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Wiesław Kurc

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Characterizations of Some Monotonicity Properties of a Lattice Norm in Musielak-Orlicz Spaces

W. KURC

Poznań*)

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Let X be a Banach lattice with a (monotone) norm $\|\cdot\|$. We call X and the norm $\|\cdot\|$ strictly monotone (STM) if $x, y \geqq 0$, $y \neq 0$ imply $\|x\| < \|x + y\|$. Following to [1] we call X and the norm $\|\cdot\|$ uniformly monotone (UM) if for every $\varepsilon > 0$ there holds $\delta_+(\varepsilon) \equiv \inf_{z \in U^+_\varepsilon} (\|z\| - 1) > 0$, where $U^+_\varepsilon = \{z = x + y : \|x\| = 1, \|y\| \geqq \varepsilon, x, y \geqq 0\}$. It is not difficult to prove ([5]) that X is (as a Banach space) uniformly rotund (UR) precisely $\delta(\varepsilon) \equiv \inf_{x \pm y \in U_\varepsilon} (\|x + y\| \vee \|x - y\| - 1) > 0$, where $U_\varepsilon = \{x \pm y : \|x\| = 1, \|y\| \geqq \varepsilon\}$. The following inequalities are obvious: $\varepsilon \geqq \delta_+(\varepsilon) \geqq \delta(\varepsilon) \geqq 0$. Hence, it follows that UR implies UM. Also, the rotundity (R) implies STM. In fact both STM and UM can be viewed as the restriction of R and UR to the positive cone X^+ of X, respectively.

Note that for $L_1(\mu)$ we have $\delta_+(\varepsilon) = \varepsilon$, i.e. the greatest possible value is attained. For $1 < p < +\infty$ the space $L_p(\mu)$ is already UR and therefore UM space. On the other hand $L_\infty(\mu)$ is not even STM space. It appears that for Orlicz spaces $L_\Phi(\mu)$ the situation is still unchanged. Namely, either $L_\Phi(\mu)$ is UM (equivalently STM) space or it is not even STM space (μ is σ-finite). However, for Musielak-Orlicz spaces the situation is different (Theorems 2 and 3).

In [1] it was proved that every UM space ("UMB space" in [1]) is monotonically complete (Levi or $X \in (\mathrm{B})$), i.e. $0 \leqq x_\alpha \uparrow$ and $\sup \|x_\alpha\| < +\infty$ imply that $\sup (x_\alpha) \in X$. Moreover, in [1] it was also implicitely proved that in each UM Banach lattice the norm is order continuous (i.e. $X \in (\mathrm{A})$).

Thus, each UM Banach lattice is KB space, i.e. $j(X) = (X^*)^\sim_n$ under the evaluation $j: X \to (X^*)^*$, where X^* denotes the order, and hence the Banach, dual to the Banach lattice X, whereas $X^\sim_n$ consists of all order continuous functionals in X^*, (e.g. [8], Theorem 6.3). It will follow from Theorems 2 and 3, that this implication is strict for Musielak-Orlicz spaces and reduces to the equivalence in the case of the usual Orlicz spaces.

Given any σ-finite positive measure space (T, Σ, μ), the Musielak-Orlicz space

*) Institute of Mathematics, A. Mickiewicz University, Poznań, Poland

$L_\Phi(\mu)$ consists of all measurable functions (equivalence classes) f from T into $[0, +\infty]$ such that $I_\Phi(\lambda f) \equiv \int_T \Phi(|\lambda f(\lambda)|, t)\, d\mu$ is finite for some $\lambda > 0$. $\Phi(r, t)$: $\mathbb{R}_+ \times T \to \to [0, +\infty]$ is any function such that for all $r > 0$ $\Phi(r, \cdot)$ is μ-measurable, and for μ-a.a $t \in T$ $\Phi(\cdot, t)$ is convex (nontrivial), left continuous, continuous at zero and assuming zero et zero. $L_\Phi(\mu)$ endowed with the Luxemburg norm

$$\|f\|_\Phi \equiv \inf\{\lambda > 0 : I_\Phi(\tfrac{f}{\lambda}/\lambda) \leqq 1\}$$

becomes a σ-complete Banach lattice of countably type (i.e. SDC lattice) with σ-Levi property ([9]). Consequently, Musielak-Orlicz spaces $L_\Phi(\mu)$ are KB spaces precisely when the norm $\|\cdot\|_\Phi$, or any other equivalent (lattice) norm, is order continuous (cf. [8], Proposition 6.2).

For the sake of completness let us point out ([9]) that the subspace $E_\Phi(\mu)$ of $L_\Phi(\mu)$, consisting of all (measurable) functions f such that $I_\Phi(\lambda f) < +\infty$ for all $\lambda > 0$, is a closed ideal (order dense) in $L_\Phi(\mu)$ and is contained in $L^a_\Phi(\mu) \equiv \{f \in L_\Phi(\mu)$: $|f| \geqq f_n \downarrow \geqq 0$ imply $\|f_n\|_\Phi \downarrow 0\}$, i.e. in the subspace (sublattice) of the order continuity of the norm $\|\cdot\|_\Phi$ in $L_\Phi(\mu)$.

Proposition. (a) If Φ assumes finite values only ($\Phi < +\infty$), then $E_\Phi(\mu) = L^a_\Phi(\mu)$ (see [9]). (b) $E_\Phi(\mu) = L_\Phi(\mu)$ if and only if Φ satisfies Δ_2-condition if μ is non-atomic (eg. [7], [3], [2]), and δ_2-condition ([4]) if μ is the counting measure.

Recall, Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$) if there exists a set T_0 of zero measure, $K > 0$ and integrable nonnegative function h such that on $T \setminus T_0$ there holds $\Phi(2r, t) \leqq \leqq K\Phi(r, t) + h(t)$ for all $r > 0$. Also, in the case $T = \mathbb{N}$, Φ satisfies δ_2^0-condition ($\Phi \in \delta_2^0$) (cf. [4]) if there exist $K > 0$ and $a > 0$, a natural number m and a nonnegative sequence (c_n) with $(c_n)_{n \geq m}$ convergent such that for all $n \in \mathbb{N}$ and $r > 0$ satisfying $\Phi(r, n) \leqq a$ there holds $\Phi(2r, n) \leqq K\Phi(r, n) + c_n$. If the condition δ_2^0 is satisfied with $m = 1$, then Φ is said to satisfy δ_2-condition ($\Phi \in \delta_2$), [4].

Main results

Proposition 1. $L_\Phi(\mu)$ has order continuous (lattice) norm, i.e. $L_\Phi(\mu) = L^a_\Phi(\mu)$, if and only if $\Phi \in \Delta_2$ — for μ non-atomic, and $\Phi \in \delta_2^0$ in the case of the counting measure μ (see [3], [5] and [4], [6] respectively).

Let us point out that (in the case of the counting measure μ) the Δ_2-condition is stronger than the δ_2-condition ([6]).

Theorem 1. ([5]) Let μ be non-atomic. T.F.A.E.

(a) $L_\Phi(\mu)$ is STM space.

(b) $L_\Phi(\mu)$ is UM space.

(c) (i) $\Phi > 0$ (i.e. $\Phi(r, t) = 0$ iff $r = 0$, for μ-a.a. $t \in T$).

(ii) $\Phi \in \Delta_2$ (equivalently: any lattice norm in $L_\Phi(\mu)$ is order continuous, $L_\Phi(\mu)$ is KB space).

It folows that for STM Musielak-Orlicz spaces $L_\Phi(\mu)$ we have $\Phi < +\infty$ (i.e. $\Phi(r, t) < < +\infty$, for $r \geqq 0$ and μ-a.a. $t \in T$).

In the case of the counting measure μ we customary write l_Φ instead of $L_\Phi(\mu)$. In this case the notions STM and UM spaces does not coincide in general, and this will be seen from the example below.

Theorem 2. ([6]) Let μ be the counting measure. T.F.A.E.

(a) l_Φ is STM space.

(b) (i) $\Phi > 0$.

(ii) Φ assumes 1 (i.e. $\forall n \in \mathbb{N}$, $\Phi(r, n) = 1$, for some $r > 0$).

(iii) $\Phi = \delta_2^0$ (equivalently: any lattice norm in l_Φ is order continuous norm, l_Φ is KB space).

Let us consider the following modulus of behaviour of Φ at the points r_n, where $\Phi(\cdot, n)$ attains 1:

$$\eta_\Phi(\varepsilon) = \sup_n \frac{\Phi^{-1}(1 - \varepsilon, n)}{\Phi^{-1}(1, n)}.$$

Roughly speaking, the assumption $\eta_\Phi(\varepsilon) < 1$, for $\varepsilon > 0$ and small, means that the graph of $\Phi(\cdot, n)$ below the level $1 - \varepsilon$ is uniformly (with respect to n) far from the vertical line at r_n. In the case of functions Φ not depending on n, it is seen that the condition $\eta_\Phi(\varepsilon) < 1$ is always satisfied.

Theorem 3. ([6]) Let μ be the counting measure. T.F.A.E.

(a) l_Φ is UM space.

(b) (i) $\Phi > 0$.

(ii) Φ assumes 1.

(iii) Φ satisfies the δ_2^0-condition.

(iv) $\eta_\Phi(\varepsilon) < 1$ for each $\varepsilon \in (0, 1)$.

Example. Let $\Phi(r, n) = r/(2n - 1)$, if $0 \leqq r \leqq n - 1/2$, and $\Phi(r, n) = r + 1 - n$ otherwise ($r \geqq 0$). Clearly $\Phi > 0$ and it is easily seen, that Φ assumes 1 (i.e. $\Phi(n, n) = = 1$, for $n \in \mathbb{N}$) and $\Phi \in \delta_2^0$ (take $K = 2$, $a = 1/2$ and $c_n \equiv 0$). However, $r_n' \equiv \equiv \Phi^{-1}(1 - \varepsilon, n) = n - \varepsilon$ and $r_n = \Phi^{-1}(1, n) = 1$. Consequently, for each $\varepsilon \in \in (0, 1)$ we have $\eta_\Phi(\varepsilon) = 1$. Thus, in view of Theorems 2 and 3 the corresponding (sequential) Musielak-Orlicz space l_Φ is STM but it is not UM space.

Corollary. Let $L_\Phi(\mu)$ be Orlicz space (i.e. $\Phi(r, t) \equiv \Phi(r)$), where μ is non-atomic or counting measure. Then the notions of STM and UM space coincide. Moreover, $L_\Phi(\mu)$ is UM space precisely when the conditions (i)–(iii) from the above Theorems 1 and 2, respectively, are satisfied.

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