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GAUGE-NATURAL FIELD THEORIES AND NOETHER THEOREMS: CANONICAL COVARIANT CONSERVED CURRENTS

MARCELLA PALESE AND EKKEHART WINTERROTH

ABSTRACT. We specialize in a new way the Second Noether Theorem for gauge-natural field theories by relating it to the Jacobi morphism and show that it plays a fundamental role in the derivation of canonical covariant conserved quantities. In particular we show that Bergmann–Bianchi identities for such theories hold true covariantly and canonically only along solutions of generalized gauge-natural Jacobi equations. *Vice versa* all vertical parts of gauge-natural lifts of infinitesimal principal automorphisms lying in the kernel of generalized Jacobi morphisms satisfy Bergmann–Bianchi identities and thus are generators of canonical covariant currents and superpotentials. As a consequence of the Second Noether Theorem, we further show that there exists a covariantly conserved current associated with the Lagrangian obtained by contracting the Euler–Lagrange morphism with a gauge-natural Jacobi vector field. We use as fundamental tools an invariant decomposition formula of vertical morphisms due to Kolář and the theory of iterated Lie derivatives of sections of fibered bundles.

1. INTRODUCTION

Our general framework is the calculus of variations on finite order jets of *gauge-natural bundles* (i.e. jet prolongations of fiber bundles associated to some gauge-natural prolongation of a principal bundle P [5, 16]). Such geometric structures have been widely recognized to suitably describe so-called gauge-natural field theories, i.e. physical theories in which right-invariant infinitesimal automorphisms of the structure bundle P uniquely define the transformation laws of the fields themselves (see e.g. [5, 6, 13, 16] and references quoted therein). In particular, we shall work within the differential setting of *finite order variational sequences on gauge-natural bundles*. In fact, it become evident that the passage from Lagrangians to Euler–Lagrange equations can be seen as a differential of a complex (see e.g. [27, 29, 30, 18]): the theory of finite order variational sequences provides then a suitable geometric framework for

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the Calculus of Variations. In this theory the Euler–Lagrange operator is a differential morphism in a sequence of sheaves of vector spaces. Geometric objects like Lagrangians, momenta, Poincaré–Cartan forms, Helmholtz conditions, Jacobi equations, find a nice interpretation in the quotient spaces of the sequence of a given order.

In the beginning of the second half of the past Century, to conveniently derive conserved quantities for covariant field theories, it appeared necessary to define in a functorial and unique way the lift of infinitesimal transformations of the basis manifolds to the bundle of fields (namely bundles of tensor fields or tensor densities as suitable representations of the action of infinitesimal space-time transformations on frame bundles of a given order [23]) [1, 3, 4]. Such theories were also called geometric or *natural* [28]. An important generalization of natural theories to gauge fields theories passed through the concept of jet prolongation of a principal bundle and the introduction of a very important geometric construction, namely the *gauge-natural bundle functor* [5, 16].

In particular, P.G. Bergmann in [3] introduced what he called *generalized Bianchi identities* for geometric field theories to get (after an integration by parts procedure) a consistent equation involving local divergences within the first variation formula. It is well known that, following the Noether theory [21], in the classical Lagrangian formulation of field theories the description of symmetries and conserved quantities consists in deriving from the invariance of the Lagrangian the existence of suitable conserved currents; in most relevant physical theories this currents are found to be the divergence of skew-symmetric (tensor) densities, which are called *superpotentials* for the conserved currents themselves. It is also well known that the importance of superpotentials relies on the fact that they can be integrated to provide conserved quantities associated with the conserved currents *via* the Stokes Theorem (see e.g. [6, 20] and references therein). Generalized Bergmann–Bianchi identities are in fact necessary and locally sufficient conditions for a Noether conserved current to be not only closed but also the divergence of a a superpotential along solutions of the Euler–Lagrange equations. However, the problem of the general covariance of such identities exists and it was already posed and partially investigated by Anderson and Bergmann in [1], where the invariance with respect to time coordinate transformations was studied. This problem reflects obviously on the covariance of conserved quantities (see Remark 4 below). Here we propose a way to deal with such open problems concerning globality aspects. For the relevance of the latter ones also in quantum field theories, see e.g. the preprints [2].

In [8] a representation of symmetries in finite order variational sequences was provided by means of the introduction of the *variational Lie derivative*, i.e. the induced quotient operator acting on equivalence classes of forms in the variational sequence. In [6] the theory of Noether conserved currents and superpotentials was tackled by using such representations for natural and gauge-natural Lagrangian field theories. Recently, further developments have been achieved concerning a canonical covariant derivation of Noether conserved quantities and global superpotentials [24, 25]. On the other hand the *second variation* of the action functional can be conveniently represented in the finite order variational sequence framework in terms of iterated variational Lie derivatives of the Lagrangian with respect to vertical parts of gauge-natural lifts of principal infinitesimal automorphisms. In particular, in [7, 9] the second variational derivative

has been represented and related with the *generalized Jacobi morphism*. Furthermore, the gauge-natural structure of the theories under consideration enables us to define the *generalized gauge-natural Jacobi morphism* where the *variation vector fields* are Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts.

In this paper we use representations of the Noether Theorems given in [8]; in particular we specialize in a new way the Second Noether Theorem for gauge-natural theories by means of the Jacobi morphism [24, 25] and show that the Second Noether Theorem plays a fundamental role in the derivation of canonical covariant conserved quantities in gauge-natural field theories (see Remark 5 below). In fact, the *indeterminacy* appearing in the derivation of gauge-natural conserved charges (for a review, see the interesting papers [11, 20]) - i.e. the difficulty of relating in a natural way infinitesimal gauge transformations with infinitesimal transformations of the basis manifold - can be solved by requiring the second variational derivative to vanish *as well* [25]. Moreover, for gauge-natural field theories, here we stress that generalized Bergmann–Bianchi identities hold true in a canonical covariant way *if and only if* the second variational derivative - with respect to vertical parts of gauge-natural lifts - of the Lagrangian vanishes [24]. As a quite strong consequence, for any gauge-natural invariant field theory we find that the above mentioned indeterminacy can be *always* solved canonically.

As a consequence of the Second Noether Theorem, we further show that there exists a covariantly conserved current associated with the Lagrangian obtained by contracting the Euler–Lagrange morphism with a gauge-natural Jacobi vector field.

2. FINITE ORDER JETS OF GAUGE-NATURAL BUNDLES

We recall some basic facts about jet spaces [16, 26]. Our framework is a fibered manifold $\pi : Y \rightarrow X$, with $\dim X = n$ and $\dim Y = n + m$.

For $s \geq q \geq 0$ integers we are concerned with the s -jet space $J_s Y$ of s -jet prolongations of (local) sections of π ; in particular, we set $J_0 Y \equiv Y$. We recall the natural fiberings $\pi_q^s : J_s Y \rightarrow J_q Y$, $s \geq q$, $\pi^s : J_s Y \rightarrow X$, and, among these, the *affine* fiberings π_{s-1}^s . We denote by VY the vector subbundle of the tangent bundle TY of vectors on Y which are vertical with respect to the fibering π .

Greek indices σ, μ, \dots run from 1 to n and they label basis coordinates, while Latin indices i, j, \dots run from 1 to m and label fibre coordinates, unless otherwise specified. We denote multi-indices of dimension n by boldface Greek letters such as $\alpha = (\alpha_1, \dots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \dots, n$; by an abuse of notation, we denote by σ the multi-index such that $\alpha_\mu = 0$, if $\mu \neq \sigma$, $\alpha_\mu = 1$, if $\mu = \sigma$. We also set $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $\alpha! := \alpha_1! \dots \alpha_n!$. The charts induced on $J_s Y$ are denoted by (x^σ, y_α^i) , with $0 \leq |\alpha| \leq s$; in particular, we set $y_0^i \equiv y^i$. The local vector fields and forms of $J_s Y$ induced by the above coordinates are denoted by (∂_i^σ) and (d_α^i) , respectively.

For $s \geq 1$, we consider the natural complementary fibered morphisms over $J_s Y \rightarrow J_{s-1} Y$ (see e.g. [18, 19, 31]):

$$D : J_s Y \times_X TX \rightarrow TJ_{s-1} Y, \quad \vartheta : J_s Y \times_{J_{s-1} Y} TJ_{s-1} Y \rightarrow VJ_{s-1} Y,$$

with coordinate expressions, for $0 \leq |\alpha| \leq s - 1$, given by

$$D = d^\lambda \otimes D_\lambda = d^\lambda \otimes (\partial_\lambda + y_{\alpha+\lambda}^j \partial_j^\alpha), \vartheta = \vartheta_\alpha^j \otimes \partial_j^\alpha = (d_\alpha^j - y_{\alpha+\lambda}^j d^\lambda) \otimes \partial_j^\alpha.$$

The morphisms above induce the following natural splitting (and its dual):

$$(1) \quad J_s Y \times_{J_{s-1} Y} T^* J_{s-1} Y = \left(J_s Y \times_{J_{s-1} Y} T^* X \right) \oplus C_{s-1}^*[Y],$$

where $C_{s-1}^*[Y] := \text{im } \vartheta_s^*$ and $\vartheta_s^* : J_s Y \times_{J_{s-1} Y} V^* J_{s-1} Y \rightarrow J_s Y \times_{J_{s-1} Y} T^* J_{s-1} Y$.

If $f : J_s Y \rightarrow \mathbb{R}$ is a function, then we set $D_\sigma f := \mathcal{D}_\sigma f$, $D_{\alpha+\sigma} f := D_\sigma D_\alpha f$, where D_σ is the standard *formal derivative*. Given a vector field $\Xi : J_s Y \rightarrow T J_s Y$, the splitting (1) yields $\Xi \circ \pi_s^{s+1} = \Xi_H + \Xi_V$ where, if $\Xi = \Xi^\gamma \partial_\gamma + \Xi_\alpha^i \partial_i^\alpha$, then we have $\Xi_H = \Xi^\gamma D_\gamma$ and $\Xi_V = (\Xi_\alpha^i - y_{\alpha+\gamma}^i \Xi^\gamma) \partial_i^\alpha$. We shall call Ξ_H and Ξ_V the horizontal and the vertical part of Ξ , respectively.

The splitting (1) induces also a decomposition of the exterior differential on Y , $(\pi_{s-1}^s)^* \circ d = d_H + d_V$, where d_H and d_V are defined to be the *horizontal* and *vertical differential*. The action of d_H and d_V on functions and 1-forms on $J_s Y$ uniquely characterizes d_H and d_V (see, e.g., [26, 31] for more details). A *projectable vector field* on Y is defined to be a pair (u, ξ) , where $u : Y \rightarrow TY$ and $\xi : X \rightarrow TX$ are vector fields and u is a fibered morphism over ξ . If there is no danger of confusion, we will denote simply by u a projectable vector field (u, ξ) . A projectable vector field (u, ξ) can be conveniently prolonged to a projectable vector field $(j_s u, \xi)$; coordinate expression can be found e.g. in [18, 26, 31].

2.1. Gauge-natural bundles. Let $P \rightarrow X$ be a principal bundle with structure group G . Let $r \leq k$ be integers and $W^{(r,k)} P := J_r P \times_X L_k(X)$, where $L_k(X)$ is the bundle of k -frames in X [5, 16], $W^{(r,k)} G := J_r G \odot GL_k(n)$ the semidirect product with respect to the action of $GL_k(n)$ on $J_r G$ given by the jet composition and $GL_k(n)$ is the group of k -frames in \mathbb{R}^n . Here we denote by $J_r G$ the space of (r, n) -velocities on G [16]. The bundle $W^{(r,k)} P$ is a principal bundle over X with structure group $W^{(r,k)} G$. Let F be any manifold and $\zeta : W^{(r,k)} G \times F \rightarrow F$ be a left action of $W^{(r,k)} G$ on F . There is a naturally defined right action of $W^{(r,k)} G$ on $W^{(r,k)} P \times F$ so that we can associate in a standard way to $W^{(r,k)} P$ the bundle, on the given basis X , $Y_\zeta := W^{(r,k)} P \times_\zeta F$.

Definition 1. We say $(Y_\zeta, X, \pi_\zeta; F, G)$ to be the *gauge-natural bundle* of order (r, k) associated to the principal bundle $W^{(r,k)} P$ by means of the left action ζ of the group $W^{(r,k)} G$ on the manifold F [5, 16].

Remark 1. A principal automorphism Φ of $W^{(r,k)} P$ induces an automorphism of the gauge-natural bundle by:

$$(2) \quad \Phi_\zeta : Y_\zeta \rightarrow Y_\zeta : [(j_r^x \gamma, j_k^0 t), \hat{f}]_\zeta \mapsto [\Phi(j_r^x \gamma, j_k^0 t), \hat{f}]_\zeta,$$

where $\hat{f} \in F$ and $[\cdot, \cdot]_\zeta$ is the equivalence class induced by the action ζ .

Definition 2. We define the *vector bundle* over X of right-invariant infinitesimal automorphisms of P by setting $\mathcal{A} = TP/G$.

We also define the *vector bundle* over X of right invariant infinitesimal automorphisms of $W^{(r,k)}P$ by setting $\mathcal{A}^{(r,k)} := TW^{(r,k)}P/W^{(r,k)}G$ ($r \leq k$).

Denote by \mathcal{T}_X and $\mathcal{A}^{(r,k)}$ the sheaf of vector fields on X and the sheaf of right invariant vector fields on $W^{(r,k)}P$, respectively. A functorial mapping \mathfrak{G} is defined which lifts any right-invariant local automorphism (Φ, ϕ) of the principal bundle $W^{(r,k)}P$ into a unique local automorphism (Φ_ζ, ϕ) of the associated bundle Y_ζ . Its infinitesimal version associates to any $\Xi \in \mathcal{A}^{(r,k)}$, projectable over $\xi \in \mathcal{T}_X$, a unique *projectable* vector field $\hat{\Xi} := \mathfrak{G}(\Xi)$ on Y_ζ , the *gauge-natural lift*, in the following way:

$$(3) \quad \mathfrak{G} : Y_\zeta \times_X \mathcal{A}^{(r,k)} \rightarrow TY_\zeta : (\mathbf{y}, \Xi) \mapsto \hat{\Xi}(\mathbf{y}),$$

where, for any $\mathbf{y} \in Y_\zeta$, one sets: $\hat{\Xi}(\mathbf{y}) = \frac{d}{dt}[(\Phi_\zeta t)(\mathbf{y})]_{t=0}$, and $\Phi_\zeta t$ denotes the (local) flow corresponding to the gauge-natural lift of Φ_t .

This mapping fulfils the following properties (see [16]):

- (1) \mathfrak{G} is linear over id_{Y_ζ} ;
- (2) we have $T\pi_\zeta \circ \mathfrak{G} = id_{TX} \circ \bar{\pi}^{(r,k)}$, where $\bar{\pi}^{(r,k)}$ is the natural projection $Y_\zeta \times_X \mathcal{A}^{(r,k)} \rightarrow TX$;
- (3) for any pair $(\bar{\Lambda}, \bar{\Xi}) \in \mathcal{A}^{(r,k)}$, we have $\mathfrak{G}([\bar{\Lambda}, \bar{\Xi}]) = [\mathfrak{G}(\bar{\Lambda}), \mathfrak{G}(\bar{\Xi})]$.

2.2. Lie derivative of sections.

Definition 3. Let γ be a (local) section of Y_ζ , $\Xi \in \mathcal{A}^{(r,k)}$ and $\hat{\Xi}$ its gauge-natural lift. Following [16] we define the *generalized Lie derivative* of γ along the vector field $\hat{\Xi}$ to be the (local) section $\mathcal{L}_{\hat{\Xi}}\gamma : X \rightarrow VY_\zeta$, given by $\mathcal{L}_{\hat{\Xi}}\gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma$.

Remark 2. The Lie derivative operator acting on sections of gauge-natural bundles satisfies the following properties:

- (1) for any vector field $\Xi \in \mathcal{A}^{(r,k)}$, the mapping $\gamma \mapsto \mathcal{L}_{\hat{\Xi}}\gamma$ is a first-order quasilinear differential operator;
 - (2) for any local section γ of Y_ζ , the mapping $\Xi \mapsto \mathcal{L}_{\hat{\Xi}}\gamma$ is a linear differential operator;
 - (3) we can regard $\mathcal{L}_{\hat{\Xi}} : J_1Y_\zeta \rightarrow VY_\zeta$ as a morphism over the basis X . By using the canonical isomorphisms $VJ_sY_\zeta \simeq J_sVY_\zeta$ for all s , we have $\mathcal{L}_{\hat{\Xi}}[j_s\gamma] = j_s[\mathcal{L}_{\hat{\Xi}}\gamma]$, for any (local) section γ of Y_ζ and for any (local) vector field $\Xi \in \mathcal{A}^{(r,k)}$. Furthermore, for gauge-natural lifts, the fundamental relation hold true:
- $$(4) \quad \hat{\Xi}_V := \mathfrak{G}(\Xi)_V = -\mathcal{L}_{\hat{\Xi}}.$$

3. VARIATIONAL SEQUENCES AND NOETHER THEOREMS

For the sake of simplifying notation, sometimes, we will omit the subscript ζ , so that all our considerations shall refer to Y as a gauge-natural bundle as defined above.

For convenience of the reader, we sketch the connection of the purely differential setting of variational sequences with the classical integral presentation of Calculus of Variations, although the two approaches (differential and integral one) are completely independent, even if the latter provided the motivation to the former from an historical viewpoint.

In the formulation of variational problems on jet spaces of a fibered manifold $Y \rightarrow X$, with $n = \dim X$ and $m = \dim Y - n$ (see e.g. [12, 14, 22, 26]), it is well known that, given an s -th order Lagrangian $\lambda \in \mathcal{H}_s^n$, the *action* of λ along a section $\gamma : U \rightarrow Y$, on an oriented open subset U of X with compact closure and regular boundary, is defined to be the real number

$$\int_U (j_s \gamma)^* \lambda.$$

A *variation vector field* is a vertical vector field $u : Y \rightarrow TY$ defined along $\gamma(U)$. A local section $\gamma : U \rightarrow Y$ is said to be *critical* if, for each variation vector field with flow ϕ_t , we have

$$\delta \int_U (j_s \phi_t \circ j_s \gamma)^* \lambda = 0,$$

where δ is the Fréchet derivative with respect to the parameter t , at $t = 0$. It is easy to see that the previous integral expression is equal to $\int_U (j_s \gamma)^* L_{j_s u} \lambda = 0$ for each variation vector field u , where $L_{j_s u}$ is the Lie derivative operator. For each variation vector field u satisfying suitable boundary conditions, since $L_{j_s u} \lambda = i_{j_s u} d\lambda$, as an application of the Stokes Theorem, we find that the above equation is equivalent to $\int_U (j_{2s} \gamma)^* (i_u E_{d\lambda}) = 0$, where $E_{d\lambda}$ is the *generalized Euler-Lagrange operator* associated with λ (see later). Finally, by virtue of the fundamental Lemma of the Calculus of Variations the above condition is equivalent to $E_{d\lambda} \circ j_{2s} \gamma = 0$, known as the Euler-Lagrange equations (see e.g. the review in [18]).

Let us now construct the Krupka's finite order variational sequence.

According to [18, 31], the fibered splitting (1) yields the *sheaf splitting* $\mathcal{H}_{(s+1,s)}^p = \bigoplus_{i=0}^p C_{(s+1,s)}^{p-i} \wedge \mathcal{H}_{s+1}^i$, which restricts to the inclusion $\Lambda_s^p \subset \bigoplus_{i=0}^p C^{p-i}_s \wedge \mathcal{H}_{s+1}^i$, where $\mathcal{H}_{s+1}^{p,h} := h(\Lambda_s^p)$ for $0 < p \leq n$ and the surjective map h is defined to be the restriction to Λ_s^p of the projection of the above splitting onto the non-trivial summand with the highest value of i . By an abuse of notation, let us denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$ in the standard way. We set $\Theta_s^* := \ker h + d \ker h$.

In [18] it was proved that the following s -th order *variational sequence* associated with the fibered manifold $Y \rightarrow X$ is an exact resolution of the constant sheaf \mathbb{R}_Y over Y :

$$0 \rightarrow \mathbb{R}_Y \rightarrow \Lambda_s^0 \xrightarrow{\mathcal{E}_0} \Lambda_s^1 / \Theta_s^1 \xrightarrow{\mathcal{E}_1} \Lambda_s^2 / \Theta_s^2 \xrightarrow{\mathcal{E}_2} \dots \xrightarrow{\mathcal{E}_{I-1}} \Lambda_s^I / \Theta_s^I \xrightarrow{\mathcal{E}_I} \Lambda_s^{I+1} \xrightarrow{d} 0,$$

where the integer I depends on the dimension of the fibers of Y (see [18]).

For practical purposes we shall limit ourselves to consider the truncated variational sequence introduced by Vitolo in [31]:

$$0 \rightarrow \mathbb{R}_Y \rightarrow \mathcal{V}_s^0 \xrightarrow{\mathcal{E}_0} \mathcal{V}_s^1 \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{\mathcal{E}_n} \mathcal{V}_s^{n+1} \xrightarrow{\mathcal{E}_{n+1}} \mathcal{E}_{n+1}(\mathcal{V}_s^{n+1}) \xrightarrow{\mathcal{E}_{n+2}} 0,$$

where, following [31], the sheaves $\mathcal{V}_s^p := C_s^{p-n} \wedge \mathcal{H}_{s+1}^{n,h} / h(d \ker h)$ with $0 \leq p \leq n + 2$ are suitable representations of the corresponding quotient sheaves in the variational sequence by means of sheaves of sections of tensor bundles.

Let $\alpha \in C_s^1 \wedge \mathcal{H}_{s+1}^{n,h} \subset \mathcal{V}_{s+1}^{n+1}$. Then there is a unique pair of sheaf morphisms ([14, 17, 31])

$$(5) \quad E_\alpha \in C_{(2s,0)}^1 \wedge \mathcal{H}_{2s+1}^{n,h}, \quad F_\alpha \in C_{(2s,s)}^1 \wedge \mathcal{H}_{2s+1}^{n,h},$$

such that $(\pi_{s+1}^{2s+1})^* \alpha = E_\alpha - F_\alpha$ and F_α is locally of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in C^1_{(2s-1, s-1)} \wedge \mathcal{H}^{n-1}_{2s}$.

We shall now introduce a - for our purposes - fundamental morphism, denoted by K_η , represented by Vitolo in [31] and further studied by Kolář and Vitolo in [17].

Let then $\eta \in C^1_s \wedge C^1_{(s,0)} \wedge \mathcal{H}^{n,h}_{s+1} \subset \mathcal{V}^{n+2}_{s+1}$; then there is a unique morphism

$$K_\eta \in C^1_{(2s,s)} \otimes C^1_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}$$

such that, for all $\Xi : Y \rightarrow VY$, $E_{j_s \Xi} \eta = C^1_1(j_{2s} \Xi \otimes K_\eta)$, where C^1_1 stands for tensor contraction on the first factor and \lrcorner denotes inner product (see [17, 31]). Furthermore, there is a unique pair of sheaf morphisms

$$(6) \quad H_\eta \in C^1_{(2s,s)} \wedge C^1_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}, \quad G_\eta \in C^2_{(2s,s)} \wedge \mathcal{H}^{n,h}_{2s+1},$$

such that $(\pi_{s+1}^{2s+1})^* \eta = H_\eta - G_\eta$ and $H_\eta = \frac{1}{2} A(K_\eta)$, where A stands for antisymmetrisation. Moreover, G_η is locally of the type $G_\eta = d_H q_\eta$, where $q_\eta \in C^2_{(2s-1, s-1)} \wedge \mathcal{H}^{n-1}_{2s}$; hence $[\eta] = [H_\eta]$ [17, 31].

Remark 3. A section $\lambda \in \mathcal{V}_s^n$ is just a Lagrangian of order $(s + 1)$ of the standard literature. Furthermore $\mathcal{E}_n(\lambda) \in \mathcal{V}_s^{n+1}$ coincides with the standard higher order Euler-Lagrange morphism associated with λ . Let $\gamma \in \Lambda_s^{n+1}$. The morphism $H_{hd\gamma} \equiv H_{[\mathcal{E}_{n+1}(\gamma)]}$, where square brackets denote equivalence class, is called the *generalized Helmholtz morphism*; its kernel coincides with Helmholtz conditions of local variationality. We shall integrate by parts the morphism K_η to provide a suitable representation of the *generalized Jacobi morphism* associated with λ [7, 9, 24, 25].

The standard Lie derivative of fibered morphisms with respect to a projectable vector field $j_s \Xi$ passes to the quotient in the variational sequence, so defining a new quotient operator (introduced in [8]), the *variational Lie derivative* $\mathcal{L}_{j_s \Xi}$, acting on equivalence classes of fibered morphisms which are sections of the quotient sheaves in the variational sequence. Thus variational Lie derivatives of generalized Lagrangians or Euler-Lagrange morphisms can be conveniently represented as equivalence classes in \mathcal{V}_s^n and \mathcal{V}_s^{n+1} . In particular, the following two results hold true [8], to which for evident reasons we will refer as the First and the Second Noether Theorem, respectively.

Theorem 1. Let $[\alpha] = h(\alpha) \in \mathcal{V}_s^n$. Then we have locally (up to pull-backs)

$$\mathcal{L}_{j_s \Xi}(h(\alpha)) = \Xi_V \lrcorner \mathcal{E}_n(h(\alpha)) + d_H(j_{2s} \Xi_V \lrcorner p_{d_V h(\alpha)} + \xi) \lrcorner h(\alpha).$$

Theorem 2. Let $\alpha \in \Lambda_s^{n+1}$. Then we have globally (up to pull-backs)

$$\mathcal{L}_{j_s \Xi}[\alpha] = \mathcal{E}_n(j_{s+1} \Xi_V \lrcorner h(\alpha)) + C^1_1(j_s \Xi_V \otimes K_{hd\alpha}).$$

Notice that the Second Noether Theorem as formulated above, is represented in terms of the morphism $K_{hd\alpha}$.

3.1. Noether conserved currents. In the following we assume that the field equations are generated by means of a variational principle from a Lagrangian which is gauge-natural invariant, i.e. invariant with respect to any gauge-natural lift of infinitesimal right invariant vector fields. Both the Noether Theorems take a quite particular form in the case of gauge-natural Lagrangian field theories (see e.g. [6, 20]) due to the fact that the generalized Lie derivative of sections of the gauge-natural bundles has

specific linearity properties recalled in Subsection 2.2 and it is related with the vertical part of gauge-natural lifts by Eq. (4).

Definition 4. Let $(\hat{\Xi}, \xi)$ be a projectable vector field on Y_ζ . Let $\lambda \in \mathcal{V}_s^n$ be a generalized Lagrangian. We say $\hat{\Xi}$ to be a *symmetry* of λ if $\mathcal{L}_{j_{s+1}\hat{\Xi}}\lambda = 0$.

We say λ to be a *gauge-natural invariant Lagrangian* if the gauge-natural lift $(\hat{\Xi}, \xi)$ of any vector field $\Xi \in \mathcal{A}^{(r,k)}$ is a symmetry for λ , i.e. if $\mathcal{L}_{j_{s+1}\Xi}\lambda = 0$. In this case the projectable vector field $\hat{\Xi} \in \mathfrak{O}(\Xi)$ is called a *gauge-natural symmetry* of λ .

In the following we rephrase the First Noether Theorem in the case of gauge-natural Lagrangians.

Proposition 1. Let $\lambda \in \mathcal{V}_s^n$ be a gauge-natural Lagrangian and $(\hat{\Xi}, \xi)$ a gauge-natural symmetry of λ . Then we have $0 = -\mathcal{L}_{\hat{\Xi}}\mathcal{E}_n(\lambda) + d_H(-j_s\mathcal{L}_{\hat{\Xi}}|p_{d_V}\lambda + \xi|\lambda)$. Suppose that $(j_{2s+1}\sigma)^*(-\mathcal{L}_{\hat{\Xi}}\mathcal{E}_n(\lambda)) = 0$. Then, the $(n-1)$ -form $\epsilon = -j_s\mathcal{L}_{\hat{\Xi}}|p_{d_V}\lambda + \xi|\lambda$ fulfills the equation $d((j_{2s}\sigma)^*(\epsilon)) = 0$.

If σ is a critical section for $\mathcal{E}_n(\lambda)$, i.e. $(j_{2s+1}\sigma)^*\mathcal{E}_n(\lambda) = 0$, the above equation admits a physical interpretation as a so-called *weak conservation law* for the density associated with ϵ and the associated sheaf morphism $\epsilon : J_{2s}Y_\zeta \times_V J_{2s}\mathcal{A}^{(r,k)} \rightarrow C_{2s}^*[\mathcal{A}^{(r,k)}] \otimes$

$C_0^*[\mathcal{A}^{(r,k)}] \wedge (\wedge^{n-1} T^*X)$ is said to be a *gauge-natural weakly conserved current*.

Remark 4. We stress that such a Noether conserved current is *not* uniquely defined, *even up to divergences*. In fact, it depends on the choice of $p_{d_V}\lambda$, which in general is not unique - even up to divergences - *depending on the fixing of suitable connections* used to derive it in an invariant way (see [14, 31] and references quoted therein).

4. VARIATIONS AND GENERALIZED JACOBI MORPHISMS

We consider *formal variations* of a morphism as *multiparameter deformations* and relate the second variational derivative of the Lagrangian λ to the Lie derivative of the associated Euler-Lagrange morphism and in turn to the generalized Bergmann-Bianchi morphism; see [24] for details.

Let $\alpha : J_sY \rightarrow \hat{\Lambda}T^*J_sY$ and let $L_{j_s\Xi_k}$ be the Lie derivative operator acting on differential fibered morphism. Let $\Xi_k, 1 \leq k \leq i$, be (vertical) variation vector fields on Y in the sense of [7, 9, 24]. We define the i -th formal variation of the morphism α to be the operator: $\delta^i\alpha = L_{j_s\Xi_1} \dots L_{j_s\Xi_i}\alpha$.

Definition 5. Let $\alpha \in (\mathcal{V}_s^n)_Y$ and \mathcal{L}_{Ξ_i} the *variational Lie derivative* [8] operator with respect to the variation vector field Ξ_i .

We define the i -th *variational derivative* operator as follows:

$$\delta^i[\alpha] := [\delta^i\alpha] = [L_{\Xi_i} \dots L_{\Xi_1}\alpha] = \mathcal{L}_{\Xi_i} \dots \mathcal{L}_{\Xi_1}[\alpha].$$

It is clear that the first variational derivative is nothing but the variational Lie derivative with respect to vertical parts of (gauge-natural lifts) of vector fields. Analogously, the second variational derivative is nothing but the iterated (twice) variational Lie derivative; thus it can be expressed by means of the Noether Theorems. As a straightforward consequence the following characterization of the second variational derivative of a generalized Lagrangian in the variational sequence holds true [24].

Proposition 2. *Let $\lambda \in (\mathcal{V}_s^n)_Y$ and let Ξ be a variation vector field; then we have*

$$(7) \quad \delta^2 \lambda = [\mathcal{L}_n(j_{2s}\Xi)h\delta\lambda] + C_1^1(j_{2s}\Xi \otimes K_{hd\delta\lambda}).$$

4.1. Generalized gauge-natural Jacobi morphisms. Let now $\hat{\Xi} \equiv \mathfrak{G}(\bar{\Xi})$ be a variation vector field associated to some $\bar{\Xi} \in \mathcal{A}^{r,k}$. Let us consider $j_s \hat{\Xi}_V$, i.e. the vertical part according to the splitting (1). We shall denote by $j_s \bar{\Xi}_V$ the induced section of the vector bundle $\mathcal{A}^{(r+s,k+s)}$. The set of all sections of this kind defines a vector subbundle of $J_s \mathcal{A}^{(r,k)}$, which by a slight abuse of notation (since we are speaking about vertical parts with respect to the splitting (1)), we shall denote by $VJ_s \mathcal{A}^{(r,k)}$.

Let λ be a Lagrangian and $\bar{\Xi}$ a variation vector field. Let us set $\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) := C_1^1(j_{2s} \hat{\Xi} \otimes K_{hd\mathcal{L}_{j_{2s+1}\bar{\Xi}_V}} \lambda) \equiv E_{j_s \hat{\Xi}} j_{hd\mathcal{L}_{j_{2s+1}\bar{\Xi}_V}} \lambda$. Let D_H be the horizontal differential on $Y_\zeta \times_V \mathcal{A}^{(r,k)}$. Since $D_H \chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$, by applying a global decomposition formula for vertical morphisms due to Kolář [14], as a consequence of linearity properties of both $\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ and the Lie-derivative operator \mathcal{L} , from Proposition 2 we deduce what follows.

Lemma 1. *We have:*

$$(\pi_{2s+1}^{4s+1})^* \chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) = E_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)} + F_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)},$$

where

$$E_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)} : J_{4s} Y_\zeta \times_V J_{4s} \mathcal{A}^{(r,k)} \rightarrow C_0^*[\mathcal{A}^{(r,k)}] \otimes C_0^*[\mathcal{A}^{(r,k)}] \wedge (\wedge^{\bar{n}} T^* X),$$

and locally, $F_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$, with

$$M_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)} : J_{4s-1} Y_\zeta \times_V J_{4s-1} \mathcal{A}^{(r,k)} \rightarrow C_{2s-1}^*[\mathcal{A}^{(r,k)}] \otimes C_0^*[\mathcal{A}^{(r,k)}] \wedge (\wedge^{\bar{n}-1} T^* X).$$

Definition 6. We call the morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ the *gauge-natural generalized Jacobi morphism* associated with the Lagrangian λ and the variation vector field $\mathfrak{G}(\bar{\Xi})_V$.

The morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ is a linear morphism with respect to the projection $J_{4s} Y_\zeta \times_V J_{4s} \mathcal{A}^{(r,k)} \rightarrow J_{4s} Y_\zeta$.

As a consequence of Theorem 2 and Proposition 2 we have the following characterization of the Second Noether Theorem for gauge-natural invariant Lagrangian field theories in terms of the second variational derivative (see [24] for the proof in detail).

Theorem 3. *Let $\delta_{\mathfrak{G}}^2 \lambda$ be the variation of λ with respect to vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. We have:*

$$(8) \quad \mathcal{L}_{\mathfrak{G}(\bar{\Xi})_V} \mathcal{L}_{\mathfrak{G}(\bar{\Xi})_V} := \delta_{\mathfrak{G}}^2 \lambda = \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V).$$

Furthermore:

$$(9) \quad \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \mathfrak{G}(\bar{\Xi})_V | \mathcal{E}_n(\mathfrak{G}(\bar{\Xi})_V) | \mathcal{E}_n(\lambda) = \mathcal{E}_n(\mathfrak{G}(\bar{\Xi})_V) | h(d\delta\lambda).$$

This result generalizes a classical result due to Goldschmidt and Sternberg [12] relating the Hessian with the Jacobi morphism for first order field theories; in addition here the gauge-natural structure of the theories under consideration enables us to define the *generalized gauge-natural Jacobi morphism* where the *variation vector fields* are Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts.

4.2. The Bergmann–Bianchi morphism. It is a well known procedure to perform suitable integrations by parts to decompose the conserved current ϵ into the sum of a conserved current $\tilde{\epsilon}$ vanishing along solutions of the Euler–Lagrange equations, the so-called *reduced current*, and the formal divergence of a skew-symmetric (tensor) density ν called a *superpotential* (which is then defined modulo a divergence). Within such a procedure, the generalized Bergmann–Bianchi identities are in fact necessary and (locally) sufficient conditions for the conserved current ϵ to be not only closed but also the divergence of a skew-symmetric (tensor) density along solutions of the Euler–Lagrange equations. In [24, 25], for the first time, the relation of the kernel of the gauge-natural Jacobi morphism with the kernel of the *Bergmann–Bianchi* morphism has been explicated in order to characterize generalized Bianchi identities in terms of a special class of gauge-natural lifts, namely those which have their vertical part in the kernel of the generalized gauge-natural Jacobi morphism.

Let now consider the term $\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V) := -\mathcal{L}_{\tilde{\Xi}} \mathcal{E}_n(\lambda)$ appearing in the formulation of the First Noether Theorem given in Proposition 1. We stress that along sections which are not critical such a term is not vanishing, in general. In the following we shall manipulate it to derive - under precise conditions - a strongly conserved current, i.e. a current satisfying a Noether conservation law also along non critical sections, considered for the first time by Bergmann in [3].

In fact, as a further application of the global decomposition formula of vertical morphisms due to Kolář [14], following essentially the procedure proposed by Bergmann in [3], we can integrate by parts $\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)$ to define the *generalized Bergmann–Bianchi morphism*.

Lemma 2. *We have globally*

$$(\pi_{s+1}^{4s+1})^* \omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\tilde{\Xi})_V) + F_{\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)},$$

where $\beta(\lambda, \mathfrak{G}(\tilde{\Xi})_V) \equiv E_{\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)}$, and locally, $F_{\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)} = D_H M_{\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)}$.

Coordinate expressions for the morphisms $\beta(\lambda, \mathfrak{G}(\tilde{\Xi})_V)$ and $M_{\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)}$ can be found by a backwards procedure (see e.g. [14]). In particular, $\beta(\lambda, \mathfrak{G}(\tilde{\Xi})_V)$ is nothing but the Euler–Lagrange morphism associated with the *new* Lagrangian $\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)$ defined on the fibered manifold $J_{2s} \mathcal{Y}_\zeta \times_V J_{2s} \mathcal{A}^{(r,k)} \rightarrow \mathbf{X}$. In particular, we get the following

local decomposition of $\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)$:

$$(10) \quad \omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\tilde{\Xi})_V) + D_H \tilde{\epsilon}(\lambda, \mathfrak{G}(\tilde{\Xi})_V),$$

where we put $\tilde{\epsilon}(\lambda, \mathfrak{G}(\tilde{\Xi})_V) \equiv M_{\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)}$.

Definition 7. We call the global morphism $\beta(\lambda, \mathfrak{G}(\tilde{\Xi})_V) := E_{\omega(\lambda, \mathfrak{G}(\tilde{\Xi})_V)}$ the *generalized Bergmann–Bianchi morphism* associated with the Lagrangian λ and the variation vector field $\mathfrak{G}(\tilde{\Xi})_V$.

As mentioned in the Introduction, the problem of the general covariance of generalized Bianchi identities for field theories was posed by Anderson and Bergmann already in 1951 (see [1]). This problem reflects obviously on the covariance of conserved quantities (see Remark 4 above). Here we propose a way to deal with such open problems concerning globality aspects.

In fact, let now $\mathfrak{K} := \text{Ker}_{\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_{\mathcal{V}})}$ be the *kernel* of the generalized gauge-natural morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_{\mathcal{V}})$. As a consequence of Theorem 3 and of Lemma 2, we have the following covariant characterization of the kernel of generalized Bergmann–Bianchi morphism, the detailed proof of which will appear in [24].

Theorem 4. *The generalized Bianchi morphism is globally vanishing if and only if $\delta_{\mathfrak{G}}^2 \lambda \equiv \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_{\mathcal{V}}) = 0$, i.e. if and only if $\mathfrak{G}(\bar{\Xi})_{\mathcal{V}} \in \mathfrak{K}$.*

The gauge-natural invariance of the variational principle *in its whole* enables us to solve the *intrinsic indeterminacy* in the conserved charges associated with gauge-natural symmetries of Lagrangian field theories (in [20], for example, the special case of the gravitational field coupled with fermionic matter is considered and the Kosmann lift is then invoked as an *ad hoc* choice to recover the well known expression of the Komar superpotential). This is well known to be of great importance within the theory of Lie derivative of sections of a gauge-natural bundle and notably for the Lie derivative of spinors (see e.g. the review given in [20]). As a quite strong consequence of the above Theorem, for any gauge-natural invariant field theory we find that the above mentioned indeterminacy can be *always* solved canonically as shown by the following.

Corollary 1. *Let $\lambda \in \mathcal{V}_s^n$ be a gauge-natural invariant generalized Lagrangian and let $\mathfrak{G}(\bar{\Xi})$ be a gauge-natural lift of the principal infinitesimal automorphism $\bar{\Xi} \in \mathcal{A}^{r,k}$, i.e. a gauge-natural symmetry of λ . Then $\mathfrak{G}(\bar{\Xi})_{\mathcal{V}}$ satisfies the invariant condition*

$$(-1)^{|\sigma|} D_{\sigma} \left(D_{\mu} \hat{\Xi}^j \left(\partial_j (\partial_i^{\mu} \lambda) - \sum_{|\alpha|=0}^{s-|\mu|} (-1)^{|\mu+\alpha|} \frac{(\mu+\alpha)!}{\mu! \alpha!} D_{\alpha} \partial_j^{\alpha} (\partial_i^{\mu} \lambda) \right) \right) = 0.$$

In particular, the condition $j_s \bar{\Xi}_{\mathcal{V}} = D_{\alpha} (\bar{\Xi}_{\mathcal{V}}^i) \partial_i^{\alpha} \in \mathfrak{K}$ implies, of course, that the components $\bar{\Xi}_{\alpha}^i$ and $\bar{\Xi}^{\gamma}$ are not independent, but they are *related* in such a way that $j_s \hat{\Xi}_{\mathcal{V}} = D_{\alpha} (\hat{\Xi}^i - y_j^i \hat{\Xi}^{\gamma}) \partial_i^{\alpha}$ must be a solution of generalized gauge-natural Jacobi equations for the Lagrangian λ .

According with the above Corollary, we shall refer to *canonical* covariant currents or to corresponding superpotentials by stressing their dependence on \mathfrak{K} ; i.e. by Theorem 4 in correspondence of gauge-natural lifts satisfying covariant Bergmann–Bianchi identities.

Remark 5. Let then $\lambda \in \mathcal{V}_s^n$ be a gauge-natural Lagrangian and $j_s \hat{\Xi}_{\mathcal{V}} \in \mathfrak{K}$ a gauge-natural symmetry of λ . Being $\beta(\lambda, \mathfrak{K}) \equiv 0$, we have, *globally*, $\omega(\lambda, \mathfrak{K}) = D_H \epsilon(\lambda, \mathfrak{K})$, then from the First Noether Theorem we have $D_H(\epsilon(\lambda, \mathfrak{K}) - \tilde{\epsilon}(\lambda, \mathfrak{K})) = 0$, which is a so-called gauge-natural ‘strong’ conservation law for the *global canonical* density $\epsilon(\lambda, \mathfrak{K}) - \tilde{\epsilon}(\lambda, \mathfrak{K})$.

As an important application, we recall that recently the existence of *canonical gauge-natural superpotential* associated with λ and \mathfrak{K} has been accordingly established in the framework of variational sequences [24, 25]. In fact, let $\lambda \in \mathcal{V}_s^n$ be a gauge-natural Lagrangian and $(j_s \hat{\Xi}, \xi)$ a gauge-natural symmetry of λ . Then there exists a canonical global sheaf morphism $\nu(\lambda, \mathfrak{K}) \in (\mathcal{V}_{2s-1}^{n-2})_{Y_{\zeta} \times_X \mathfrak{K}}$ such that $D_H \nu(\lambda, \mathfrak{K}) = \epsilon(\lambda, \mathfrak{K}) - \tilde{\epsilon}(\lambda, \mathfrak{K})$. Notice that by the exactness of the variational sequence, the existence of a local

superpotential can be deduced as a section of $(\mathcal{V}_{2s-1}^{n-2})_{Y_{\zeta \times X}^{\mathcal{A}(r,k)}}$. This local section can be always globalized by fixing gauge-natural prolongations of principal connections on P which are prolongations of principal connections with respect to linear symmetric connections on the basis manifold [6]. However, such a globalization depends on the choice of the connection itself. Furthermore, although this choice can be always done geometrically, connections are generally the unknown to be determined in field theories and then they *should not* be fixed *a priori* in a consistent truly covariant field theory. Our result enables us to get global sections of the *reduced sheaf* $(\mathcal{V}_{2s-1}^{n-2})_{Y_{\zeta \times X}^{\mathcal{R}'}}$, without fixing any connection *a priori*.

4.3. Generalized symmetries and Bergmann–Bianchi identities. As well known, the Second Noether Theorem deals with invariance properties of the Euler–Lagrange equations (so-called generalized symmetries or also Bessel–Hagen symmetries, see e.g. the fundamental papers [28]). Although symmetries of a Lagrangian turn out to be also symmetries of the Euler–Lagrange morphism the converse is not true, in general.

In particular, although for a gauge-natural invariant Lagrangian λ we always have $\mathcal{L}_{j_s \bar{\Xi}} \lambda = 0$, $\mathcal{L}_{j_s \bar{\Xi}_V} \lambda$ does not need to be zero in principle; however when the second variation $\delta_{\mathcal{G}}^2 \lambda$ is required to vanish then $\mathcal{L}_{j_s \bar{\Xi}_V} \mathcal{E}_n(\lambda)$ surely vanishes, i.e. $j_s \bar{\Xi}_V$ is a generalized or Bessel–Hagen symmetry. The symmetries of the Euler–Lagrange morphism (Second Noether Theorem) impose some constraints on the conserved quantities associated with gauge-natural symmetries of λ (see e.g. [1]).

Symmetries of the Euler–Lagrange morphism are clearly related with invariance properties of $\omega(\lambda, \mathcal{G}(\bar{\Xi})_V) := -\mathcal{L}_{\bar{\Xi}} \mathcal{E}_n(\lambda)$. We stress that, because of linearity properties of \mathcal{L} , $\omega(\lambda, \mathcal{G}(\bar{\Xi})_V)$ can be considered as a new Lagrangian, defined on an extended space; thus Theorems 3 and 4 can provide us with some kind of Noether conservation law associated with the induced invariance properties of $\omega(\lambda, \mathcal{G}(\bar{\Xi})_V)$.

First of all let us make the following important consideration.

Proposition 3. *For each $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ such that $\bar{\Xi}_V \in \mathcal{R}$, we have*

$$(11) \quad \mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathcal{R}) = -D_H(-j_s \mathcal{L}_{\bar{\Xi}_V} \rfloor p_{D_V \omega(\lambda, \mathcal{R})}).$$

Proof. The horizontal splitting gives us $\mathcal{L}_{j_s \bar{\Xi}} \omega(\lambda, \mathcal{R}) = \mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathcal{R}) + \mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathcal{R})$. Furthermore, $\omega(\lambda, \mathcal{R}) \equiv -\mathcal{L}_{\bar{\Xi}} \mathcal{E}_n(\lambda) = \mathcal{L}_{j_s \bar{\Xi}} \lambda - d_H(-j_s \mathcal{L}_{\bar{\Xi}} \rfloor p_{d_V \lambda} + \xi \rfloor \lambda)$; so that $\mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathcal{R}) = \mathcal{L}_{j_s \bar{\Xi}_V} \mathcal{L}_{j_s \bar{\Xi}} \lambda = \mathcal{L}_{j_s [\bar{\Xi}_V, \bar{\Xi}_H]} \lambda$. On the other hand we have $\mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathcal{R}) = \mathcal{L}_{j_s [\bar{\Xi}_H, \bar{\Xi}_V]} \lambda = -\mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathcal{R})$.

Recall now that $\bar{\Xi}_V \in \mathcal{R}$ if and only if $\beta(\lambda, \mathcal{R}) = 0$. Since

$$\begin{aligned} \mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathcal{R}) &= -\mathcal{L}_{\bar{\Xi}_V} \mathcal{E}_n(\omega(\lambda, \mathcal{R})) + D_H(-j_s \mathcal{L}_{\bar{\Xi}_V} \rfloor p_{D_V \omega(\lambda, \mathcal{R})}) \\ &= \beta(\lambda, \mathcal{R}) + D_H(-j_s \mathcal{L}_{\bar{\Xi}_V} \rfloor p_{D_V \omega(\lambda, \mathcal{R})}), \end{aligned}$$

we get the assertion. □

It is easy to realize that, because of the gauge-natural invariance of the generalized Lagrangian λ , the new generalized Lagrangian $\omega(\lambda, \mathcal{R})$ is gauge-natural invariant too, i.e. $\mathcal{L}_{j_s \bar{\Xi}} \omega(\lambda, \mathcal{R}) = 0$. However, a stronger result holds true. In fact, we can state the following naturality property for $\omega(\lambda, \mathcal{R})$, which provides some more information concerning the Hamiltonian structure of gauge-natural field theories [10].

Proposition 4. *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have $\mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathfrak{K}) = 0$.*

Proof. In fact, when $\bar{\Xi}_V \in \mathfrak{K}$, by the theory of iterated Lie derivatives of sections [16], we have $\mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathfrak{K}) = [\bar{\Xi}_V, \bar{\Xi}_V] \mathcal{E}_n(\lambda) + \bar{\Xi}_V \mathcal{L}_{j_s \bar{\Xi}_V} \mathcal{E}_n(\lambda) = 0$. Thus $\mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathfrak{K}) + \mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathfrak{K}) = 0$. \square

As a consequence of Propositions 3 and 4, corresponding to any $\mathfrak{G}(\bar{\Xi})_H$, we get the existence of a generalized Noether conserved current (which could be interpreted as a *generalized energy-momentum tensor* for $\omega(\lambda, \mathfrak{K})$).

Corollary 2. *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have the covariant conservation law*

$$(12) \quad D_H(-j_s \mathcal{L}_{\bar{\Xi}_V} \mathcal{P}_{D_V \omega(\lambda, \mathfrak{K})}) = 0.$$

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