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SYMMETRIZATION OF BRACE ALGEBRAS

MARILYN DAILY AND TOM LADA

ABSTRACT. We show that the symmetrization of a brace algebra structure yields the structure of a symmetric brace algebra. We also show that the symmetrization of the natural brace structure on $\bigoplus_{k \geq 1} \text{Hom}(V^{\otimes k}, V)$ coincides with the natural symmetric brace structure on $\bigoplus_{k \geq 1} \text{Hom}(V^{\otimes k}, V)^{as}$, the direct sum of spaces of antisymmetric maps $V^{\otimes k} \rightarrow V$.

1. INTRODUCTION

Brace algebras were first studied in the context of multilinear operations on the Hochschild complex of an associative algebra [3, 2, 1]. Symmetric brace algebras, in which the brace operations possess the property of graded symmetry, were subsequently introduced in [5]. Just as one may construct L_∞ algebra structures by anti (skew) symmetrizing A_∞ algebra structures [4], we show in this note that the symmetrization of a brace algebra structure yields a symmetric brace algebra structure. We prove in Section 5 that one may define a symmetric brace operation $\langle \cdot, \cdot \rangle$ on a graded vector space with a given non symmetric brace operation $\{ \cdot, \cdot \}$ by

$$f\{g_1, \dots, g_n\} := \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\}.$$

The motivating example of a brace algebra is $\bigoplus_{k \geq 1} \text{Hom}(V^{\otimes k}, V)$, and the fundamental example of a symmetric brace algebra is the subspace of anti symmetric maps, $\bigoplus_{k \geq 1} \text{Hom}(V^{\otimes k}, V)^{as}$.

In Section 6, we show that these algebras are related by

$$\sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) as(f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\}) = as(f)\langle as(g_1), \dots, as(g_n) \rangle,$$

where $f, g_i \in \text{Hom}(V^{\otimes k}, V)$, $as(f)(v_1, \dots, v_k) := \sum_{\sigma \in \mathcal{S}_k} (-1)^\sigma \epsilon(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$, and $\epsilon(\sigma)$ is just the Koszul sign of the permutation.

In Sections 2 and 3, we review the definitions and fundamental examples of brace algebras and symmetric brace algebras respectively. Section 4 contains a collection of

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technical lemmas that are needed to prove the main theorems in the final two sections. Throughout this article, S_n will denote the symmetric group on n symbols.

2. BRACE ALGEBRAS

Definition 1. A *brace structure* on a graded vector space consists of a collection of degree 0 multilinear braces $x, x_1, \dots, x_n \mapsto x\{x_1, \dots, x_n\}$ which satisfy the identity, $x\{\} = x$, and in which $x\{x_1, \dots, x_n\}\{y_1, \dots, y_r\}$ is equal to

$$\sum \epsilon \cdot x\{y_1, \dots, y_{i_1}, x_1\{y_{i_1+1}, \dots, y_{j_1}\}, y_{j_1+1}, \dots, y_{i_n}, x_n\{y_{i_n+1}, \dots, y_{j_n}\}, y_{j_n+1}, \dots, y_r\}.$$

In the above formula, the sum is over all sequences $0 \leq i_1 \leq j_1 \leq \dots \leq i_n \leq j_n \leq r$, and ϵ is the Koszul sign of the permutation which maps $(x_1, \dots, x_n, y_1, \dots, y_r)$ to

$$(y_1, \dots, y_{i_1}, x_1, y_{i_1+1}, \dots, y_{j_1}, y_{j_1+1}, \dots, y_{i_n}, x_n, y_{i_n+1}, \dots, y_{j_n}, y_{j_n+1}, \dots, y_r).$$

The motivating example for a brace algebra structure is the space $\bigoplus \text{Hom}(V^{\otimes k}, V)$ with the natural brace operation of degree $-n$ given by the composition

$$f\{g_1, \dots, g_n\} = \sum_{k_0 + \dots + k_n = N-n} f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \dots \otimes g_n \otimes 1^{\otimes k_n}),$$

where $f \in \text{Hom}(V^{\otimes N}, V)$. This operation arises from the endomorphism operad of V considered in [1]. This operation was also utilized in the context of the Hochschild complex of an associative algebra V in [3] and [2]. For maps of arbitrary degree p , we have

Example 2. Let V be a graded vector space and consider the graded vector space $B_*(V)$ where

$$B_s(V) := \bigoplus_{p-k+1=s} \text{Hom}(V^{\otimes k}, V)_p$$

and where $\text{Hom}(V^{\otimes k}, V)_p$ denotes the space of k -multilinear maps of degree p . Given $f \in \text{Hom}(V^{\otimes N}, V)_p$ and $g_i \in \text{Hom}(V^{\otimes a_i}, V)_{q_i}$, define

$$f\{g_1, \dots, g_n\} \in \text{Hom}(V^{\otimes r}, V)_{p+q_1+\dots+q_n}$$

where $r = a_1 + \dots + a_n + N - n$ by

$$f\{g_1, \dots, g_n\} = \sum_{k_0 + \dots + k_n = N-n} (-1)^\beta f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \dots \otimes 1^{\otimes k_{n-1}} \otimes g_n \otimes 1^{\otimes k_n}),$$

where

$$\beta = \sum_{j < i} [a_i - 1][k_j + a_j] + \sum_i (N - i) q_i + \sum_{j < i} q_i a_j.$$

Remark 3. In Example 2, suppose that there exists a collection of maps

$$\mu_k \in \text{Hom}(V^{\otimes k}, V)_{k-2} \in B_{-1}(V).$$

If we let $\mu = \mu_1 + \mu_2 + \dots$, then an A_∞ algebra structure on V may be described by the brace relation $\mu\{\mu\} = 0$ [5].

3. SYMMETRIC BRACE ALGEBRAS

Definition 4. An n -unshuffle of N elements is a partition $\sum_{i=1}^n a_i = N$ and a permutation $\gamma \in S_N$ such that

$$\gamma(1) < \dots < \gamma(a_1), \gamma(1 + a_1) < \dots < \gamma(a_2 + a_1), \dots, \gamma\left(1 + \sum_{i=1}^{n-1} a_i\right) < \dots < \gamma(N).$$

If we wish to emphasize the partition, we call such a permutation an $(a_1|a_2|\dots|a_n)$ unshuffle.

Definition 5. A *symmetric brace algebra* is a graded vector space together with a collection of degree zero multilinear braces $f\langle g_1, \dots, g_n \rangle$ which are graded symmetric in g_1, \dots, g_n . In a symmetric brace algebra, it is also required that $f\langle \rangle = f$, and that $f\langle g_1, \dots, g_n \rangle\langle x_1, \dots, x_r \rangle$ be equal to

$$\sum_{\substack{\gamma \text{ is } (n+1) \\ \text{unshuffle}}} \epsilon \cdot f\langle g_1\langle x_{\gamma(1)}, \dots, x_{\gamma(a_1)} \rangle, \dots, g_n\langle x_{\gamma(1+\sum_{i=1}^{n-1} a_i)}, \dots, x_{\gamma(\sum_{i=1}^n a_i)} \rangle, x_{\gamma(1+\sum_{i=1}^n a_i)}, \dots, x_{\gamma(r)} \rangle,$$

where ϵ is the Koszul sign of the permutation which maps $(g_1, \dots, g_n, x_1, \dots, x_r)$ to

$$(g_1, x_{\gamma(1)}, \dots, x_{\gamma(a_1)}, g_2, \dots, x_{\gamma(1+\sum_{i=1}^{n-1} a_i)}, \dots, x_{\gamma(\sum_{i=1}^n a_i)}, g_n, x_{\gamma(1+\sum_{i=1}^n a_i)}, \dots, x_{\gamma(r)}).$$

Just as with brace algebras, the fundamental example of a symmetric brace algebra is provided by the space of antisymmetric maps of degree p , $\bigoplus \text{Hom}(V^{\otimes k}, V)_p^{as}$. To be precise, we have

Example 6. Let V be a graded vector space and $B_*(V)$ be the graded vector space given by

$$B_s(V) = \bigoplus_{p-k+1=s} \text{Hom}(V^{\otimes k}, V)_p^{as},$$

Given $f \in \text{Hom}(V^{\otimes N}, V)_p^{as}$ and $g_i \in \text{Hom}(V^{\otimes a_i}, V)_{q_i}^{as}$, $1 \leq i \leq n$, define the symmetric brace

$$f\langle g_1, \dots, g_n \rangle\langle x_1, \dots, x_r \rangle = (-1)^\delta \sum_{\substack{\gamma \text{ is an} \\ (a_1|a_2|\dots|a_{n+1}) \\ \text{unshuffle}}} \chi(\gamma) f(g_1 \otimes \dots \otimes g_n \otimes 1^{\otimes N-n})(x_{\gamma(1)}, \dots, x_{\gamma(r)}),$$

where

$$\delta = \sum_i^n (N-i)q_i + \sum_{j<i} q_i a_j + \sum_{j<i} a_i a_j + \sum_i (n-i)a_i,$$

and $\chi(\gamma) := \text{sgn}(\gamma)e(\gamma)$ is the antisymmetric Koszul sign of the permutation γ .

Remark 7. Suppose that in Example 6 we have maps

$$l_k \in \text{Hom}(V^{\otimes k}, V)_{k-2}^{as} \in B_{-1}(V).$$

If we let $l = l_1 + l_2 + \dots$, then an L_∞ algebra structure on V is given by the symmetric brace relation $l\langle l \rangle = 0$.

4. SOME LEMMAS

Although the expressions in this paper involve many sums, permutations, and anti-symmetrizations, we will be able to simplify things considerably with the help of the following lemmas. Lemma 8 provides a decomposition of $as(f)$ which will be useful later.

Lemma 8. $as(f) = f \circ \Phi_{nm} \circ \Psi_n \circ \Theta_m \quad \forall f \in \text{Hom}(V^{\otimes n+m}, V)$, where

$$\begin{aligned}\Theta_m(y_1, \dots, y_n, z_1, \dots, z_m) &= \sum_{\pi \in S_m} \chi(\pi)(y_1, \dots, y_n, z_{\pi(1)}, \dots, z_{\pi(m)}), \\ \Psi_n(y_1, \dots, y_n, z_1, \dots, z_m) &= \sum_{\sigma \in S_n} \chi(\sigma)(y_{\sigma(1)}, \dots, y_{\sigma(n)}, z_1, \dots, z_m), \\ \Phi_{nm}(y_1, \dots, y_n, z_1, \dots, z_m) &= \sum_{k_0 + \dots + k_n = m} (-1)^\eta (z_1, \dots, z_{k_0}, y_1, z_{1+k_0}, \dots, y_n, z_{1+k_0} \\ &\quad + \dots + k_{n-1}, \dots, z_m),\end{aligned}$$

$$\text{and } \eta = \sum_{i=1}^n \{y_i [z_1 + \dots + z_{(k_0+k_1+\dots+k_{i-1})}] + (n-i)k_i\}.$$

Proof. Since Ψ_n does all permutations of the first n inputs, Θ_m provides all permutations of the last m inputs, and Φ_{nm} distributes the last n variables between the first m in every possible way, the composition is clearly a sum of all permutations of the original $n+m$ variables. A moment's reflection also reveals that the sign of each summand in the composition is the Koszul sign together with the sign of the permutation. \square

Lemma 9 states that if we sum over all (signed) $(a_1 | \dots | a_n)$ unshuffles, and then sum over all (signed) permutations of the a_i variables in each piece, then this is equivalent to just summing over all signed permutations of the original $a_1 + \dots + a_n$ variables.

Lemma 9. If $N = a_1 + \dots + a_n$, then $\sum_{\pi \in S_N} \chi(\pi)(x_{\pi(1)}, \dots, x_{\pi(N)})$ is equal to

$$\begin{aligned}\sum_{\substack{\gamma \text{ is} \\ (a_1 | \dots | a_n) \\ \text{unshuffle}}} \chi(\gamma) \sum_{\pi_1 \in S_{a_1}} \chi(\pi_1) \cdots \sum_{\pi_n \in S_{a_n}} \chi(\pi_n)(x_{\gamma(\pi(1))}, \dots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \\ \dots, x_{\gamma(\pi_n(a_n)+\sum_{i=1}^{n-1} a_i)}).\end{aligned}$$

Proof. Clearly, the right hand side is the sum of distinct permutations of the x terms with the correct sign. Furthermore, since there are $\frac{N!}{(a_1)! \dots (a_n)!}$ unshuffles γ and $(a_i)!$ permutations π_i , there are $N!$ summands in the right hand side, which agrees with the number of summands on the left hand side. \square

Lemma 10. Suppose $k_0 + a_1 + k_1 + \dots + a_n + k_n = r$, $\sigma \in S_n$, and $\pi \in S_r$. Let $A = a_1 + \dots + a_n$, denote $X_i = x_{\pi(1+a_1+\dots+a_{i-1})}, \dots, x_{\pi(a_1+\dots+a_i)}$, and also denote $X_\pi = x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, X_{\sigma(1)}, x_{\pi(1+k_0+A)}, \dots, X_{\sigma(n)}, x_{\pi(1+k_0+\dots+k_{n-1}+A)}, \dots, x_{\pi(r)}$. Then we

can define $\hat{\pi} \in S_r$ by

$$\hat{\pi}(i) = \begin{cases} \pi\left(i + A - \sum_{j \leq m} a_{\sigma(j)}\right) & \text{if } \sum_{j < m} k_j + \sum_{j \leq m} a_{\sigma(j)} < i \\ & \leq \sum_{j \leq m} k_j + \sum_{j \leq m} a_{\sigma(j)}; \\ \pi\left(i - \sum_{j < m} k_j + \sum_{j < \sigma(m)} a_j\right) & \text{if } \sum_{j < m} k_j + \sum_{j < m} a_{\sigma(j)} < i \\ & \leq \sum_{j < m} k_j + \sum_{j \leq m} a_{\sigma(j)}. \end{cases}$$

Furthermore, given this notation,

$$X_\pi = x_{\hat{\pi}(1)}, \dots, x_{\hat{\pi}(r)} \quad \text{and} \quad \epsilon(\hat{\pi}) = \epsilon(\pi)(-1)^{\alpha_1} \quad \text{and} \quad \chi(\hat{\pi}) = \chi(\pi)(-1)^{\alpha_2},$$

where

$$\alpha_1 = \sum_{i < j \ \& \ \sigma(i) > \sigma(j)} |X_{\sigma(i)}| |X_{\sigma(j)}| + \sum_{i=1}^n |X_{\sigma(i)}| [x_{\pi(1+A)} + \dots + x_{\pi(k_0 + \dots + k_{i-1} + A)}]$$

and

$$\alpha_2 = \alpha_1 + \sum_{i < j \ \& \ \sigma(i) > \sigma(j)} a_{\sigma(i)} a_{\sigma(j)} + \sum_{j < i} a_{\sigma(i)} k_j.$$

Proof. Careful examination of the definition of $\hat{\pi}$ reveals that the first formula moves “free” strings of the form $x_{\pi(1+k_0+\dots+k_{i-1})}, \dots, x_{\pi(k_0+\dots+k_i)}$ into place (for $0 \leq m \leq n$), and the second formula relocates the strings $X_{\sigma(i)}$ (for $1 \leq m \leq n$). Thus $X_\pi = x_{\hat{\pi}(1)}, \dots, x_{\hat{\pi}(r)}$.

Furthermore, when $x_{\pi(1)}, \dots, x_{\pi(r)}$ are permuted to yield $x_{\hat{\pi}(1)}, \dots, x_{\hat{\pi}(r)}$, the Koszul sign is $(-1)^{\alpha_1}$, where the first sum in α_1 comes from σ permuting the X_i strings, and the second sum comes from moving the “free” strings into place. Finally, the additional sums in α_2 count the transpositions, yielding the correct antisymmetric Koszul sign. \square

Lemma 11. Suppose that $\sigma \in S_n$ permutes $\{v_1 \dots v_n\}$ and $\{w_1 \dots w_n\}$. Then

- (1)
$$\sum_{i > j} v_i w_j + \sum_{i < j \ \& \ \sigma(i) > \sigma(j)} \{w_{\sigma(i)} v_{\sigma(j)} + v_{\sigma(i)} w_{\sigma(j)}\} + \sum_{i > j} v_{\sigma(i)} w_{\sigma(j)} \equiv 0 \pmod{2};$$
- (2)
$$\sum_{i < j \ \& \ \sigma(i) > \sigma(j)} \{v_{\sigma(i)} + v_{\sigma(j)}\} \equiv \sum_i (i-1)v_i + \sum_i (i-1)v_{\sigma(i)} \pmod{2}.$$

Proof. To prove the first assertion, we note that

$$\begin{aligned} \sum_{i < j \ \& \ \sigma(i) > \sigma(j)} \{v_{\sigma(i)} w_{\sigma(j)} + w_{\sigma(i)} v_{\sigma(j)}\} + \sum_{i > j} v_{\sigma(i)} w_{\sigma(j)} &= \sum_{i < j \ \& \ \sigma(i) > \sigma(j)} v_{\sigma(i)} w_{\sigma(j)} \\ &+ \sum_{i > j \ \& \ \sigma(i) < \sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i > j} v_{\sigma(i)} w_{\sigma(j)}, \end{aligned}$$

which is congruent (mod 2) to

$$\sum_{i < j \ \& \ \sigma(i) > \sigma(j)} v_{\sigma(i)} w_{\sigma(j)} + \sum_{i > j \ \& \ \sigma(i) > \sigma(j)} v_{\sigma(i)} w_{\sigma(j)} = \sum_{\sigma(i) > \sigma(j)} v_{\sigma(i)} w_{\sigma(j)} = \sum_{i > j} v_i w_j.$$

To prove the second statement, suppose that all w_i are odd. Then

$$\begin{aligned} \sum_{i < j \text{ \& } \sigma(i) > \sigma(j)} \{v_{\sigma(i)} + v_{\sigma(j)}\} &\equiv \sum_{i < j \text{ \& } \sigma(i) > \sigma(j)} \{v_{\sigma(i)}w_{\sigma(j)} + w_{\sigma(i)}v_{\sigma(j)}\} \\ &\equiv \sum_{i > j} \{v_i w_j + v_{\sigma(i)} w_{\sigma(j)}\} \end{aligned}$$

(by the first assertion). Since all w -terms are odd, this is congruent to

$$\sum_{j=1}^n \sum_{i=j+1}^n (v_i + v_{\sigma(i)}) = \sum_j (j-1)v_j + \sum_j (j-1)v_{\sigma(j)}. \quad \square$$

5. SYMMETRIZATION OF BRACE ALGEBRAS

Given a (non-symmetric) brace structure $\{ , \}$ on a graded vector space, we can define a symmetric brace structure \langle , \rangle via

$$f\langle g_1, \dots, g_n \rangle := \sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\}.$$

Clearly, this satisfies the first symmetric brace axiom, since $f\langle \rangle = f\{ \} = f$. We show in Theorem 15 that it satisfies the second symmetric brace axiom given in Definition 5, so this does in fact induce a symmetric brace structure. First, however, we need the following two lemmas, which are analogous to Lemmas 8 and 9.

Lemma 12. $\sum_{\rho \in S_{n+m}} \epsilon(\rho) f\{x_{\rho(1)}, \dots, x_{\rho(n)}\} = \tilde{f}_n \circ \theta_m(x_1, \dots, x_{n+m})$, where

$$\Theta_m(y_1, \dots, y_n, z_1, \dots, z_m) = \sum_{\pi \in S_n} \epsilon(\pi) (y_1, \dots, y_n, z_{\pi(1)}, \dots, z_{\pi(m)}) \text{ and}$$

$$\begin{aligned} &\tilde{f}_n(y_1, \dots, y_n, z_1, \dots, z_m) \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{k_0 + \dots + k_n = m} (-1)^\eta f\{z_1, \dots, z_{k_0}, y_{\sigma(1)}, z_{1+k_0}, \dots, y_{\sigma(n)}, z_{1+k_0+\dots+k_{n-1}}, \dots, z_m\}, \end{aligned}$$

with a Koszul sign given by $\eta = \sum_{i=1}^n y_{\sigma(i)} [z_1 + \dots + z_{(k_0+k_1+\dots+k_{i-1})}]$.

Lemma 13. If $N = a_1 + \dots + a_n$, then $\sum_{\pi \in S_N} \epsilon(\pi) (x_{\pi(1)}, \dots, x_{\pi(N)})$ is equal to

$$\begin{aligned} &\sum_{\substack{\gamma \text{ is} \\ (a_1 | \dots | a_n) \\ \text{unshuffle}}} \epsilon(\gamma) \sum_{\pi_1 \in S_{a_1}} \epsilon(\pi_1) \dots \sum_{\pi_{a_n} \in S_{a_n}} \epsilon(\pi_n) (x_{\gamma(\pi(1))}, \dots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \\ &\dots, x_{\gamma(\pi_n(a_n)+\sum_{i=1}^{n-1} a_i)}). \end{aligned}$$

Remark 14. Although a brace structure allows operators g which accept an arbitrary number of inputs, it will be convenient in the proof of the following theorem to let g^a denote the restriction of g which accepts only exactly a inputs.

Theorem 15. Given a (non-symmetric) brace structure $\{ , \}$ on a graded vector space, define \langle , \rangle via

$$f\langle g_1, \dots, g_n \rangle := \sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\}.$$

Then $f\langle g_1, \dots, g_n \rangle(x_1, \dots, x_r)$ is equal to

$$\sum_{\substack{\gamma \text{ is } (n+1) \\ \text{unshuffle}}} \epsilon \cdot f\langle g_1 \rangle x_{\gamma(1)}, \dots, x_{\gamma(a_1)}, \dots, g_n \langle x_{\gamma(1+\sum_{i=1}^{n-1} a_i)}, \dots, x_{\gamma(\sum_{i=1}^n a_i)}, x_{\gamma(1+\sum_{i=1}^n a_i)}, \dots, x_{\gamma(r)} \rangle,$$

where ϵ is the Koszul sign of the permutation which maps $(g_1, \dots, g_n, x_1, \dots, x_r)$ to

$$\left(g_1, x_{\gamma(1)}, \dots, x_{\gamma(a_1)}, g_2, \dots, x_{\gamma(1+\sum_{i=1}^{n-1} a_i)}, \dots, x_{\gamma(\sum_{i=1}^n a_i)}, g_n, x_{\gamma(1+\sum_{i=1}^n a_i)}, \dots, x_{\gamma(r)} \right).$$

Proof. First, we will look at the right hand side.

If we temporarily denote

$$\begin{aligned} h_k &= g_k \langle x_{\gamma(1+a_1+\dots+a_{k-1})}, \dots, x_{\gamma(a_1+\dots+a_k)} \rangle \\ &= \sum_{\pi_k \in S_{a_k}} \epsilon(\pi_k) g_k \{ x_{\gamma(\pi_k(1)+a_1+\dots+a_{k-1})}, \dots, x_{\gamma(\pi_k(a_k)+a_1+\dots+a_{k-1})} \}, \end{aligned}$$

and denote $A = \sum_{i=1}^n a_i$, then the right hand side is equal to

$$\sum_{\substack{a_1+\dots+a_{n+1}=r \ \& \\ \gamma \text{ is } (a_1|\dots|a_{n+1}) \text{ unshuffle}}} (-1)^\nu \epsilon(\gamma) f\langle h_1, \dots, h_n, x_{\gamma(1+A)}, \dots, x_{\gamma(a_{n+1}+A)} \rangle,$$

where $\nu = \sum_{i=2}^n g_i[x_{\gamma(1)} + \dots + x_{\gamma(a_1+\dots+a_{i-1})}]$ is a Koszul sign. After applying Lemma 12, this is equal to

$$\sum_{\substack{a_1+\dots+a_{n+1}=r, \\ \gamma \text{ is unshuffle}}} (-1)^\nu \epsilon(\gamma) \tilde{f}_n \left(\sum_{\pi_{n+1} \in S_{a_{n+1}}} \epsilon(\pi_{n+1}) (h_1, \dots, h_n, x_{\gamma(\pi_{n+1}(1)+A)}, \dots, x_{\gamma(\pi_{n+1}(a_{n+1})+A)}) \right),$$

where \tilde{f}_n is as defined in Lemma 12. Now, we will pull all of the x terms back out, in order to apply Lemma 13. Note that the Koszul signs from this transformation merely cancel out $(-1)^\nu$. We then have the following long formula:

$$\begin{aligned} \sum_{(a_i), \gamma} \epsilon(\gamma) \sum_{\pi_1 \in S_{a_1}} \epsilon(\pi_1) \dots \sum_{\pi_{(n+1)} \in S_{a_{n+1}}} \epsilon(\pi_{n+1}) \tilde{f}_n(g_1^{a_1}, \dots, g_n^{a_n}, 1^{a_{n+1}}) \\ \times (x_{\gamma(\pi_1(1))}, \dots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \\ \dots, x_{\gamma(\pi_n(A))}, x_{\gamma(\pi_{n+1}(1)+A)}, \dots, x_{\gamma(\pi_{n+1}(a_{n+1})+A)}). \end{aligned}$$

Now, though, we can apply Lemma 13, which yields the much shorter formula,

$$\sum_{(a_i)} \sum_{\pi \in S_r} \epsilon(\pi) \tilde{f}_n(g_1^{a_1}, \dots, g_n^{a_n}, 1^{a_{n+1}})(x_{\pi(1)}, \dots, x_{\pi(r)}).$$

Before continuing, we need to pull all of the x terms back inside. In order to make our expressions a bit shorter, let X_i denote the input to g_i . In other words, define

$$X_i = x_{\pi(1+a_1+\dots+a_{i-1})}, \dots, x_{\pi(a_1+\dots+a_i)} \quad \text{for } i \in \{1 \dots n\}.$$

It will also be convenient to let $|X_i|$ denote the sum of the degrees of the variables in X_i . When we pull the x -terms inside and use the more concise notation just defined, the formula for the right hand side becomes

$$\sum_{(a_i)} \sum_{\pi \in S_r} \epsilon(\pi) (-1)^{\tilde{\nu}} \tilde{f}_n(g_1(X_1), \dots, g_n(X_n), x_{\pi(1+A)}, \dots, x_{\pi(r)}),$$

where $\tilde{\nu} = \sum_{j < i} \dot{g}_i |X_j|$. After expanding \tilde{f}_n , the right hand side is equal to

$$\begin{aligned} \sum_{(a_i)} \sum_{\pi \in S_r} \epsilon(\pi) (-1)^{\tilde{\nu}} \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{k_0+\dots+k_n=a_n+1} (-1)^\eta f\{x_{\pi(1+A)}, \\ \dots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \\ \dots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)}\}. \end{aligned}$$

Here, $\eta = \sum_{i=1}^n (g_{\sigma(i)} + |X_{\sigma(i)}|) [x_{\pi(1+A)} + \dots + x_{\pi(k_0+\dots+k_{i-1}+A)}]$ and $\epsilon(\sigma) = (-1)^\lambda$, where

$$\lambda = \sum_{i < j \text{ \& } \sigma(i) > \sigma(j)} (g_{\sigma(i)} + |X_{\sigma(i)}|) (g_{\sigma(j)} + |X_{\sigma(j)}|).$$

Now, we will look at the left hand side. $f\langle g_1, \dots, g_n \rangle \langle x_1, \dots, x_r \rangle$ is equal to $\sum_{\sigma \in S_n} \epsilon(\sigma) f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\} \langle x_1, \dots, x_r \rangle$, which is equal to

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{\pi \in S_r} \epsilon(\pi) f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\} \{x_{\pi(1)}, \dots, x_{\pi(r)}\}.$$

If we apply Definition 1 and let $g_i^{a_i}$ denote the restriction of g_i which accepts exactly a_i inputs, then the left hand side is equal to

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{\pi \in S_r} \epsilon(\pi) \sum_{k_0+\dots+k_n+a_1+\dots+a_n=r} f\{1^{k_0}, g_{\sigma(1)}^{a_{\sigma(1)}}, 1^{k_1}, \dots, g_{\sigma(n)}^{a_{\sigma(n)}}, 1^{k_n}\} \langle x_{\pi(1)}, \dots, x_{\pi(r)} \rangle.$$

After applying Lemma 10, this is equal to

$$\begin{aligned} \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{(k_i, a_i)} \sum_{\pi \in S_r} \epsilon(\pi) (-1)^{\alpha_1} f\{1^{k_0}, g_{\sigma(1)}^{a_{\sigma(1)}}, 1^{k_1}, \dots, g_{\sigma(n)}^{a_{\sigma(n)}}, 1^{k_n}\} \\ \times \langle x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, X_{\sigma(1)}, x_{\pi(1+k_0+A)}, \\ \dots, X_{\sigma(n)}, x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)} \rangle, \end{aligned}$$

where α_1 is given in Lemma 10. Finally, when the x -terms are moved inside, the left hand side is equal to

$$\begin{aligned} \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{(k_i, a_i)} \sum_{\pi \in S_r} \epsilon(\pi) (-1)^{\alpha_1+\mu} f\{x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \\ \dots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)}\}. \end{aligned}$$

Here, $\mu = \sum_i g_{\sigma(i)} [x_{\pi(1+A)} + \dots + x_{\pi(k_0+\dots+k_{i-1}+A)}] + \sum_{j<i} g_{\sigma(i)} |X_{\sigma(j)}|$ and $\epsilon(\sigma) = (-1)^\zeta$, where $\zeta = \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} g_{\sigma(i)} g_{\sigma(j)}$.

Now that the terms on both sides are easy to compare, it is clear that the two sides are equal if and only if $\tilde{\nu} + \lambda + \eta + \zeta + \alpha_1 + \mu \equiv 0 \pmod{2}$.

After making the most obvious cancellations, we see that $\tilde{\nu} + \lambda + \eta + \zeta + \alpha_1 + \mu$ is congruent to

$$\sum_{j<i} g_i |X_j| + \sum_{i<j \ \& \ \sigma(i)>\sigma(j)} (g_{\sigma(i)} |X_{\sigma(j)}| + g_{\sigma(j)} |X_{\sigma(i)}|) + \sum_{j<i} g_{\sigma(i)} |X_{\sigma(j)}|,$$

which is congruent to zero $\pmod{2}$ by Lemma 11. □

6. SYMMETRIZATION OF THE BRACE STRUCTURE ON $\bigoplus_{k \geq 1} \text{Hom}(V^{\otimes k}, V)$

In this section, we will demonstrate a nice relationship between the the brace defined in Example 2 and the symmetric brace defined in Example 6, by showing that the symmetrization of the non symmetric brace structure on $\text{Hom}(V^{\otimes k}, V)$ is equal to the symmetric brace of the anti-symmetrized maps. Specifically, we have

Theorem 16. $\sum_{\sigma \in S_n} \epsilon(\sigma) as(f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\}) = as(f)(as(g_1), \dots, as(g_n))$.

Proof. First, we will manipulate the right hand side. Using the symmetric brace structure defined in Example 6, $as(f)(as(g_1), \dots, as(g_n))(x_1, \dots, x_r)$ is equal to

$$(-1)^\delta \sum_{\substack{\gamma \text{ is an} \\ (a_1|a_2|\dots|a_{n+1}) \\ \text{unshuffle}}} \chi(\gamma) as(f)(as(g_1) \otimes \dots \otimes as(g_n) \otimes 1^{\otimes N-n})(x_{\gamma(1)}, \dots, x_{\gamma(r)}),$$

where δ is given in Example 6.

When we substitute the x terms using the Koszul convention and suppress the tensor notation, this is equal to

$$(-1)^\delta \sum_{\gamma} \chi(\gamma) (-1)^\nu as(f)(h_1, \dots, h_n, x_{\gamma(1+\sum_{i=1}^n a_i)}, \dots, x_{\gamma(r)}),$$

where

$$\nu = \sum_{i=2}^n q_i [x_{\gamma(1)} + \dots + x_{\gamma(a_1+\dots+a_{i-1})}]$$

and

$$\begin{aligned} h_k &= as(g_k) (x_{\gamma(1+a_1+\dots+a_{k-1})}, \dots, x_{\gamma(a_1+\dots+a_k)}) \\ &= \sum_{\pi_k \in S_{a_k}} \chi(\pi_k) g_k (x_{\gamma(\pi_k(1)+a_1+\dots+a_{k-1})}, \dots, x_{\gamma(\pi_k(a_k)+a_1+\dots+a_{k-1})}) \cdot \end{aligned}$$

If we denote $A = \sum_{i=1}^n a_i$ and apply Lemma 8, this is equal to

$$\sum_{\gamma} \chi(\gamma) (-1)^{\delta+\nu} f \circ \Phi_{na} \circ \Psi_n \left(\sum_{\pi_{a_n+1} \in S_{a_n+1}} \chi(\pi_{n+1})(h_1, \dots, h_n, x_{\gamma(\pi_{a_n+1}(1)+A)}, \dots, x_{\gamma(\pi_{a_n+1}(a_n+1)+A)}) \right).$$

Now, we will pull all of the x terms back out, in order to apply Lemma 9. Note that the Koszul signs from this transformation merely cancel out $(-1)^\nu$. We then have the following long formula,

$$(-1)^\delta \sum_{\gamma} \chi(\gamma) \sum_{\pi_1 \in S_{a_1}} \chi(\pi_1) \dots \sum_{\pi_{(a_n+1)} \in S_{a_n+1}} \chi(\pi_{n+1}) f \circ \Phi_{na} \circ \Psi_n(g_1, \dots, g_n, 1^{a_n+1}) (x_{\gamma(\pi_1(1))}, \dots, x_{\gamma(\pi_1(a_1))}, x_{\gamma(\pi_2(1)+a_1)}, \dots, x_{\gamma(\pi_{a_n}(A)}, x_{\gamma(\pi_{a_n+1}(1)+A)}, \dots, x_{\gamma(\pi_{a_n+1}(a_n+1)+A)}).$$

Now, though, we can apply Lemma 9, which yields the much shorter formula,

$$(-1)^\delta \sum_{\pi \in S_r} \chi(\pi) f \circ \Phi_{na} \circ \Psi_n(g_1, \dots, g_n, 1^{a_n+1}) (x_{\pi(1)}, \dots, x_{\pi(r)}).$$

Before continuing, we need to pull all of the x terms back inside. In order to make our expressions a bit shorter, let X_i denote the input to g_i , and let X_{n+1} denote the free x terms (letting $a_{n+1} = N - n$). In other words, define

$$X_i = x_{\pi(1+a_1+\dots+a_{i-1})}, \dots, x_{\pi(a_1+\dots+a_i)}.$$

It will also be convenient to let $|X_i|$ denote the sum of the degrees of the variables in X_i . When we pull the x -terms inside and use the more concise notation just defined, the formula for the right hand side becomes

$$(-1)^\delta \sum_{\pi \in S_r} \chi(\pi) (-1)^{\tilde{\nu}} f \circ \Phi_{n, N-n} \circ \Psi_n(g_1(X_1), \dots, g_n(X_n), X_{n+1}),$$

where $\tilde{\nu} = \sum_{j < i} q_i |X_j|$. After expanding Ψ_n , the right hand side is equal to

$$(-1)^\delta \sum_{\pi \in S_r} \chi(\pi) (-1)^{\tilde{\nu}} f \circ \Phi_{n, N-n} \left(\sum_{\sigma \in S_n} \chi(\sigma) (g_{\sigma(1)}(X_{\sigma(1)}), \dots, g_{\sigma(n)}(X_{\sigma(n)}), X_{n+1}) \right).$$

In the above expression, $\chi(\sigma)$ is equal to $(-1)^\lambda$, where

$$\lambda = \sum_{\substack{i < j \leq n, \\ \sigma(i) > \sigma(j)}} [(q_{\sigma(i)} + |X_{\sigma(i)}|)(q_{\sigma(j)} + |X_{\sigma(j)}|) + 1].$$

Now, if we expand $\Phi_{n, N-n}$, we get

$$\sum_{\substack{\pi \in S_r, \sigma \in S_n, \\ k_0 + \dots + k_n = N-n}} \chi(\pi) (-1)^{\delta + \tilde{\nu} + \lambda + \eta} f(x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \dots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)}),$$

where

$$\eta = \sum_{i=1}^n \{ (q_{\sigma(i)} + |X_{\sigma(i)}|) (x_{\pi(1+A)} + \dots + x_{\pi(k_0 + \dots + k_{i-1} + A)}) + (n - i)k_i \}.$$

Now, we will work with the left hand side of the equation. Using the brace defined in Example 2, $\sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma} (f\{g_{\sigma(1)}, \dots, g_{\sigma(n)}\})(x_1, \dots, x_r)$ is equal to

$$\sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma} \left(\sum_{k_0 + \dots + k_n = N - n} (-1)^{\beta} f(1^{\otimes k_0} \otimes g_{\sigma(1)} \otimes 1^{\otimes k_1} \otimes \dots \otimes 1^{\otimes k_{n-1}} \otimes g_{\sigma(n)} \otimes 1^{\otimes k_n}) \right) (x_1, \dots, x_r),$$

where β is given in Example 2. Note also that the Koszul sign $\epsilon(\sigma)$ must be calculated using the degree of g_i as an element of the symmetric brace algebra (so $|g_i| = q_i + a_i - 1$). Thus $\epsilon(\sigma) = (-1)^{\zeta}$, where

$$\zeta = \sum_{i < j \ \& \ \sigma(i) > \sigma(j)} (q_{\sigma(i)} + a_{\sigma(i)} - 1)(q_{\sigma(j)} + a_{\sigma(j)} - 1).$$

If we now antisymmetrize by taking all signed permutations of the x 's, and suppress the tensor notation, this is equal to

$$\sum_{\sigma \in S_n} \sum_{k_0 + \dots + k_n = N - n} (-1)^{\beta + \zeta} f(1^{k_0}, g_{\sigma(1)}, 1^{k_1}, \dots, 1^{k_{n-1}}, g_{\sigma(n)}, 1^{k_n}) \left(\sum_{\pi \in S_r} \chi(\pi) (x_{\pi(1)}, \dots, x_{\pi(r)}) \right).$$

After applying Lemma 10, the left hand side is equal to

$$\sum_{\substack{\pi \in S_r, \sigma \in S_n, \\ k_0 + \dots + k_n = N - n}} (-1)^{\beta + \zeta + \alpha_2} \chi(\pi) f(1^{k_0}, g_{\sigma(1)}, 1^{k_1}, \dots, g_{\sigma(n)}, 1^{k_n}) (x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, X_{\sigma(1)}, x_{\pi(1+k_0+A)}, \dots, X_{\sigma(n)}, x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)}),$$

where α_2 is given in Lemma 10.

Finally, when the variables are moved inside, the left hand side is equal to

$$\sum_{\substack{\pi \in S_r, \sigma \in S_n, \\ k_0 + \dots + k_n = N - n}} (-1)^{\beta + \zeta + \alpha + \mu} \chi(\pi) f(x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \dots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)}),$$

where $\mu = \sum_i q_{\sigma(i)} [x_{\pi(1+A)} + \dots + x_{\pi(k_0+\dots+k_{i-1}+A)}] + \sum_{j < i} q_{\sigma(i)} |X_{\sigma(j)}|$.

Since the right hand side is equal to

$$\sum_{\substack{\pi \in S_r, \sigma \in S_n, \\ k_0 + \dots + k_n = N - n}} \chi(\pi) (-1)^{\delta + \tilde{\nu} + \lambda + \eta} f(x_{\pi(1+A)}, \dots, x_{\pi(k_0+A)}, g_{\sigma(1)}(X_{\sigma(1)}), x_{\pi(1+k_0+A)}, \dots, g_{\sigma(n)}(X_{\sigma(n)}), x_{\pi(1+k_0+\dots+k_n+A)}, \dots, x_{\pi(r)}),$$

we see that the two sides are equal if and only if

$$\beta + \zeta + \alpha_2 + \mu + \delta + \tilde{\nu} + \lambda + \eta \equiv 0 \pmod{2}.$$

After cancelling the most obvious terms, $\beta + \zeta + \alpha_2 + \mu + \delta + \tilde{\nu} + \lambda + \eta$ is congruent to

$$\begin{aligned} & \sum_i (n-i)a_{\sigma(i)} + \sum_i (N-i)q_{\sigma(i)} + \sum_{j<i} q_{\sigma(i)}a_{\sigma(j)} \\ & + \sum_{i<j \text{ \& } \sigma(i)>\sigma(j)} [q_{\sigma(i)}a_{\sigma(j)} + q_{\sigma(i)} + a_{\sigma(i)}q_{\sigma(j)} + a_{\sigma(i)} + q_{\sigma(j)} + a_{\sigma(j)}] \\ & + \sum_{j<i} q_{\sigma(i)}|X_{\sigma(j)}| + \sum_i (N-i)q_i + \sum_{j<i} q_i a_j + \sum_i (n-i)a_i \\ & + \sum_{j<i} q_i |X_j| + \sum_{i<j' \text{ \& } \sigma(i)>\sigma(j)} [q_{\sigma(i)}|X_{\sigma(j)}| + |X_{\sigma(i)}|q_{\sigma(j)}]. \end{aligned}$$

After applying Lemma 11, this is congruent to

$$\sum_i \{(n-i)a_{\sigma(i)} + (N-i)q_{\sigma(i)} + (i-1)[a_i + a_{\sigma(i)} + q_i + q_{\sigma(i)}] + (N-i)q_i + (n-i)a_i\},$$

which is equal to $\sum_i \{(n-1)[a_{\sigma(i)} + a_i] + (N-1)[q_{\sigma(i)} + q_i]\} \equiv 0 \pmod{2}$. \square

As a corollary, we obtain Theorem 3.1 of [4]:

Corollary 17. *The anti-symmetrization $l := as(\mu)$ of an A_∞ -algebra structure μ yields an L_∞ -algebra structure.*

Proof. Given $\mu\{\mu\} = 0$ (recall Remarks 3 and 7) we have

$$0 = as(\mu\{\mu\}) = as(\mu)\langle as(\mu) \rangle = l(l). \quad \square$$

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