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Continuous Restrictions of Linear Maps between Banach Spaces

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In this Note, which emerged from a question of the first-named author at the 17th Winter school on Abstract Analysis, we investigate under what circumstances linear mappings between Banach spaces have continuous restrictions to infinite dimensional subspaces. Two typical cases for which the answer is positive are: 1) If the source space is l^1 or 2) if the source space and the target space both are Hilbert spaces. On the other hand, for separable spaces X not containing l^1 isomorphically, there is a linear map $T: X \to l^1$ such that for no infinite dimensional subspace Z of X the restriction T/Z is continuous. Some additional remarks and questions are also included.

1. We denote by X and Y Banach spaces and by T a linear (not necessarily continuous) map from X to Y. In [1] the question was analysed under which conditions there is an infinite dimensional (not necessarily closed) subspace Z of X such that T restricted to Z is continuous. It is shown in [1] that if X does not contain l^1 then there always is a linear map T from X to some Banach space Y such that there does not exist an infinite dimensional subspace Z of X such that the restriction of T to Z is continuous. See **Prop 5** below for a strengthening of this result. In the subsequent proposition we show that the condition that X does not contain l^1 is also necessary for the above result to hold true:

2. Proposition. Let T be a linear map from l^1 to a Banach space Y. Then there is an infinite dimensional subspace Z of l^1 such that the restriction of T to Z is continuous.

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First note the subsequent easy result whose proof is analogous to the usual proof of the Banach-Steinhaus theorem and left to the reader.

3. Lemma. Let $T: X \to Y$ be linear. Then there is $M \in R^+$ such that the set $\{x \in X;\ \|T(x)\| \leqq M\}$ is **dense** in the unit ball of X.

Now we return to the **Proof of Proposition 2:**

Let M be chosen according to **Lemma 3** and let $\{e_i\}_{i=1}^{\infty}$ denote the canonical basis in l^1. For each $i \geqq 1$ we can select $x_i \in l^1$ with $\|x_i - e_i\| < 2^{-i-1}$ and $\|T(x)\| \leqq$ $\leqq M$. Hence

$$\sum_{i=1}^{\infty} \|x_i - e_i\| < 1/2$$

and by the **Perturbation**-theorem, see [2; Prop. 1. a. 9], there exists a positive number c such that

$$\Big\|\sum_{i=1}^{n} \lambda_i x_{j_i}\Big\| \geqq c \sum_{i=1}^{n} |\lambda_i| \quad \text{if} \quad n \geqq 1\,, \quad \lambda_i \in R$$

and moreover $j_1 < \ldots < j_n$. But this implies

$$\|T_{/span\{x_i, i \geqq 1\}}\| \leqq M/c < \infty\,. \quad \square$$

4. Remark. Note that the subspace constructed above is dense in l^1. Also note – as has been pointed out by P. Holicky – that a similar reasoning as above shows that for any uncountable set I and any linear mapping $T: l^1(I) \to Y$ there is an uncountable set $J \subset I$ such that the restriction of T to $span\,\{\chi_{\{j\}};\ j \in J\} \subset\subset l^1(I)$ is continuous.

Note that the question whether for a Banach space X every linear map $T: X \to Y$ admits infinite dimensional continuous restrictions now is completely settled as to whether X does or does not contain l^1 isomorphically.

However, one thing remained open. The construction in [1] used an operator $T: X \to l^1(I)$, where I is a set of the same cardinality as X. Hence the question aroses as to whether one may do the construction, where the target space is „small". This is indeed possible.

5. Proposition. Assume that the separable Banach space X does not contain l^1 isomorphically. Then there exists a linear map $T: X \to l^1$ such that there does not exist an infinite dimensional subspace Z of X such that the restriction of T to Z is continuous.

Proof. We choose a countable dense subset $\{x_i,\ i \geqq 1\}$ of the unit ball $B_X(1)$ of X Then we define by

$$S(\alpha) = \sum_{i=1}^{\infty} \alpha_i x_i \quad \text{for} \quad \alpha = (\alpha_1, \alpha_2, \ldots) \in l^1$$

a linear map from l^1 into X with $\|S\| \leqq 1$. Since $S(B_{l^1}(1))$ is dense in $B_X(1)$ and because S is continuous, the second part of the proof of the open map theorem shows that S is **onto** X. Let $Y \subset\subset l^1$ be an (arbitrary) algebraic complement of **Ker** S, then $S_{/Y}$ is a bijective map onto X. Hence, the inverse T of $S_{/Y}$ maps X into l^1 and fulfills $\|T(x)\| \geqq \|x\|$ for any $x \in X$.

Assume that that $T_{/Z}$ is continuous, where Z is an infinite dimensional subspace of X. Then there exists an continuous linear map $\hat{T}$: $clos(Z) \to l^1$ which extends T and satisfies $\|x\| \leqq \|\hat{T}(x)\| \leqq \|T\| \cdot \|x\|$ for $x \in clos(Z)$. Therefore $\hat{T}(clos(Z))$ is an infinite dimensional closed subspace of l^1 and by [2; Prop. 2. a. 2] $\hat{T}(clos(Z))$ contains l^1 isomorphically. Since $\hat{T}$ is an isomorphism, also $clos(Z) \subset\subset X$ contains l^1 isomorphically, contradiction. □

6. Remark. The proof shows in fact the following nonseparable version of the above result. Let I be a set of cardinality equal to the density character of X. Then there is a linear map $T: X \to l^1(I)$ without infinite dimensional continuous restrictions.

This has a curious consequence, which seems worth noting. Let X be a Banach space with density character less then or equal to the continuum (e.g. l^∞, $C[0, 1]^*$). Then there is a sequence $\{f_n\}_{n=1}^\infty$ of linear functionals on X such that, for every $x \in X$, $\{f_n(x)\}_{n=1}^\infty$ stays bounded, while for every infinite dimensional subspace $Z \subset\subset X$ we have

$$\sup_{\substack{z \in Z \\ \|z\| \leqq 1}} \sup_{n \in N} |f_n(z)| = \infty .$$

Indeed, as $l^1([0, 1])$ embeds (isometrically) into $l^\infty(N)$ we deduce from the above remark that for X with $dens(X) \leqq 2^\omega$ there is linear map $T: X \to l^\infty(N)$ without continuous infinite dimensional restrictions. Letting f_n to be the n'th coordinate of T, we obtain the assertion.

Note, however, that for the sequence $\{f_n\}_{n=1}^\infty$ as above it follows from **Prop. 9** below that for every closed infinite dimensional subspace X_1 of X and every sequence $\{\alpha_n\}_{n=1}^\infty$ from c_0 there is an infinite dimensional subspace $Z \subset\subset X_1$ such that

$$\sup_{\substack{z \in Z \\ \|z\| \leqq 1}} \sup_{n \in N} |\alpha_n f_n(z)| < \infty .$$

We now turn to the following question: Fix X and Y; under which assumptions is it true that every linear operator $T: X \to Y$ admits infinite dimensional continuous restrictions?

First note that if Y equals the scalars or — more generally — is finite dimensional, than this question has an affirmative answer: Indeed, it suffices to take Z = **Ker** T.

A less trivial and more natural question is to consider the case where X and Y both are *Hilbert* spaces. In this case the answer is again positive as shown by the next result, which is fairly more general.

7. Proposition. Let $1 \leqq p \leqq q < \infty$ and $T: l^p \to l^q$ be linear. Then there is an infinite dimensional subspace Z of l^p such that $T_{/Z}$ is continuous.

Proof. We may assume $p > 1$ by **Prop. 2** above. Denote by $\{e_i\}_{i=1}^{\infty}$ the canonical unit vector basis in l^p and set $\varepsilon_i = 2^{-i-2}$ for $i \geqq 1$. By **Lemma 3** we may find $M < \infty$ and a sequence $\{f_i\}_{i=1}^{\infty} \subset l^p$ such that $\|e_i - f_i\| < \varepsilon_i$ and $\|T(f_i)\| \leqq M$ if $i \geqq 1$. As l^q is reflexive, we may assume (by passing to a subsequence if necessary) that $\{T(f_i)\}_{i=1}^{\infty}$ weakly converges in l^q. Hence we may find a strictly increasing sequence $\{i_k\}_{k=1}^{\infty} \subset N^+$ and elements $\{g_k\}_{k=1}^{\infty}$ in l^p with mutually disjoint supports such that, letting $h_k = f_{i_{2k}} - f_{i_{2k-1}}$, $\|T(h_k) - g_k\|_{l^q} < \varepsilon_k$ holds for any $k \geqq 1$.

Let $Z = span(\{h_k\}_{k=1}^{\infty})$, then the restriction of T to Z is continuous. Indeed, since

$$\|h_k/2 - (e_{i_{2k}} - e_{i_{2k-1}})/2\|_{l^p} < 2^{-i_{2k-1}-2} \leqq 2^{-k-2}$$

and since $\{(e_{i_{2k}} - e_{i_{2k-1}})/2\}_{k=1}^{\infty}$ is a normalized monotone basic sequence, there exists a positive constant c such that $\|\sum_{k=1}^{n} \lambda_k h_k/2\|_{l^p} \geqq c\sqrt[p]{(\sum_{k=1}^{n} |\lambda_k|^p)}$ for every $n \geqq 1$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in R$ (see [2; Prop. 1. a. 9]). Moreover

$$\|T(\sum_{k=1}^{n} \lambda_k h_k)\|_{l^q} \leqq \|\sum_{k=1}^{n} (T(h_k) - g_k)\,\lambda_k\|_{l^q} + \|\sum_{k=1}^{n} \lambda_k g_k\|_{l^q} \leqq$$

$$\leqq \sum_{k=1}^{n} \|(T(h_k) - g_k)\,\lambda_k\|_{l^q} + \sqrt[q]{(\sum_{k=1}^{n} |\lambda_k|^q\,(\|g_k\|_{l^q})^q} \leqq$$

$$\leqq \max_{k \leqq n} |\lambda_k| \sum_{k=1}^{n} \|(T(h_k) - g_k\|_{l^q} + (M + 1)\sqrt[q]{(\sum_{k=1}^{n} |\lambda_k|^q} \leqq$$

$$\leqq \sqrt[p]{(\sum_{k=1}^{n} |\lambda_k|^p)}\, 1 + (M + 1)\sqrt[p]{(\sum_{k=1}^{n} |\lambda_k|^p)}, \quad p \leqq q\,!$$

Conseqüently,

$$\|T(\sum_{k=1}^{n} \lambda_k h_k)\|_{l^q} \leqq \frac{M + 2}{2c} \|\sum_{k=1}^{n} \lambda_k h_k\|_{l^p} .$$

Since $n \geqq 1$ and $\lambda_1, \ldots, \lambda_n$ are arbitrary, the proof is finished. □

8. Remark. First note that the above result remains true for linear maps $T: l^p(I) \to$ $\to l^q(J)$ where I and J are arbitrary sets.

Less trivial is the next consequence of the above proof, which is somewhat dual to **Prop. 2** above:

9. Proposition. Let X be an arbitrary infinite dimensional Banach space and $T: X \to$ $\to c_0$ linear. Then T admits an infinite dimensional continuous restriction.

Proof. Indeed, it suffices to choose a basic sequence $\{e_n\}_{n=1}^{\infty}$ in X (see [2; Thm. 1. a. 5.]) and to mimic the above proof.

Hence, the following questions naturally appear:

Question 1. Characterize those pairs (X, Y) of Banach spaces such that every linear $T: X \to Y$ has an infinite dimensional continuous restriction.

Question 2. In the situation of **Prop.** 7 or **Prop.** 9, can one find a **dense** subspace Z of l^p (resp. X) such that $T_{/Z}$ is continuous?

Finally we present a result, which says that the continuous restrictions can not have too nice domains (even if the target space is c_0).

10. Proposition. Let X be a separable Banach space not containing l^1 isomorphically. If Y is an arbitrary infinite dimensional Banach space then there exists a linear map $T: X \to Y$ such that there does not exist any **closed** infinite dimensional subspace Z of X such that the restriction of T to Z is continuous.

Proof. Let $\{e_n\}_{n=1}^{\infty} \subset Y$ be a basis sequence and define

$$\Phi: l^1 \to Y \quad \text{by} \quad \Phi(\alpha_1, \alpha_2, \ldots) = \sum_{k=1}^{\infty} \alpha_k e_k .$$

Then Φ is injective and continuous. According to Prop. 5 there exists $S: X \to l^1$ without any infinite dimensional continuous restriction. We define $T: X \to Y$ by $T = \Phi \circ S$.

Now assume that $T_{/Z}$ is continuous, where $Z \subset\subset X$ is an infinite dimensional closed subspace of X. Then $x_n \to x$ in Z and $Sx_n \to y$ in l^1 implies $Tx_n \to Tx$ and $Tx_n = \Phi(Sx_n) \to \Phi(y)$. We conclude $\Phi(y) = \Phi(Sx)$ and $y = Sx$. Hence $S_{/Z}$ is closed and because Z is a Banach space, $S_{/Z}$ is continuous, contradiction. □

11. Remark. However, in [1] a more satisfactory answer is given in the case that Y is a countable union of finite dimensional spaces. But there is still an open problem.

Question 3. Let $T: l^1 \to X$ be a linear map into the Banach space X. Can one find a closed infinite dimensional subspace $Z \subset\subset l^1$ such that $T_{/Z}$ is continuous?

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