2). In that case, regular normal parabolic geometries are equivalent to a contact analog of projective structures, see [20].

(4) As an example of general parabolic geometries, we discuss *generalized path geometries.* These correspond to the $|2|$-grading on $\mathfrak{sl}(n+2,\mathbb{R})$ corresponding to the first and second simple root. In block form, this decomposition has the form

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^L & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^L & \mathfrak{g}_0 & \mathfrak{g}_1^R \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^R & \mathfrak{g}_0 \end{pmatrix},$$

where the blocks are of size 1, 1, and n. We have met this grading for $n=2$ in 2.4. For later use, we have indicated decomposition of $\mathfrak{g}_{\pm 1}$ into a one-dimensional part $\mathfrak{g}_{\pm 1}^L$ and an n-dimensional part $\mathfrak{g}_{\pm 1}^R$. Evidently, this decomposition is invariant under the adjoint action of $\mathfrak{g}_0$. For an appropriate choice of G, the subgroup G_0 consists of all automorphisms of the graded Lie algebra $\mathfrak{g}_-$ which in addition preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^L \oplus \mathfrak{g}_{-1}^R$.

From this description, we can directly read off the geometric meaning of a regular infinitesimal flag structure of type (G,P) on a smooth manifold M of dimension $2n+1$: One has two transversal subbundles $L,R \subset TM$ of rank 1 and n, respectively, such that for $\xi,\eta \in \Gamma(R)$ we have $[\xi,\eta] \in \Gamma(L \oplus R)$ while the Lie bracket induces an isomorphism $L \otimes R \to TM/(L \oplus R)$.

Examples of such structures come from path geometries. Let N be a manifold of dimension $n+1$ and consider the projectivized tangent bundle $M := \mathcal{P}TN$, the space of lines through the origin in TN. Take R to be the vertical bundle of the projection $\mathcal{P}TN \to N$. Since M is a projectivized tangent bundle, there is a tautological subbundle $H \subset TM$ of rank $n+1$. The fiber of H in a point consists of those tangent vectors whose image in TN lies in the line determined by the point. Hence R is contained in H and a *path geometry* on N is given by the choice of a line subbundle $L \subset H$ such that $H = L \oplus R$. A path geometry on N is equivalent to a family of unparametrized curves in N, with exactly one curve through each point in each direction. In particular, a system of second order ODE's on a manifold Y can be equivalently described as a path geometry on $Y \times \mathbb{R}$ by considering the unparametrized curves describing the graphs of solutions, see [21, 19].

For $n \neq 2$, the data (M,L,R) corresponding to a regular infinitesimal flag structure as above are locally isomorphic to a path geometry. Namely, for $n \neq 2$ the subbundle $R \subset TM$ turns out to be automatically integrable, and one defines N to be a local leaf space for the corresponding foliation. Then for an open subset $U \subset M$, there is a surjective submersion $\psi : U \to N$ such that $\ker(T_x\psi) = R_x$ for all $x \in U$. Under $T_x\psi$, the line L_x gives rise to a line in $T_{\psi(x)}N$, hence defining a lift $\tilde{\psi} : U \to \mathcal{P}TN$. Possibly shrinking U, $\tilde{\psi}$ is an open embedding. By construction, $T\tilde{\psi}$ maps R to the vertical subbundle and $L \oplus R$ to the tautological subbundle.

2.9. **Harmonic curvature.** There is a last element of the general theory of parabolic geometries that we have to discuss. The Cartan curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ as defined in 2.2 and 2.7 is a fairly complicated object. In particular, to understand it geometrically, one needs the adjoint tractor bundle, which is an equivalent encoding of the principal Cartan bundle. An important feature of regular normal parabolic geometries is that one may pass to the harmonic curvature κ_H, which is much easier to handle, but as powerful as κ.

In 2.7 we have defined the operators $\partial^* : \Lambda^\ell T^*M \otimes \mathcal{A}M \to \Lambda^{\ell-1}T^*M \otimes \mathcal{A}M$ and noted that $\partial^* \circ \partial^* = 0$. For a normal geometry the curvature κ by definition is a section of the subbundle $\ker(\partial^*) \subset \Lambda^2 T^*M \otimes \mathcal{A}M$. Hence we can project it to a section κ_H of the quotient $\ker(\partial^*)/\operatorname{im}(\partial^*)$. As we have noted in 2.7, this quotient bundle can be identified with $\mathcal{G}_0 \times_{G_0} H_2(\mathfrak{p}_+, \mathfrak{g})$, so it admits a direct interpretation in terms of the underlying structure and is algorithmically computable.

We have also seen that $H_2(\mathfrak{p}_+, \mathfrak{g})$ splits into a direct sum of G_0-irreducible components. Correspondingly, we obtain a splitting of κ_H into fundamental curvature quantities. There are several general tools to describe (parts of) κ_H in terms of the underlying structure.

The following result shows that κ_H still is a complete obstruction to local flatness, and indeed, it contains the full information about κ.

Theorem. *Let $(p : \mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, P) with curvature κ and harmonic curvature κ_H.*
(1) *(Tanaka) If κ_H vanishes identically, then κ vanishes identically.*
(2) *(Calderbank–Diemer) There is a natural linear differential operator L such that $L(\kappa_H) = \kappa$.*

The first part is a rather easy application of the Bianchi-identity for Cartan connections. The second part is much more difficult. It follows from the general machinery of BGG-sequences, see [14, 6].

3. Correspondence spaces and twistor spaces

Now we switch to the discussion of constructions relating parabolic geometries of different type. We start with the constructions of correspondence spaces and twistor spaces, which is related to different parabolic subgroups of the same group G. The basic reference for this chapter is [7].

3.1. **Correspondence spaces.** Consider a semisimple Lie group G with nested parabolic subgroups $Q \subset P \subset G$. For the homogeneous models, we have the simple observation that G/Q naturally fibers over G/P. Moreover, we can interpret G/Q as $G \times_P (P/Q)$, so this is the total space of a natural fiber bundle over G/P. It turns out that the fiber P/Q can be equivalently viewed as the quotient of the semisimple part of $G_0 \subset P$ by its intersection with Q. This intersection turns out to be parabolic, so P/Q again is a generalized flag manifold. The situations covered by this constructions are easy to describe in the Dynkin (or Satake) diagram notation: The diagram corresponding to $\mathfrak{q}$ is obtained from the one corresponding to $\mathfrak{p}$ by changing dots into crosses. The fiber P/Q can then be directly read off the two diagrams, see [2].

Carrying this over to curved Cartan geometries is easy. Given a geometry $(p : \mathcal{G} \to N, \omega)$ of type (G, P) the subgroup $Q \subset P$ acts freely on $\mathcal{G}$. Hence the *correspondence space* $\mathcal{C}N := \mathcal{G}/Q$ is a smooth manifold, and the obvious map $\mathcal{G} \to \mathcal{C}N$ is a Q-principal bundle. Moreover, $\mathcal{C}N = \mathcal{G} \times_P (P/Q)$, so $\pi : \mathcal{C}N \to N$ is a natural fiber bundle with fiber a generalized flag manifold. In particular, this fiber is always compact. By definition, $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ can also be viewed as a Cartan connection on the principal Q-bundle $\mathcal{G} \to \mathcal{C}N$.

The next obvious question is whether this construction is compatible with regularity and normality. At this point, the uniform algebraic construction of the normalization condition pays off:

Proposition. *If $(\mathcal{G} \to N, \omega)$ is a normal parabolic geometry of type (G, P) then the parabolic geometry $(\mathcal{G} \to \mathcal{C}N, \omega)$ of type (G, Q) is normal, too.*

As we shall see in an example below, regularity is not preserved by the construction in general. However, finding conditions which are equivalent to regularity is usually very easy.

3.2. **Example.** Let $Q \subset G := SL(n+2, \mathbb{R})$ be the parabolic subgroup corresponding to generalized path geometries as in Example (4) of 2.8. Then Q is the stabilizer of the flag consisting of the line spanned by the first vector sitting inside the plane spanned by the first two vectors of the standard basis of $\mathbb{R}^{n+2}$. Hence we can write it as the intersection $P_1 \cap P_2$ for parabolics P_1 and P_2 (the stabilizers of the line respectively the plane). Let us start by analyzing the nested parabolics $Q \subset P_1 \subset G$.

Parabolic geometries of type (G, P_1) correspond to classical projective structures on $(n+1)$-dimensional manifolds, see Example (1) of 2.8. Such a structure on a manifold Z is given by the choice of a projective equivalence class $[\nabla]$ of torsion free linear connections on TZ. Two linear connections ∇ and $\hat{\nabla}$ on TZ are called *projectively equivalent* if there is a one form $\Upsilon \in \Omega^1(Z)$ such that

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$$

for all vector fields $\xi, \eta \in \mathfrak{X}(Z)$. Evidently, projectively equivalent connections have the same torsion. Alternatively, projective equivalence can be characterized as having the same torsion and the same geodesics up to parametrization. The harmonic curvature for this geometry is the projective Weyl curvature, i.e. the totally tracefree part of the curvature of any connection in the class.

Since ω is a Cartan connection on $\mathcal{G} \to Z$, we have $TZ = \mathcal{G} \times_{P_1} (\mathfrak{g}/\mathfrak{p}_1)$. One easily verifies that $Q \subset P_1$ can be described as the stabilizer of a line in $\mathfrak{g}/\mathfrak{p}_1$. Since P_1 acts transitively on the projective space $\mathcal{P}(\mathfrak{g}/\mathfrak{p}_1)$, see that $P/Q \cong \mathcal{P}(\mathfrak{g}/\mathfrak{p}_1)$. Hence $\mathcal{C}Z = \mathcal{G} \times_{P_1} P/Q$ can be naturally identified with the projectivized tangent bundle $\mathcal{P}TZ$. Since projective structures are torsion free, the curvature κ of ω has values in $\mathfrak{p}_1$, which immediately implies that ω is regular as a Cartan connection on $\mathcal{G} \to \mathcal{C}Z$. From Example (4) of 2.8 we conclude that $(\mathcal{G} \to \mathcal{C}Z, \omega)$ can be interpreted as a path geometry on Z. One verifies that the paths described in that way are exactly the unparametrized geodesics of the connections from the projective class.

Let us now switch to the nested parabolic subgroups $Q \subset P_2 \subset G$. A normal parabolic geometry $(\mathcal{G} \to N, \omega)$ of type (G, P_2) exists only for $\dim(N) = 2n$ and is equivalent to an almost Grassmannian structure. Essentially, such a geometry is given by two auxiliary vector bundles E and F over N of rank 2 and n, respectively, and an isomorphism $E \otimes F \to TN$. The subgroup $Q \subset P_2$ can be characterized as the stabilizer of a line in the representation inducing E, which similarly as above implies that $\mathcal{C}N$ can be identified with the projectivization $\mathcal{P}E$ of $E \to N$.

Here ω is not regular as a Cartan connection on $\mathcal{G} \to \mathcal{C}N$ in general. Regularity turns out to be equivalent to the fact that the structure on N is Grassmannian rather than almost Grassmannian. This can be characterized by vanishing of a certain torsion or equivalently by the fact that there is a torsion free connection compatible with the structure. If this is satisfied, then we obtain a generalized path geometry on $\mathcal{P}E$. The subbundle L which is one of the ingredients of that structure is simply the vertical bundle of $\mathcal{P}E \to N$. In particular, the manifold N can be viewed the space of all paths of the induced path geometry. The subbundle $L \oplus R \subset T\mathcal{C}N$ is again a tautological subbundle. The splitting of this tautological subbundle as $L \oplus R$ comes from the torsion free connections compatible with the Grassmannian structure.

Suppose that $n > 2$ (the case $n = 2$ will be discussed later). Then starting from a Grassmannian structure on N, we obtain a generalized path geometry on $\mathcal{C}N := \mathcal{P}E$. We know that the resulting subbundle $R \subset T\mathcal{C}N$ is involutive, so for sufficiently small open subsets $U \subset \mathcal{C}N$ we can form a local leaf space $\psi : U \to Z$. With a bit more work, one shows that one may take $U = \pi^{-1}(V)$, for sufficiently small and convex open subsets $V \subset N$, where $\pi : \mathcal{C}N \to N$ is the natural projection. One then obtains a *correspondence*

$$Z \xleftarrow{\psi} \pi^{-1}(V) \xrightarrow{\pi} V,$$

which is the basis for *twistor theory* for Grassmannian structures.

3.3. **Characterizing correspondence spaces.** A central feature of the general theory of correspondence spaces is that one can completely characterize parabolic geometries which are locally isomorphic to correspondence spaces. This characterization is uniform for all the structures.

Let us return to the general setting of nested parabolics $P \subset Q \subset G$. The question we want to address is when a regular normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, Q) is locally isomorphic to the correspondence space $\mathcal{C}N$ for a parabolic geometry of type (G, P). There is a fairly obvious necessary condition: The subspace $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is Q-invariant, thus giving rise to a subbundle $\mathcal{V} \subset TM$. For a correspondence space $\mathcal{C}N$, this subbundle is the vertical subbundle of the natural projection $\mathcal{C}N \to N$. Since the Cartan connections for N and $\mathcal{C}N$ are the same, so are their curvatures. Since vectors from $\mathcal{V}$ are vertical from the point of view of N, they must hook trivially into the Cartan curvature of $\mathcal{C}N$.

It turns out that this condition is also sufficient:

Theorem. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, Q) with Cartan curvature κ, and let $\mathcal{V} \subset TM$ be the distribution corresponding to $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. Then M admits an open covering $\{U_i\}$ such that the restriction of $(\mathcal{G} \to M, \omega)$ to each U_i is isomorphic to the correspondence space of some parabolic geometry of type (G, P) if and only if $i_\xi \kappa = 0$ for all $\xi \in \mathcal{V}$.*

The proof of this theorem is not specifically "parabolic" and uses only principal bundle geometry. One first shows that the curvature condition in the theorem implies that the distribution $\mathcal{V} \subset TM$ is involutive. Hence $\mathcal{V}$ gives rise to a foliation of M, and one considers a local leaf space for this foliation, i.e. an open subset $U \subset M$ together with a surjective submersion $\psi : U \to N$ such that $\ker(T_x\psi) = \mathcal{V}_x$ for all $x \in U$. For sufficiently small U, one next constructs a diffeomorphism from an open subset of $p^{-1}(U) \subset \mathcal{G}$ onto an open subset of the trivial principal bundle $N \times P \to N$, which satisfies a certain equivariancy condition. This diffeomorphism is then used to carry over ω to this open subset of $N \times P$, and one proves that the resulting form uniquely extends to all of $N \times P$ by equivariancy. It is easy to see that this not only gives a parabolic geometry of type (G, P) on N but also an isomorphism (of parabolic geometries) between U and an open subset of $\mathcal{C}N$.

While this result is very satisfactory from a conceptual point of view, it is difficult to apply in concrete cases, since the Cartan curvature is a complicated object. From part (2) of Theorem 2.9 we know that for regular normal geometries there is a natural differential operator L which computes the Cartan curvature from the harmonic curvature κ_H, which is much easier to handle. This operator is constructed using the machinery of BGG sequences and the construction is explicit enough to lead to relations between algebraic properties of κ and κ_H.

Proposition. *Let $(\mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, Q) with Cartan curvature κ and harmonic curvature κ_H, and let $\mathcal{V} \subset TM$ be as above. If $i_\xi \kappa_H = 0$ for all $\xi \in \mathcal{V}$, then $i_\xi \kappa = 0$ for all $\xi \in \mathcal{V}$.*

Combining this result with the theorem above, one obtains a very efficient local characterization of correspondence spaces. From another point of view, these are equivalent conditions for the existence of natural geometric structures on twistor spaces. It has to be pointed out here that usually the structure of the harmonic curvature can be understood without detailed knowledge of the canonical Cartan connection.

3.4. **Example.** Let us interpret the results on local characterization of correspondence spaces in the example discussed in 3.2. So we start with a generalized path geometry (M, L, R) and the associated regular normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, Q). For $n > 2$ (which we will still assume throughout this subsection), the harmonic curvature κ_H splits into two irreducible components:

$$T : L \wedge TM/(L \oplus R) \to R \qquad \text{Torsion}$$
$$\rho : R \wedge TM/(L \oplus R) \to R^* \otimes R \qquad \text{Curvature}$$

The types of these components can be deduced from the structure of the homology group $H_2(\mathfrak{q}_+, \mathfrak{g})$, which can be determined algorithmically using Kostant's version of the Bott–Borel–Weil theorem. There are general procedures how to obtain explicit formulae for the two components, say in terms of a local non-vanishing section of L.

Let us first consider the characterization of correspondence spaces coming from the inclusion $Q \subset P_1 \subset G$. From 3.2 we know that these are exactly the path geometries associated to the unparametrized geodesics of a projective class of connections. The distribution $\mathcal{V}$ corresponding to $\mathfrak{p}_1/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ evidently is the subbundle $R \subset TM$. The results from 3.3 now show that M is locally isomorphic to a correspondence spaces if and only if ρ vanishes identically.

As we have noted in 2.8, the subbundle $R \subset TM$ is involutive (since $n > 2$). For a local leaf space $\psi : U \to Z$ of the corresponding foliation, the subset U then is naturally diffeomorphic to an open subset in the projectivized tangent bundle $\mathcal{P}TZ$. Then our result shows that the generalized path geometry on M induces a projective structure on Z if and only if ρ vanishes identically. If this is the case, then the torsion T is directly related to the projective Weyl curvature of the induced structures on the local leaf spaces. In particular, the path geometry on M is locally flat if and only if the induced projective structures on all local leaf spaces are locally projectively flat.

Another interesting application of this criterion is to the path geometry associated to a system of second order ODE's as described in 3.2. This reproduces a result of [19]:

Theorem. *A system of second order ODE's is locally equivalent to a geodesic equation if and only if the curvature ρ of the associated path geometry vanishes identically.*

Now we switch to the characterization of correspondence spaces with respect to the inclusion $Q \subset P_2 \subset G$. The distribution $\mathcal{V}$ corresponding to $\mathfrak{p}_2/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is the subbundle $L \subset TM$. This is always involutive and local leaf spaces for the associated foliation locally parametrize the paths of the path geometry. Hence here the main interpretation of the characterization result is a criterion when a generalized path geometry locally descends to a Grassmannian structure on the space of all paths. From 3.3 we see that this is the case if and only if $T = 0$, which is equivalent to the generalized path geometry being torsion free.

Again there is an interesting application to the theory of systems of second order ODE's: One defines such a system to be torsion free if and only if the associated path geometry is torsion free. For such a systems we obtain an induced Grassmannian structure on the space of solutions of the system. The curvature of this Grassmannian structure can be constructed from the curvature ρ of the path geometry. Of course, this curvature descends to the space of solutions and hence is constant along each solution. Using this, D. Grossman proved in [21] the following result.

Theorem. *For generic torsion free systems of second order ODE's, the curvature of the associated path geometry can be used to solve the system explicitly.*

3.5. **The case $n = 2$.** Let us briefly discuss how the examples related to generalized path geometries discussed in 2.9, 3.2, and 3.4 change in the case $n = 2$. The ingredients are projective structures on three manifolds, generalized path geometries in dimension five, and four dimensional almost Grassmannian structures. The main point is that an almost Grassmannian structure in dimension four is equivalent to a conformal pseudo-Riemannian spin structure of split signature $(2,2)$. The auxiliary bundles E and F whose tensor product is isomorphic to the tangent bundle both have rank two. They are exactly the two spinor bundles.

The structure of harmonic curvatures for $n = 2$ is also different from the case $n > 2$. For almost Grassmannian structures the more symmetric situation leads to the fact that there are two curvatures rather than one curvature and one torsion. These two components are exactly the self dual and the anti self dual part of the Weyl curvature of the corresponding conformal structure.

On the level of path geometries, a third irreducible component in the harmonic curvature shows up. This component is represented by a torsion $\tau : \Lambda^2 R \to L$, which is the obstruction to involutivity of the subbundle R. (For $n > 2$, there also is a corresponding component in the homology $H_2(\mathfrak{q}_+, \mathfrak{g})$, but this sits in homogeneity zero. By regularity, this component cannot contribute to the harmonic curvature.)

Starting from a conformal four manifold, the correspondence space is a projectivized spinor bundle, which inherits a generalized path geometry. The torsion τ on this space corresponds exactly to the self dual part of the Weyl curvature downstairs. Vanishing of this part, i.e. anti self duality, is equivalent to existence of local leaf spaces for the bundle R on the correspondence space. This is the basis for twistor theory for anti self dual four manifolds in split signature. The Riemannian version of twistor theory can be either obtained from the complex version of this construction or by an analog of the correspondence space construction (for a subgroup which is not parabolic).

4. Analogs of the Fefferman construction

We now switch to a second general construction relating parabolic geometries of different types. The basic example for this is Fefferman's construction which relates CR structures to conformal structures. This construction is of different nature to the ones discussed in section 3 since it involves two different semisimple groups. More details on the contents of this section can be found in [8] and [11].

4.1. **The Fefferman construction.** We start by reviewing Fefferman's original construction from [18] and its interpretation in terms of Cartan geometries. He started from a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$ with smooth boundary $M := \partial\Omega$. This boundary naturally inherits a CR structure (see below). Studying the Bergman kernel of Ω, Fefferman was led to consider the *ambient metric*: Put $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and consider $M_\# = M \times \mathbb{C}^*$ and $\Omega_\# = \Omega \times \mathbb{C}^*$. A defining function r for M induces a defining function $r_\#$ for $M_\#$. Since M is strictly pseudoconvex, $r_\#$ can be used as the potential for a pseudo-Kähler metric $g_\#$ of signature $(n+1,1)$. Fefferman showed that one may always chose r to be an approximate solution of a Monge–Ampère equation and doing this a certain jet of $g_\#$ along $M_\#$ is invariant under biholomorphisms of Ω. Otherwise put, this jet is a CR invariant of M.

Hence it is a natural idea to look at the restriction of $g_\#$ to $M_\#$. This turns out to be degenerate but only in the real directions within the vertical subspaces of the projection $M_\# \to M$. To get rid of these directions, one passes to the space $\tilde{M} = M \times (\mathbb{C}^*/\mathbb{R}^*) \cong M \times S^1$. Using a section of the evident projection $M_\# \to \tilde{M}$, one can pull back $g_\#$ to a non-degenerate Lorentz metric on $\tilde{M}$. Changing the sections leads to a conformal change of the metric, so one obtains a well defined conformal class of metrics of signature $(2n+1,1)$ on $\tilde{M}$. This conformal class is invariant under biholomorphisms of Ω and hence depends only on the CR structure of M.

CR structures fit into the general concept of parabolic geometries as the parabolic contact structures associated to $\mathfrak{g} = \mathfrak{su}(p+1,q+1)$. In fact, one obtains a more general concept: A *partially integrable almost CR structure* on a smooth manifold M of dimension $2n+1$ is a contact structure $H \subset TM$ together with an almost complex structure $J : H \to H$ such that the Levi bracket $\mathcal{L}$ (see 2.6) satisfies $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi,\eta)$ for all ξ,η. Under this assumption, $\mathcal{L}$ is the imaginary part of a Hermitian form (with values in the real line bundle TM/H), the *Levi form*, which has a signature (p,q). Since there is an ambiguity of sign, we require $p \geq q$ to have the signature well defined.

The compatibility of $\mathcal{L}$ and J, which is usually referred to as partial integrability, can also be nicely formulated in terms of complexifications. The almost complex structure J leads to a splitting of $H \otimes \mathbb{C} \subset TM \otimes \mathbb{C}$ into the direct sum of the holomorphic part $H^{1,0}$ and the anti holomorphic part $H^{0,1}$, which are conjugate to each other. Partial integrability is equivalent to the fact that the Lie bracket of two sections of $H^{0,1}$ is a section of $H \otimes \mathbb{C}$. An almost CR structure is called *integrable* or a *CR structure* if the subbundle $H^{0,1}M \subset TM \otimes \mathbb{C}$ is involutive, so the Lie bracket of two sections of $H^{0,1}$ even is a section of $H^{0,1}$.

Partially integrable almost CR structures of signature (p,q) are then equivalent to regular normal parabolic geometries associated to the group $PSU(p+1,q+1)$. For many applications it is better to extend the group to $G := SU(p+1,q+1)$. Let P be the stabilizer of an isotropic complex line ℓ in $\mathbb{V} := \mathbb{C}^{p+q+2}$. Then a regular normal parabolic geometry of type (G,P) on a manifold M is equivalent to a partially integrable almost CR structure of signature (p,q) plus a choice of a complex line bundle, which is an $(n+2)$nd root of the so-called canonical bundle. While such a choice need not exist in general, it is always possible locally. The integrability condition turns out to be equivalent to torsion freeness of the associated parabolic geometry.

If M is the boundary of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$, then one defines H_xM as the maximal complex subspace of $T_xM \subset T_x\mathbb{C}^{n+1}$. This evidently has an almost complex structure and it defines a contact structure by strict pseudoconvexity. The latter condition also implies that the signature is $(n,0)$. Looking at the complexified tangent bundle, we see that $H^{0,1}M = (TM \otimes \mathbb{C}) \cap T^{0,1}\mathbb{C}^{n+1}$. Since $\mathbb{C}^{n+1}$ is a complex manifold, the subbundle $T^{0,1}\mathbb{C}^{n+1} \subset T\mathbb{C}^{n+1} \otimes \mathbb{C}$ is involutive, so we obtain a CR structure on M. Triviality of the tangent bundle of $\mathbb{C}^{n+1}$ implies that the canonical bundle of M is canonically trivial, so there is no problem in choosing an $(n+2)$nd root.

Now it is easy to obtain the Fefferman construction for the homogeneous model: The real part of the Hermitian form on $\mathbb{V}$ defines an inner product of signature $(2p+2,2q+2)$ on the underlying real vector space $\mathbb{V}_{\mathbb{R}}$. Since elements of G preserve this real part, we obtain an injection $G \hookrightarrow SO(2p+2,2q+2)$. Analyzing the induced homomorphism between the fundamental groups one even shows that this naturally lifts to an inclusion into the spin group $\tilde{G} := Spin(2p+2,2q+2)$. Choose a real line $\ell_{\mathbb{R}}$ in the isotropic complex line ℓ and let $\tilde{P} \subset \tilde{G}$ be the stabilizer of $\ell_{\mathbb{R}}$. The intersection $Q := G \cap \tilde{P}$ is the stabilizer of $\ell_{\mathbb{R}}$ in G, so it is evidently contained in P and $P/Q \cong \mathbb{R}P^1$. Elementary linear algebra shows that G acts transitively on the space of real null lines in $\mathbb{V}_{\mathbb{R}}$. Hence the inclusion $G \hookrightarrow \tilde{G}$ induces a diffeomorphism $G/Q \to \tilde{G}/\tilde{P}$. The latter space is well known to be the homogeneous model of conformal spin structures of signature $(2p+1,2q+1)$. Hence we obtain such a structure (which by construction is invariant under the action of G) on G/Q which is the total space of a circle bundle over G/P.

Passing to curved geometries is easy: Looking at the tangent spaces at the base points, the diffeomorphism $G/Q \to \tilde{G}/\tilde{P}$ induces a linear isomorphism $\mathfrak{g}/\mathfrak{q} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ which is equivariant over the inclusion $Q \hookrightarrow \tilde{P}$. Here $\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}}$ is the Lie algebra of Q. In particular, we obtain a conformal class of inner products of signature $(2p+1,2q+1)$ on $\mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}})$ which is invariant under the natural action of Q. Given a partially integrable almost CR structure (M,H,J), let $(\mathcal{G} \to M, \omega)$ be the associated regular normal parabolic geometry. The subgroup $Q \subset P$ acts freely on $\mathcal{G}$, so the *Fefferman space* $\tilde{M} := \mathcal{G}/Q$ is a smooth manifold and the total space of the natural fiber bundle $\mathcal{G} \times_P P/Q$ over M. On the other hand, the evident projection $\mathcal{G} \to \tilde{M}$ is a principal Q-bundle and $\omega \in \Omega^1(\mathcal{G},\mathfrak{p})$ defines a Cartan connection on that bundle. In particular, $T\tilde{M} \cong \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$ so the Q-invariant class of inner products on $\mathfrak{g}/\mathfrak{q}$ gives rise to a conformal structure on $\tilde{M}$, which by construction depends only on the CR structure on M.

It is easy to give a more explicit description of $\tilde{M}$. Namely, one shows that $\tilde{M}$ can be naturally identified with the space of real lines in a natural complex line bundle, which is closely related to the chosen root of the canonical bundle. One can also construct explicitly a metric from the conformal class in terms of a choice of contact form on M (usually called a *pseudo Hermitian structure*) and the associated Weyl connection (see [13]) on a complex line bundle.

4.2. **Cartan geometry interpretation.** The construction of the canonical conformal class on $\tilde{M}$ from above can be easily reformulated in terms of Cartan geometries. As we know from 4.1, we have the Q-principal bundle $\mathcal{G} \to \tilde{M}$ and we can view the canonical CR Cartan connection ω as a Cartan connection on that bundle. Now via the inclusion $Q \hookrightarrow \tilde{P}$, we can extend the structure group of this bundle. Define a principal $\tilde{P}$-bundle $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P} \to \tilde{M}$. Mapping $u \in \mathcal{G}$ to the class of (u, e) in $\tilde{\mathcal{G}}$ defines an injective smooth map $j : \mathcal{G} \to \tilde{\mathcal{G}}$ which is equivariant over the inclusion $Q \hookrightarrow \tilde{P}$. It is easy to show that there is a unique Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $\tilde{\omega}|_{Tj(T\mathcal{G})} = \omega$ (viewing $\mathfrak{g}$ as a Lie subalgebra of $\tilde{\mathfrak{g}}$).

As a Cartan connection on a principal $\tilde{P}$-bundle $\tilde{\omega}$ is automatically regular and hence it induces a conformal spin structure on the base $\tilde{M}$. From the construction it is evident this this leads to the conformal structure described in 4.1.

Now one might expect that $\tilde{\omega}$ is the normal Cartan connection associated to this conformal spin structure, but this is *not* true in general:

Theorem. *Let (M, H, J) be a partially integrable almost CR structure with Fefferman space $\tilde{M}$. Then the Cartan connection $\tilde{\omega}$ on the extended principal bundle $\tilde{\mathcal{G}} \to \tilde{M}$ is normal if and only if the almost CR structure is integrable.*

The necessity of integrability follows rather easily from the fact that normal conformal Cartan connections are automatically torsion free. The proof of sufficiency of this condition is much more subtle. The result does not follow from algebraically comparing the normalization conditions for the two geometries in question but one has to prove additional properties of the curvature of a torsion free geometry. In that respect, the situation is very different from the case of correspondence spaces discussed in the last section.

For some applications of the Fefferman construction, the question of normality of $\tilde{\omega}$ is not relevant. For example, conformal invariants of the Fefferman space are always invariants of the underlying partially integrable almost CR structure. However, we will show below that normality of $\tilde{\omega}$ leads to many other and deeper results.

If the structure on M is not integrable, then the canonical Cartan connection for the conformal spin structure on $\tilde{M}$ can be obtained by normalizing $\tilde{\omega}$. The difference of $\tilde{\omega}$ from the normal Cartan connection is given by a one form on $\tilde{M}$ with values in the conformal adjoint tractor bundle $\tilde{\mathcal{G}} \times_{\tilde{P}} \tilde{\mathfrak{g}}$. One may try to imitate some of the developments described below taking into account the change caused by this form. To my knowledge, this has not been explored up to now.

4.3. **Applications of normality to CR geometry.** We want to discuss a few results which are based on normality of the Cartan connection $\tilde{\omega}$ in the case of a CR structure.

The first of these was the main application in Fefferman's original article [18] as well as in the first version for abstract CR structures in [5].

• *Chern–Moser chains are the projections to M of null geodesics in $\tilde{M}$.*
Chern–Moser chains in M can be obtained as the projections of flow lines of vector fields on $\mathcal{G}$ which are mapped to certain constant functions by ω. Likewise, conformal circles on $\tilde{M}$ are the projections of flow lines of vector fields on $\tilde{\mathcal{G}}$ which are mapped to certain constant functions by $\tilde{\omega}$. For initial directions which are null, conformal circles are just null geodesics which, as unparametrized curves, are conformally invariant. The initial direction of a chain is always transversal to the contact subbundle, and such a direction always admits a lift to a null direction in $\tilde{M}$. Then the result easily follows from the fact that $\tilde{\omega}$ is obtained from ω by equivariant extension.

• *Relations between CR tractor calculus on M and conformal tractor calculus on $\tilde{M}$.*
Standard tractors are probably the nicest way to relate a CR manifold to its Fefferman space. The CR standard tractor bundle $\mathcal{T}$ of M is by definition the associated bundle $\mathcal{G} \times_P \mathbb{V}$, where $\mathbb{V}$ denotes the standard representation of G. By construction, this is a rank $n+2$ complex vector bundle endowed with Hermitian inner product h of signature $(p+1, q+1)$, and a complex line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ which is isotropic for h. This subbundle corresponds to the complex line in $\mathbb{V}$ which is stabilized by P. The canonical Cartan connection ω on $\mathcal{G}$ induces a Hermitian linear connection on $\mathcal{T}$, called the *normal standard tractor connection.*

Likewise, the conformal standard tractor bundle $\tilde{\mathcal{T}}$ of the Fefferman space $\tilde{M}$ is the bundle $\tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{V}$. This is a real bundle of rank $2n+4$ endowed with a Euclidean bundle metric $\tilde{h}$ of signature $(2p+2, 2q+2)$ and an a real line subbundle $\tilde{\mathcal{T}}^1$ which is isotropic for $\tilde{h}$. The Cartan connection $\tilde{\omega}$ induces the normal standard tractor connection on $\tilde{\mathcal{T}}$.

The relation between the Cartan bundles and the Cartan connections discussed above can be interpreted as the fact that $\tilde{\mathcal{T}}$ (including the additional structures) can also be obtained as $\mathcal{G} \times_{G\cap\tilde{P}} \mathbb{V}$ and the normal tractor connection on $\tilde{\mathcal{T}}$ is induced by ω, viewed as a Cartan connection on $\mathcal{G} \to \tilde{M}$.

Both for conformal and for CR structures, the standard tractor bundle and the standard tractor connection lead to an efficient calculus. Hence we obtain a close relation between CR tractor calculus on a CR manifold and conformal tractor calculus on its Fefferman space.

• *Conformally invariant differential operators on $\tilde{M}$ descend to families of CR invariant differential operators on M.*
The relations between the standard tractor bundles of M and $\tilde{M}$ can be extended to other bundles, for example other tractor bundles and density bundles. One can then interpret sections of some bundle over M as a subset of sections of some other bundle over $\tilde{M}$, which usually are characterized as solutions of some differential equation. It often happens that this works for a whole family of bundles over M (with different weights) and the same bundle on $\tilde{M}$. Based on the relations between tractor calculi discussed above, one shows that in several cases conformally invariant differential operators preserve the subspaces of "downstairs" sections and hence descend to (families of) CR invariant differential operators.

• *Interpretation of solutions of certain CR invariant differential equations.*

The solutions of certain CR invariant differential equations admit a natural interpretation in terms of the conformal geometry of the Fefferman space. An example for this will be given in the discussion of conformal isometries of the Fefferman space below.

4.4. **Conformal geometry of Fefferman spaces.** The second interesting line of applications is towards Fefferman spaces as an interesting subclass of conformal structures.

• *Fefferman spaces have a parallel orthogonal complex structure on the standard tractor bundle and are locally characterized by that.*

We have seen above that for the Fefferman space $\tilde{M}$ of a CR manifold M, the conformal standard tractor bundle can be interpreted as $\tilde{\mathcal{T}} = \mathcal{G} \times_{G\cap\tilde{P}} \mathbb{V}$, and the tractor connection on that bundle is induced by the CR Cartan connection ω. Since $\mathbb{V}$ is a complex vector space, we obtain an almost complex structure J on $\tilde{\mathcal{T}}$, which is orthogonal (or equivalently skew symmetric) with respect to the tractor metric, and parallel for the connection on $L(\tilde{\mathcal{T}},\tilde{\mathcal{T}})$ induced by the standard tractor connection.

This can be interpreted as the fact that the holonomy of the standard tractor connection is contained in $SU(p+1,q+1) \subset SO(2p+2,2q+2)$. Conversely, one can show that a conformal structure of signature $(2p+2,2q+2)$ which admits such a holonomy reduction, is locally conformally isometric to a Fefferman space. This shows that the role of Fefferman spaces among general conformal structures is similar to the role of Calabi–Yau manifolds among general Riemannian manifolds.

• *Fefferman spaces admit nontrivial Twistor spinors and conformal Killing forms of all odd degrees.*

Several conformally invariant differential equations which are overdetermined (and thus do not have solutions in general) always admit nontrivial solutions on Fefferman spaces. The simplest example of this situation is that one constructs a nowhere vanishing conformal Killing field j on $\tilde{M}$, which spans the vertical subbundle of $\tilde{M} \to M$. The most conceptual interpretation of this is via the almost complex structure J on the standard tractor bundle $\tilde{\mathcal{T}} \to \tilde{M}$. Since this is skew symmetric with respect to the tractor metric, it can be interpreted as a parallel section of the adjoint tractor bundle $\tilde{\mathcal{A}} = \tilde{\mathcal{G}} \times_{\tilde{P}} \tilde{\mathfrak{g}}$. It is well known that there is a natural projection $\Pi : \tilde{\mathcal{A}} \to T\tilde{M}$ and the image of a parallel section under this projection is automatically a conformal Killing field (which in addition hooks trivially into the Cartan curvature).

Viewing $\tilde{\mathcal{A}}$ as $\Lambda^2\tilde{\mathcal{T}}$, we can form the k-fold wedge product of J with itself, which defines a nonzero parallel section of the tractor bundle $\Lambda^{2k}\tilde{\mathcal{T}}$. This bundle naturally projects onto the bundle $\Lambda^{2k-1}T^*\tilde{M}$ (twisted by an appropriate density bundle) and the image of a parallel section is a conformal Killing form (with additional properties), see [22]. These conformal Killing forms can be explicitly expressed in terms of the conformal Killing field j from above. In contrast to the simple algebraic formula on the tractor level, these expressions involve covariant derivatives of j.

We have noted in 4.1, the Fefferman space $\tilde{M}$ carries a natural spin structure. In particular, we can consider the tractor bundle $\tilde{\mathcal{S}} \to \tilde{M}$ corresponding to the spin representation of $\tilde{G} = Spin(2p+2,2q+2)$. Now it is well known that as a representation of the subgroup $G = SU(p+1, q+1)$ this spin representation decomposes and in particular contains a two dimensional trivial subrepresentation. Using the relation between the tractor calculi discussed above, one shows that this leads to a decomposition of the spin tractor bundle $\tilde{\mathcal{S}} \to \tilde{M}$, and in particular one obtains a two parameter family of parallel sections of that bundle. The bundle $\tilde{\mathcal{S}}$ comes with a canonical projection to the spinor bundle of $\tilde{M}$, which maps parallel sections to twistor spinors. Hence any Fefferman space admits a two parameter family of twistor spinors.

• *Decomposition of conformal Killing fields.*

By naturality of the construction of the Fefferman space, any CR automorphism of M lifts to a conformal isometry of $\tilde{M}$. Likewise, an infinitesimal automorphism of M induces a conformal Killing field on $\tilde{M}$. As the example of the homogeneous model shows, there may be other conformal Killing fields on $\tilde{M}$. It turns out that one can completely describe the space of all conformal Killing fields on $\tilde{M}$ in terms of the CR geometry of M.

Infinitesimal automorphisms of parabolic geometries can be described in general in terms of sections of the adjoint tractor bundle. For the case of conformal structures, this means that any conformal Killing field is the image of a uniquely determined section of the adjoint tractor bundle $\tilde{\mathcal{A}}$ which satisfies a certain conformally invariant differential equation.

As a representation of $G = SU(\mathbb{V})$, the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{so}(\mathbb{V})$ is not irreducible, but decomposes as $\mathfrak{su}(\mathbb{V}) \oplus \mathbb{R} \oplus \Lambda^2_{\mathbb{C}}\mathbb{V}$. Here the first two summands correspond to complex linear maps, while the last one corresponds to conjugate linear maps, and the trivial summand consists of purely imaginary multiples of the identity. This induces an analogous splitting of the conformal adjoint tractor bundle $\tilde{\mathcal{A}} \to \tilde{M}$.

We can use this splitting to decompose any section of $\tilde{\mathcal{A}}$ into a sum of three terms. Via tractor calculus one shows that for a section corresponding to a conformal Killing field, each of the three parts satisfies the infinitesimal automorphism equation. Thus one concludes that any conformal Killing fields $\xi \in \mathfrak{X}(\tilde{M})$ decomposes uniquely into a sum $\xi_1 + \xi_2 + \xi_3$ of conformal Killing fields. One further shows that ξ_1 descends to an infinitesimal automorphism of the underlying CR manifold M and ξ_2 is a constant multiple of j. The summand ξ_3 descends to a section of $\Lambda^2_{\mathbb{C}}T \to M$ which solves a certain CR invariant differential equation. Likewise, appropriate solutions of this equation give rise to conformal Killing fields on $\tilde{M}$.

4.5. **Analogs of the Fefferman construction.** From the discussion in 4.1 it is pretty evident what is needed to obtain an analog of the Fefferman construction: One starts with an inclusion $G \hookrightarrow \tilde{G}$ of semisimple Lie groups and chooses a parabolic subgroup $\tilde{P} \subset \tilde{G}$ such that the G orbit of $e\tilde{P}$ in $\tilde{G}/\tilde{P}$ is open. Finally, one needs a parabolic subgroup $P \subset G$ which contains $G \cap \tilde{P}$.

Suppose that $(p : \mathcal{G} \to M, \omega)$ is a parabolic geometry of type (G, P). The define $\tilde{M} := \mathcal{G}/(G \cap \tilde{P})$, which is a smooth manifold and the total space of the natural fiber bundle $\mathcal{G} \times_P P/(G \cap \tilde{P}) \to M$. To obtain an explicit description of $\tilde{M}$, it suffices to give a good description of the subgroup $G \cap \tilde{P} \subset P$. As before, one can view $\mathcal{G} \to \tilde{M}$ as a principal bundle with structure group $G \cap \tilde{P}$ and $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ as a Cartan connection on this bundle. In particular, this identifies $T\tilde{M}$ with the associated bundle $\mathcal{G} \times_{G\cap\tilde{P}} \mathfrak{g}/(\mathfrak{g}\cap\tilde{\mathfrak{p}})$.

Since the G-orbit of $e\tilde{P}$ in $\tilde{G}/\tilde{P}$ is open, the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ induces a linear isomorphism $\mathfrak{g}/(\mathfrak{g}\cap\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Clearly, this isomorphism is equivariant under the inclusion $G\cap\tilde{P} \hookrightarrow \tilde{P}$. Hence we can carry over $\tilde{P}$-invariant objects related to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ to $(G\cap\tilde{P})$-invariant objects related to $\mathfrak{g}/(\mathfrak{g}\cap\tilde{\mathfrak{p}})$ and hence to natural geometric objects on $\tilde{M}$. In most examples discussed below, this already suffices to obtain the underlying structure of a regular normal parabolic geometry of type $(\tilde{G},\tilde{P})$ on $\tilde{M}$. In more complicated situations one in addition has to check that the map $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}$ induced by the curvature of ω is regular, but this usually is very easy.

It is a much more difficult problem to check whether ω induces the regular normal Cartan connection associated to this underlying structure. As in the classical case, one can form the extended bundle $\tilde{\mathcal{G}} := \mathcal{G} \times_{G\cap\tilde{P}} \tilde{P}$, and there is a unique Cartan connection $\tilde{\omega}$ on that bundle which restricts to ω on $T\mathcal{G} \subset T\tilde{\mathcal{G}}$. To obtain an analog of Theorem 4.2 and applications similar to the ones described in 4.3 and 4.4, one has to find conditions for $\tilde{\omega}$ being normal. To my knowledge, this has not been done for all the examples described below but for many of them there are hints coming from independent works on these structures.

Examples.

(1) Closest to the classical Fefferman construction, one may consider the group $G := Sp(p+1,q+1)$ associated to a quaternionic Hermitian form of signature $(p+1,q+1)$ on $\mathbb{H}^{p+q+2}$. Viewing this space as $\mathbb{C}^{2p+2q+4}$ gives rise to an inclusion $Sp(p+1,q+1) \hookrightarrow \tilde{G} := SU(2p+2,2q+2)$. Taking $P \subset G$ and $\tilde{P} \subset \tilde{G}$ the stabilizer of a quaternionic respectively a complex null line, one obtains $G\cap\tilde{P} \subset P$ and $P/(G\cap\tilde{P}) \cong \mathbb{C}P^1$. Parabolic geometries of type (G,P) fall into the class discussed in Example (2) of 2.8, i.e. the structures which are (essentially) determined by a filtration of the tangent bundle. The modeling Lie algebra $\mathfrak{g}_-$ is a quaternionic Heisenberg algebra of signature (p,q). This means that $\mathfrak{g}_{-1} \cong \mathbb{H}^{p+q}$ and $\mathfrak{g}_{-2} \cong \Im(\mathbb{H})$, the space of purely imaginary quaternions, in such that way that the bracket is by the imaginary part of a quaternionic Hermitian form of signature (p,q). For $q = 0$, one obtains the quaternionic contact structures introduced by Olivier Biquard, see [3, 4].

Hence we see that, up to some discrete data (related to the fact that we use the group Sp rather than PSp) our construction starts with a quaternionic contact structure of signature (p,q) on some manifold M. The Fefferman space $\tilde{M}$ is then the total space of a natural fiber bundle over M with fiber $\mathbb{C}P^1 \cong S^2$, and on $\tilde{M}$ we naturally obtain a partially integrable almost CR structure of signature $(2p+1,2q+1)$. This should be closely related to O. Biquard's construction of a twistor space for quaternionic contact structures.

(2) Consider a vector space $\tilde{\mathbb{V}}$ endowed with an inner product of signature $(p+1,q+2)$. Fixing a line ℓ on which the inner product is negative definite, the inclusion $\ell^\perp \hookrightarrow \tilde{\mathbb{V}}$ gives rise to an inclusion $G := SO(p+1,q+1) \hookrightarrow SO(p+1,q+2) =: \tilde{G}$. Choose a null plane $\mathbb{W}$ which is transversal to $\ell^\perp$ and let $\tilde{P} \subset \tilde{G}$ be the stabilizer of $\mathbb{W}$. Then $\mathbb{W}\cap\ell^\perp$ is a null line, and its stabilizer P evidently contains $G\cap\tilde{P}$. One verifies that

the G-orbit of $e\tilde{P}$ in $\tilde{G}/\tilde{P}$ consists of those null planes in $\tilde{\mathbb{V}}$ which are transversal to $\ell^\perp$, so in this case $G/(G\cap\tilde{P})$ is a proper open subset of $\tilde{G}/\tilde{P}$.

Normal parabolic geometries of type (G,P) are just conformal structures of signature (p,q). Given such a structure on M one shows that the Fefferman space $\tilde{M}$ can be identified with the open subset $\mathcal{P}_+(T^*M)$ of the projectivized cotangent bundle of M consisting of all lines in T^*M on which the conformal inner product is positive definite. In particular, for $q=0$, we obtain the full projectivized cotangent bundle. Of course, in any case $\tilde{M}$ carries a canonical contact structure and the analog of the Fefferman construction refines this to a Lie contact structure. This generalizes and explains the results of [26].

(3) Consider the inclusion of $G := Sp(2n,\mathbb{R})$ into $\tilde{G} := SL(2n,\mathbb{R})$ by the standard representation. Parabolic subgroups of G correspond to isotropic flags in the symplectic vector space $\mathbb{R}^{2n}$, which parabolic subgroups in $\tilde{G}$ correspond to arbitrary flags. Hence there is only one choice for a parabolic subgroup $\tilde{P}\subset\tilde{G}$ such that the G-orbit of $e\tilde{P}$ in $\tilde{G}/\tilde{P}$ is open. Namely, one has to use the stabilizer of a line, since for lines being isotropic is a vacuous condition. In this case $P := G\cap\tilde{P}$ is itself parabolic in G.

Hence we conclude that the analog of the Fefferman construction this time starts from a geometry of type (G,P) on M and produces the underlying structure of a geometry of type $(\tilde{G},\tilde{P})$ on the same space M. Geometries of type (G,P) are a contact analog of projective structures, and our construction extends such a structure to a classical projective structure. This has been directly obtained in [20], where more details about such structures can be found.

(4) To finish, we discuss an exotic example which however has a long history. Let G be the split real form of the exceptional Lie group G_2. It is well known that G_2 has a 7 dimensional representation, and for the split form there is an invariant inner product of signature $(3,4)$ on this representation. Hence this gives rise to an inclusion of G into $\tilde{G} := SO(3,4)$. The stabilizer $P\subset G$ of a line through a highest weight vector in this representation is one of the two maximal parabolic subgroups of G. This line is easily seen to be isotropic, so as in (3) we obtain $P = G\cap\tilde{P}$, where $\tilde{P}$ is the stabilizer of the highest weight line in $\tilde{G}$.

Geometries of type (G,P) are exactly the generic rank two distributions in dimension five which are studied in Cartan's famous "five variables paper" [15]. Given such a distribution on M, the analog of the Fefferman construction produces a canonical conformal class of split signature $(2,3)$ on M. Such a canonical conformal class was recently discovered by P. Nurowski using Cartan's method of equivalence, see [24]. Since in Nurowski's construction one obtains the same normal Cartan connection for both geometries, it is very likely that the structure described here coincides with his.

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