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# Analysis in complex quaternions and its connections with mathematical physics

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The use of complex quaternions in mathematical physics is far from being new, many relativistic notions have been naturally expressed in term of complex quaternions ( [1], [2] ). We want to describe here some new aspects of the connections between complex-quaternionic analysis and mathematical physics, especially that of twistor theory.

## 1. Introductory remarks.

a) Quaternionic analysis.

There are two interesting types of regular quaternionic functions:

(i) $f : \mathbb{Q} \to \mathbb{Q}$ is said to be regular at $x \in \mathbb{Q}$ iff

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists.}$$

It can be shown ([3]) that only linear function have this property. Such functions can be described as a solution of a differential operator $D_1 f = 0$ .

(ii) Another generalization of the notion of holomorphic functions was introduced by Fueter; let us denote by $D_2$ the operator

$$D_2 = \frac{\partial}{\partial x_0} - i_1\frac{\partial}{\partial x_1} - i_2\frac{\partial}{\partial x_2} - i_3\frac{\partial}{\partial x_3} \;;\; x = x_0 + i_1x_1 + i_2x_2 + i_3x_3 \in \mathbb{Q},$$

The solutions of the equation $D_2 f = 0$ have many nice properties in common with holomorphic functions (Cauchy integral formula, residue formula, power series expansion - see [3] ).

b) Basic twistor diagram.

Let us consider 4-dimensional complex vector space $\mathbb{T}_4$ . We shall use flag anifolds $F_1; F_2; F_{1,2}$ of vector subspaces of $\mathbb{T}_4$:

$$F_1 = \{ L_1 \subset \mathbb{T}_4 \mid \dim L_1 = 1 \} \cong \mathbb{P}^3(\mathbb{C})$$

$$F_2 = \{ L_2 \subset \mathbb{T}_4 \mid \dim L_2 = 2 \} \cong G_{2,4}(\mathbb{C}) \cong \overline{\mathbb{C}M}$$

$$F_{1,2} = \{ [L_1, L_2] \mid L_1 \subset L_2,\ \dim L_1 = 1,\ \dim L_2 = 2 \}$$

The Grasmanian $G_{2,4}$ can be considered to be the conformal compactification of complex Minkowski space $\overline{\mathbb{C}M}$ .

Set-valued maps $\varphi, \psi$ defined using natural forgetting projections in basic twistor diagram

$$\begin{tikzcd} & F_{1,2} \arrow[dl] \arrow[dr] & \\ P^3(\mathbb{C}) \arrow[rr, squiggly, shift left, "\varphi"] & & \overline{\mathbb{C}M} \arrow[ll, squiggly, shift left, "\psi"] \end{tikzcd}$$

$$\varphi(L_1) = \{L_2 \mid L_2 \supset L_1\} \sim \mathbb{P}_2(\mathbb{C}) \quad (\equiv \alpha\text{-plane in } \mathbb{C}M)$$

$$\psi(L_2) = \{L_1 \mid L_1 \subset L_2\} \sim \mathbb{P}_1(\mathbb{C})$$

are fundamental maps in twistor theory. They are used in Penrose transform, Ward's correspondence and for nonlinear gravitons ([4])

## 2. Space $\mathbb{C}\mathbb{Q}$ of complex quaternions.

From the analysists point of view, the main reason why to investigate the algebra $\mathbb{C}\mathbb{Q}$ is the fact that both possibility for regular functions are (as mappings from $\mathbb{R}_4$ to $\mathbb{R}_4$ ) real-analytic. But, in a sense, it is the nonsense to consider only such mappings for it is well-known that many vital informations on them are hidden in its holomorphic extensions to the maps from $\mathbb{C}\mathbb{Q}$ to $\mathbb{C}\mathbb{Q}$. So it is natural to consider holomorphic extensions of regular functions of quaternion variables as the basic notion of 'quaternion analysis'. To have better impression on the physical meaning of properties of the basic algebra $\mathbb{C}\mathbb{Q}$, let us consider first some of its algebraic properties in more details.

a) The algebraic structure of $\mathbb{C}\mathbb{Q}$.

We have two natural conjugations in $\mathbb{C}\mathbb{Q}$ :

$$q^+ = q_0 - i_1 q_1 - i_2 q_2 - i_3 q_3$$
$$q^* = q_0^* + i_1 q_1^* + i_2 q_2^* + i_3 q_3^*$$
$$|q|^2 = q \cdot q^+ \in \mathbb{C} \quad , \quad (q_1 q_2)^+ = q_2^+ q_1^+ .$$

The algebra $\mathbb{C}\mathbb{Q}$ is no more a field: $q^{-1}$ exists iff $|q|^2 \neq 0$; $q^{-1} = \dfrac{q^+}{|q|^2}$

Let us denote $N = \{q \mid |q|^2 = 0\}$.

Sometimes it is useful to work in a special representation of $\mathbb{C}\mathbb{Q}$:

$$q \in \mathbb{C}\mathbb{Q} \leftrightarrow \boxed{q} = q_0 \cdot \mathbb{1} - i\sigma_1 q_1 - i\sigma_2 q_2 - i\sigma_3 q_3 =$$
$$= \begin{bmatrix} q_0 - iq_3 , & -q_2 - iq_1 \\ q_2 - iq_1 , & q_0 + iq_3 \end{bmatrix} \in L(2, \mathbb{C}),$$

$|q|^2 = \det \boxed{q}$ , $q^{+*}$ ~ Hermitian conjugation

The standard ph sical interpretation of $\mathbb{C}\mathbb{Q}$ is ( [1] ):

$$q_\mu \in \mathbb{C}M \longleftrightarrow \tilde{q}_\mu = q_0 + i i_1 q_1 + i i_2 q_2 + i i_3 q_3$$

$$q_\mu q^\mu = |\tilde{q}|^2$$

Lorentz group action on $\mathbb{C}\mathbb{Q}$ is described by ( [1] ):

$q \longmapsto AqB$, $|A|^2 = |B|^2 = 1$ ... complex Lorentz group

$q \longmapsto AqA^{+*}$, $|A| = 1$ .... real Lorentz group (Minkowski space $M$ is invariant subspace)

The basic structure of any ring is the set of its ideals. The ring $\mathbb{C}\mathbb{Q}$ is not commutative, so we have two sets:

$\mathcal{L}$ ... the set of all nontrivial left ideals $L$,

$\mathcal{R}$ ... the set of all nontrivial right ideals $R$.

The following properties can be proved for $\mathcal{L}$, $\mathcal{R}$ :

1) $\dim_{\mathbb{C}} L = 2$

2) $\forall L \in \mathcal{L}$ ... $L \subset N = \{q \mid |q|^2 = 0\}$

3) $\forall x \in N$ ... $L_x = \{qx \mid q \in \mathbb{C}\mathbb{Q}\} \in \mathcal{L}$

4) either $L_x = L_y$ or $L_x \cap L_y = \{0\}$

5) $\forall L \in \mathcal{L}\ \exists x \in N$ ... $L = L_x$

6) $N = \bigcup\limits_{L \in \mathcal{L}} L = \bigcup\limits_{R \in \mathcal{R}} R = \bigcup\limits_{\substack{L \in \mathcal{L} \\ R \in \mathcal{R}}} L \cap R$ , $\dim_{\mathbb{C}} L \cap R = 1$

In the representation:

$$x \in N \iff \det \boxed{x} = 0 \iff \boxed{x} = \begin{bmatrix} aa', & ab' \\ ba', & bb' \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \otimes [a' \ b']$$

then

$$L_x = \{y \mid \boxed{y} = \begin{bmatrix} A \\ B \end{bmatrix} \otimes [a' \ b'] ;\ A, B \in \mathbb{C}\}$$

$$L_{x_1} = L_{x_2} \iff a_1' : b_1' = a_2' : b_2' .$$

So the suitable parametr set for $\mathcal{L}$ is $\mathbb{P}^1(R)$ (for any $R \in \mathcal{R}$). The numbers $[a', b']$ are homogeneous coordinate on $\mathbb{P}^1(R)$.

Consider now a function $f: \mathbb{C}\mathbb{Q} \to \mathbb{C}\mathbb{Q}$ . What is the physical interpretation of left and right ideals in this situation:

$\alpha$) on the left - in Minkowski space, where fields are living:

$L \longleftrightarrow \alpha$- planes (i.e. self-dual planes in $\mathbb{C}M$ )

$R \longleftrightarrow \beta$- planes (i.e. antiself-dual planes in $\mathbb{C}M$ )

$\beta$) on the right (values of the field):

The mappings $q \mapsto Aq$ ; $q \longmapsto qA^{+*}$

are spinor representation of Lorentz group, but they are reducible. It is easy to see that the left (right) ideals are just invariant subspaces of these representations. Hence $L \in \mathcal{L}$ can be identified with spinor space $S$ ; $R \in \mathcal{R}$ can be identified with $S'$ .

Hence the special functions $f: \mathbb{C}\mathbb{Q} \longrightarrow L \subset \mathbb{C}\mathbb{Q}$

$f: \mathbb{C}\mathbb{Q} \longrightarrow R \subset \mathbb{C}\mathbb{Q}$

can be interpretes as spinor fields.

b) The analysis on $\mathbb{C}Q$.

The basic differential operators $D_1, D_2$ from sec. 1 can be extended to holomorphic mappings from $\mathbb{C}Q$ to $\mathbb{C}Q$. After restriction to real Minkowski space $M \subset \mathbb{C}M$ nice physical interpretation can be given to these operators. The operator $D_1$ is nothing else than the Penrose's twistor operator ( [4]), the operator $D_2$ can be identify with usual (Weyl or Dirac) differential operator $\nabla_A^{A'}$ ( [4]) for massless fields. Doing analysis in $\mathbb{C}Q$ there are possibilities to mix together informations from both quaternion and complex cases. This can help to solve some problems of (real) quaternion analysis (especially connected with singularities of regular functions - see [5] ), moreover it can help in future to clarify some physical problems as well.

3. The projective space $\mathbb{P}_1(\mathbb{C}Q)$.

The main problem in quaternion analysis is to create a sufficiently rich class of quaternion manifolds. The Fueter's regular functions are not closed with respect to composition, so they can't be used as transition functions.

Looking for some models for future manifolds the best ( and simplest) ones are the projective spaces. While the space $\mathbb{P}_1(Q)$ is well-known,standard and gives no new insight, there is an unexpecting surprise hidden in the complex-quaternion version $\mathbb{P}_1(\mathbb{C}Q)$ of it.

Let us define

$$\mathbb{P}_1(\mathbb{C}Q) = \overbrace{[\mathbb{C}Q \times \mathbb{C}Q] \setminus [0,0]}^{A} /\sim$$

where $[q_1,q_2] \sim [q_1',q_2'] \iff \exists \lambda \in \mathbb{C}Q,\ |\lambda|^2 \neq 0$

$$[q_1',q_2'] = [q_1\lambda, q_2\lambda].$$

The space $\mathbb{P}_1(\mathbb{C}Q)$ is a topological space (with the factor-topology). We shall devide it into two parts

$$A = B \cup C$$
$$:= \left\{ [\mathbb{C}Q \times \mathbb{C}Q] \setminus \bigcup_{L\in\mathcal{L}} [L\times L] \right\} \cup \left\{ \bigcup_{L\in\mathcal{L}} [L\times L] \setminus [0,0] \right\}$$

and we obtain
$$\mathbb{P}_1(\mathbb{C}Q) = (B/\sim) \cup (C/\sim)$$

But after some effort we find that

$$B/\sim \;\cong\; \overline{\mathbb{C}M}\ , \quad C/\sim \;\cong\; \mathbb{P}_3(\mathbb{C})$$

so

$$\mathbb{P}^1(\mathbb{C}Q) = \overline{\mathbb{C}M} \cup \mathbb{P}_3(\mathbb{C}).$$

The topology in the whole $\mathbb{P}^1(\mathbb{C})$ is nonstandard, it is not Hausdorff.

We can prove the following facts on this topology:

$$\begin{tikzcd} & \mathbb{F}_{12} \arrow[dl] \arrow[dr] & \\ \mathbb{P}_3 \arrow[rr, shift left, squiggly, "\varphi"] & & \overline{\mathbb{C}M} \arrow[ll, shift left, "\Psi"] \end{tikzcd}$$

1) For every $\beta \in B/\sim \;\cong \overline{\mathbb{C}M}$
   it holds that $\mathrm{clos}(\beta) = \beta \cup \Psi(\beta)$.

2) For every $\gamma \in C/\sim \;\cong \mathbb{P}_3$
   it holds that $\bigcap\limits_{\sigma\ \text{open},\ \gamma\subset\sigma} \sigma = \gamma \cup \varphi(\gamma)$.

If we restrict the topology only on $\overline{\mathbb{C}M}$ ( or $\mathbb{P}_3(\mathbb{C})$ ), we shall recover the usual topology on them. So 'strangeness' of the topology is just describing twistor correspondences $\varphi, \Psi$ between $\overline{\mathbb{C}M}$ and $\mathbb{P}_3(\mathbb{C})$. The character of the topology is very closed to Zariski topology from algebraic geometry.

If we now reconsider the problem of a notion of complex-quaternion manifolds, we should (with this basic example in mind) take open subsets of $\mathbb{P}_1(\mathbb{CQ})$ with their strange topology as local models and gluing them together properly to find a new notion of such, highly nonstandard, manifold.

We hope to return to these interesting questions elsewhere.

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