Leif Mejlbro
Vitali systems of neighbouring type

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 105--119

Persistent URL: http://dml.cz/dmlcz/701802

Vitali Systems of Neighbouring Type

LEIF MEJBRO

Lyngby*)

Received 15 March 1989

A generalized Vitali theorem is proved for socalled neighbouring Vitali systems consisting of cubes. A Neighbouring Vitali system is characterized by the property that inside each cube one can find *two* separated, fairly large sets (both properties to be defined by means of the geometry and the Lebesgue measure in the following) from the Vitali system. The proof uses a bootstrap method, which cannot be applied in the simpler pointwise case.

1. Introduction

In [8] was proved a fairly simple sufficient condition for a Vitali covering result, when the underlying Vitali system consisted of cubes in $\mathbf{R}^N$ and had some uniform structure. Keeping this uniform structure it was possible in [5] to prove Vitali theorems for more complicated sets. The idea was to establish a sufficient condition, which also took care of the geometry of the sets under consideration, and yet had a structure pretty close to the necessary condition derived in [8]. The result was, that for uniform systems, fairly large families of sets in $\mathbf{R}^N$ (some of them may even have a boundary, which is a fractal set) almost behave like a Vitali system consisting only of cubes in $\mathbf{R}^N$, as long as the Lebesgue measure is considered.

When dealing with cubes the working hypothesis was for a long time that the same result should hold for pointwise Vitali systems. This conjecture proved to be wrong as demonstrated by Talagrand's counterexample (published in [2]). Nevertheless a positive result could still be obtained by introducing a logarithmic factor in the sufficient condition. This was done in [2] for systems of cubes and generalized to other sets in [6] and [7]. It is almost obvious that the sets in [6] and [7] must satisfy a stronger geometrical condition that the sets in [5] explaining why we are using different names for the geometrical properties. Note, however, that also sets of constant or variable complexity may have a fractal boundary, and yet we can obtain a Vitali theorem for such systems of sets.

*) Mathematical Institute, Building 303, The Technical University of Denmark, DK-2800 Lyngby, Denmark

In the search for pointwise Vitali systems for which the "uniform condition" (i.e. a sufficient condition of the same structure as for uniform Vitali systems) would suffice for a Vitali theorem, the*n eighbouring Vitali systems* were introduced in [2]. Roughly speaking one assumes the existence of *two* fairly large sets from the family S in every open ball $B(x, r)$ instead of just one, where furthermore these two sets should be separated in some sense. This seemingly innocent extra geometrical condition on the system became an enigma in the light of Talagrand's counterexample, and as long as one only considers systems of cubes it is not possible to understand why Vitali theorems for neighbouring systems are "stronger" than Vitali theorems for pointwise systems. The introduction of the generalizations in [6] and [7] now enables us to find this difference. This investigation will be performed in the following and the result is that the idea of *separation* is far stronger than first supposed.

The difference between the proofs of the pointwise and the neighbouring Vitali theorems lies in the mass distribution process (cf. [2], and also [6] and [7]). In the pointwise case each of the chosen sets K_{n_i} (selected by the optimistic procedure introduced by Banach [1], cf. also [2] or [6] for further details) is used as a gauge on roughly $|\log |K_{n_i}||$ levels, which causes the logarithmic factor in these papers. By assuming that we have *two* separated sets instead of just one in the neighbourhood of each x it was possible in [2] to use K_{n_i} as a gauge on a *bounded* number of levels, where this bound only depends on the separation number. In fact, we choose *two* separated sets (K_{n_i}, K_{n_j}), where $|K_{n_i}| \geqq |K_{n_j}|$. Then we use the pointwise mass distribution process on K_{n_i} through m levels, say. The neighbouring condition ensures that K_{n_j} can be paired with another selected set $K_{n_k} \neq K_{n_i}$, so the process can be repeated on the pair (K_{n_j}, K_{n_k}). Such a mass distribution process of bootstrap type will only cause a constant factor m, and as this constant m only depends on the separation number, the introduction of the logarithmic factor is not necessary in the condition for the Vitali theorem.

The comments above describe to some extent the difference between the pointwise and the neighbouring Vitali systems, as long as Lebesgue measure or measures which are absolutely continuous with respect to Lebesgue measure are considered. For completeness it should be mentioned that some of the ideas developed in connection with these general Vitali systems still can be extended to other measures in $\mathbf{R}^N$. So far Vitali theorems for general measures in $\mathbf{R}^N$ have only been obtained in the wellknown classical case for systems of centered cubes or balls and for generalized uniform Vitali systems consisting of cubes, cf. [4]. The results indicate that the measure in general must play an unexpected geometrical role, which will make the generalizations extremely difficult. This does not mean that such generalizations are impossible, only that they very hard to obtain, though they may reveal an unexpected geometrical aspect, if ever such an investigation is performed.

2. Definition of neighbouring Vitali systems

Since we are mostly interested in the difference between the pointwise and the neighbouring systems we shall here only consider the Lebesgue measure and Vitali systems consisting of cubes in $\mathbf{R}^N$. The family of all such cubes is denoted by $\mathscr{Q}$ in following. It should, however, be mentioned that the main result is immediately extended to measures which are absolutely continuous with respect to Lebesgue measure, as well as to Lebesgue's family $\mathscr{K}_\alpha$ of *regular sets*, i.e. $\alpha \in]0, 1]$ is a constant and $\mathscr{K}_\alpha$ is the family of compact sets in $\mathbf{R}^N$, for which

$$\forall K \in \mathscr{K}_\alpha \, \exists Q \in \mathscr{Q} : K \subseteq Q \wedge |K| \geqq \alpha |Q| ,$$

where $|A|$ as usual denotes the (outer) Lebesgue measure of the set A. (Cf. also [3]) These simple generalizations are left to the reader.

Definition 2.1. *By Θ^N we shall understand the class of all continuous nondecreasing functions $\theta : \mathbf{R}_+ \to]0, \frac{1}{2}]$, for which*

$$(1) \qquad \int_0^R \frac{\{\theta(r)\}^N}{|\log(r\theta(r))|} \frac{dr}{r} = +\infty \quad \textit{for all} \quad R \in]0, 1] .$$

A function $\theta \in \Theta^N$ is called a separation function.

Every constant function $\theta(r) = c \in]0, \frac{1}{2}]$ is a separation function. A nontrivial function from Θ^N is

$$(2) \qquad \theta(r) = \{\log |\log r|\}^{-1/N}$$

for r sufficiently small and suitable otherwise. When (1) is applied on (2), we see that (2) is almost the "smallest" possible function in the class Θ^N.

Definition 2.2. *Let $\varphi : \mathbf{R}^N \times [0, +\infty[\to [0, +\infty[$ be continuous and nondecreasing in $r \in [0, +\infty[$ for every fixed $x \in \mathbf{R}^N$. We say that φ is a Φ-function and we write $\varphi \in \Phi^N$, if*

$$\forall x \in \mathbf{R}^N \, \forall r \geqq 0 : 0 \leqq \varphi(x, r) \leqq (2r)^N .$$

We shall use the Φ-functions to estimate the measure of the sets under consideration. The pointwise aspect comes in because we allow $\varphi(x, r)$ to depend on $x \in \mathbf{R}^N$.

Definition 2.3. *Let $\varphi \in \Phi^N$ and $\theta \in \Theta^N$. By $\mathscr{V}^N_{nbh}[\varphi; \theta]$ we shall understand the system of all pairs (A, S), where $A \subseteq \mathbf{R}^N$ and $S \subseteq \mathscr{Q}$ satisfy the following condition:*

For every $x \in A$ there exists a constant $r_x > 0$, such that one to every $r \in]0, r_x]$ can find two sets $Q_1, Q_2 \in S$ fulfilling

$$(3) \qquad Q_i \subseteq B(x, r), \; |Q_i| \geqq \varphi(x, r), \quad i = 1, 2, \quad \|c(Q_1) - c(Q_2)\|_\infty \geqq r\,\theta(r) ,$$

where $c(Q)$ denotes the centre of the cube $Q \in \mathscr{Q}^N$ and $B(x, r)$ is the open cube of centre x and radius r.

The definitions of the functions in Φ^N and Θ^N imply that $\mathscr{V}^N_{nbh}[\varphi; \theta] \neq \emptyset$. Note that we do not require in (3) that Q_1 and Q_2 are disjoint. If $Q_1 \cap Q_2 \neq \emptyset$, then a simple geometrical consideration shows that

$$r(Q_1) + r(Q_2) \geqq r\,\theta(r),$$

hence

$$\max\{l(Q_1), l(Q_2)\} \geqq r\,\theta(r),$$

where $2\,r(Q) = l(Q)$ denotes the edge-length of the cube Q.

Following F. Topsøe [9], a *Vitali system* $\mathscr{V}$ is a class of pairs $(A, \mathscr{S})$ with $A \subseteq \mathbf{R}^N$ and $\mathscr{S}$ a family of closed sets in $\mathbf{R}^N$ satisfying

VS 1: $\forall(A, \mathscr{S}) \in \mathscr{V}\ \forall B \subseteq A : (B, \mathscr{S}) \in \mathscr{V}$;

VS 2: $\forall(A, \mathscr{S}) \in \mathscr{V}\ \forall F$ closed $: (A \setminus F, \{S \in \mathscr{S} : S \cap F = \emptyset\}) \in \mathscr{V}$.

A family $\mathscr{S}_0$ of closed sets in $\mathbf{R}^N$ is called a *packing of* $A \subseteq \mathbf{R}^N$, if the elements of $\mathscr{S}_0$ are mutually disjoint and

$$|A \setminus \bigcup\{S : S \in \mathscr{S}_0\}| = 0.$$

If $\mathscr{S}$ contains a packing $\mathscr{S}_0$ of A, we say that the pair $(A, \mathscr{S})$ has the *packing property*.

A Vitali system $\mathscr{V}$ is said to have the *Vitali property*, if every pair $(A, \mathscr{S}) \in \mathscr{V}$ has the packing property.

The following classical lemma can be traced back to H. Lebesgue [3]:

Lemma 2.1. *A Vitali system $\mathscr{V}$ has the Vitali property, if and only if there exists a positive constant c, such that whenever A is bounded and $(A, \mathscr{S}) \in \mathscr{V}$ one can select disjoint sets $\{S_n : n \in J\} \subseteq \mathscr{S}$ with*

$$|\bigcup\{S_n : n \in J\}| \geqq c|A|.$$

The proof of the following theorem is identical with the proof of the corresponding theorem in e.g. [5].

Theorem 2.1. If $\varphi \in \Phi^N$ *and* $\theta \in \Theta^N$, *then* $\mathscr{V}^N_{nbh}[\varphi; \theta]$ *is a Vitali system. It is called a* neighbouring Vitali system *in the sequel.*

The key to the Vitali theorem for the neighbouring systems is the following complicated geometrical lemma.

Lemma 2.2. *Let $(A, \mathscr{S}) \in \mathscr{V}^N_{nbh}[\varphi; \theta]$, and assume that $\{K_n\} \subseteq S$ has been selected by Banach's optimistic procedure as described in [6] and [7] with respect to $(A, \mathscr{S})$. Let $m_0(r) \in \mathrm{N}$ be the uniquely determined integer given by*

$$\tag{4} \tfrac{1}{8}\,\theta(r) \leqq 2^{-m_0(r)} < \tfrac{1}{4}\,\theta(r), \quad r \in R_+.$$

Let $x \in A$ and $r \in]0, r_x]$, where r_x is the constant given by (3). *Then at least one of the following two condition is fulfilled:*

a) *There exists an* $n \in \mathrm{N}$, *such that*

$$K_n \cap B(x, r) \neq \emptyset \wedge |K_n| \geqq 2^{-(N+1)}\{r\,\theta(r)\}^N .$$

b) *There exist* $n_1, n_2 \in \mathrm{N}$, $n_1 \neq n_2$, *such that*

$$K_{n_i} \cap B(x, r) \neq \emptyset \wedge |K_{n_i}| \geqq \tfrac{1}{2}\varphi(x, r), \quad \textit{for} \quad i = 1, 2,$$

where at least one of two sets K_{n_1} *and* K_{n_2} *is disjoint from the open cube* $B(x, 2^{-m_0(r)}r)$.

Proof. Let $x \in A$ and $r \in]0, r_x]$ be given, and suppose that the lemma does not hold. It suffices to assume that whenever $K_n \cap B(x, r) \neq \emptyset$ and $|K_n| \geqq \frac{1}{2}\varphi(x, r)$, then

$$K_n \cap B(x, 2^{-m_0(r)}r) \neq \emptyset \wedge |K_n| < 2^{-(N+1)}\{r\,\theta(r)\}^N . \tag{5}$$

Using (3) we can find $Q_1, Q_2 \in \mathscr{S}$, such that

$$Q_i \subseteq B(x, r), \quad |Q_i| \geqq \varphi(x, r), \quad i = 1, 2, \quad \|c(Q_1) - c(Q_2)\|_\infty \geqq r\,\theta(r).$$

Using a result from [6] we can find K_{n_1} and K_{n_2} from the sequence $\{K_n\}$, such that

$$|Q_i| \leqq 2|K_{n_i}| \wedge \emptyset \neq Q_i \cap K_{n_i} \subseteq K_{n_i} \cap B(x, r) \quad \text{for} \quad i = 1, 2.$$

By assumption, (5) holds for K_{n_i}, $i = 1, 2$, so

$$|Q_i| \leqq 2|K_{n_i}| < 2 \cdot 2^{-(N+1)}\{r\,\theta(r)\}^N = \left\{\frac{r}{2}\,\theta(r)\right\}^N . \tag{6}$$

Since Q_i is a cube, it follows from (6) that $l(Q_i) \leqq r\,\theta(r)/2$, and as both Q_1 and Q_2 by assumption intersect $B(x, 2^{-m_0(r)}r)$, we get the following estimate

$$r\,\theta(r) \leq \|c(Q_1) - c(Q_2)\|_\infty < \frac{1}{2}\,l(Q_1) + l(B(x, 2^{-m_0(r)})) + \frac{1}{2}\,l(Q_2) \leq$$

$$\leq \frac{1}{2}\cdot\frac{r}{2}\,\theta(r) + 2 \cdot 2^{-m_0(r)}r + \frac{1}{2}\cdot\frac{r}{2}\,\theta(r) < \frac{r}{4}\,\theta(r) + \frac{r}{2}\,\theta(r) + \frac{r}{4}\,\theta(r) = r\theta\,(r),$$

and we have obtained a contradiction.

We conclude that we can find K_n, such that $K_n \cap B(x, r) \neq \emptyset$ and either

$$K_n \cap B(x, 2^{-m_0(r)}r) = \emptyset \quad \text{or} \quad |K_n| \geqq 2^{-(N+1)}\{r\,\theta(r)\}^N . \tag{7}$$

Suppose that a) does not hold. Since the last alternative in (7) is not fulfilled, at least one of the sets K_{n_1}, K_{n_2}, $n_1 \neq n_2$, can be chosen, such that

$$K_{n_i} \cap B(x, r) \neq \emptyset, \quad |K_{n_i}| \geqq \tfrac{1}{2}\,\varphi(x, r), \quad K_{n_i} \cap B(x, 2^{-m_0(r)}r) = \emptyset,$$

proving b). □

In the terminology of [6] and [7] the function $m_0(r)$ defined by (4) will be our variable distribution number to be used in the following .When $\theta \in \Theta^N$ is a constant, then m_0 is also a constant, and we obtain the neighbouring Vitali theorem in [2] as a special case of Theorem 3.1 in the next section.

3. A Vitali theorem for neighbouring Vitali systems

By modifying the proofs in [6] and [7] we are now able to prove the main theorem:

Theorem 3.1. *Let* $\varphi \in \Phi^N$ *and* $\theta \in \Theta^N$, *and assume that there exist a constant* $C > 0$ *and, for every* $x \in \mathbf{R}^N$, *another constant* $r_x > 0$, *such that for every* $r \in \,]0, r_x]$,

$$\frac{\{r\,\theta(r)\}^N}{|\log(r\,\theta(r))|} \geqq C\,\varphi\,(x,r)\,. \tag{8}$$

If for every $R > 0$ *and almost every* $x \in \mathbf{R}^N$,

$$\int_0^R \frac{\varphi(x,r)}{|\log\theta(r)|}\,\frac{dr}{r^{N+1}} = +\infty\,, \tag{9}$$

then $\mathscr{V}^N_{nbh}[\varphi;\theta]$ *has the Vitali property, i.e. for every pair* $(A,\mathscr{S}) \in \mathscr{V}^N_{nbh}[\varphi;\theta]$ *one can find a disjointed subfamily* $\{S_n\} \subseteq \mathscr{S}$, *such that*

$$\Big|A \setminus \bigcup_n S_n\Big| = 0\,.$$

Remark. It follows from [6] that Theorem 3.1 is only interesting, when

$$\int_0^R \frac{\varphi(x,r)}{|\log\varphi(x,r)|}\,\frac{dr}{r^{N+1}} < +\infty$$

for every $x \in A$, where $|A| > 0$. When r is sufficiently small, it follows from (8) that

$$|\log\varphi(x,r)| \geqq \log C + \log|\log(r\,\theta(r))| + N\log\left(\frac{1}{r}\right) + N|\log\theta(r)| > |\log\theta(r)|\,,$$

proving that

$$\int_0^R \frac{\varphi(x,r)}{|\log\varphi(x,r)|}\,\frac{dr}{r^{N+1}} \leqq \int_0^R \frac{\varphi(x,r)}{|\log\theta(r)|}\,\frac{dr}{r^{N+1}}$$

for R sufficiently small, where the equality sign only holds, when both integrals are $+\infty$. Hence we may find neighbouring Vitali systems for which [6] does not give a Vitali theorem, while (8) and (9) hold, and in this special case Theorem 3.1 above is an extension of the previous known results.

Proof. The first steps in the proof are identical with the first steps in the proof of the corresponding theorem in [6]. The proof is reduced to the case, where $A \subseteq \,]0,1[^N$, where $r_x \geqq r_0 = \frac{1}{2}$ in (3) and (8) for every $x \in A$. This simplification uses a property of the outer Lebesgue measure. By removing a nullset, if necessary, we may futhermore assume that (9) holds for every $x \in A$. As in [6] we may assume that

$$l(Q) \leqq \tfrac{1}{2} \quad \text{and} \quad Q \subseteq \,]0,1[^N \quad \text{for every} \quad Q \in \mathscr{S}\,. \tag{10}$$

By these reductions we see that it suffices to prove the theorem, when $A \subseteq \,]0,1[^N$

and the cubes in $\mathscr{S}$ satisfy (10), and when for every $x \in A$ and every $r \in]0, \frac{1}{2}]$ there exist $Q_1, Q_2 \in \mathscr{S}$, such that

(11) $\quad Q_i \subseteq B(x, r),\ |Q_i| \geq \varphi(x, r)\,,\quad i = 1, 2\,,\quad \wedge \|c(Q_1) - c(Q_2)\|_\infty \geq r\,\theta(r)\,,$

and when there exists a constant $C > 0$, such that for every $x \in A$ and every $r \in [0, \frac{1}{2}]$,

$$\text{(12)} \qquad \frac{\{r\,\theta(r)\}^N}{|\log(r\,\theta(r))|} \geqq C\,\varphi(x, r)\,.$$

If $\theta(r) \to 0$ for $r \to 0$, then we can find $m' \in \mathrm{N}$, such that

$$\text{(13)} \qquad \frac{C}{4} \log 2 \,.\, 8^{-N} \geq 2 \,.\, 4^{-(N+1)} \left\{3 + \frac{|\log \theta(2^{-k})|}{\log 2}\right\}^{-1} \quad \text{for} \quad k \geq m'\,,$$

and

$$\text{(14)} \qquad N|\log(2^{-k}\,\theta(2^{-k}))| \geq$$
$$\geq \tfrac{1}{2}\{(N+1)\log 2 + N|\log(2^{-k}\,\theta(2^{-k}))|\} \quad \text{for} \quad k \geq m'\,.$$

These two estimates will be used at the very end of proof.

If θ is constant, replace if necessary θ by a smaller constant, such that (13) and (14) hold. This replacement will not affect the conditions (9), (11) and (12).

For every fixed $x \in A$ the function $\varphi(x, r)/|\log \theta(r)|$ is nondecreasing in r, so it follows from [6] that (2) holds for every $x \in A$, if and only if

$$\sum_{n=0}^{+\infty} 2^{nN} \,.\, \frac{\varphi(x, 2^{-n})}{|\log \theta(2^{-n})|} = +\infty \quad \text{for every} \quad x \in A\,.$$

Define for $m \in \mathrm{N}$, $m \geqq m'$,

$$\text{(15)} \qquad A_m = \left\{ x \in A \,\middle|\, \sum_{k=m'}^{m} 2^{kN} \,.\, \frac{\varphi(x, 2^{-k})}{3 + \{|\log \theta(2^{-k})|/\log 2\}} \geq 4^{N+1} \right\}.$$

Since the series above is divergent for every $x \in A$, we may using a property of the outer Lebesgue measure choose $m \in \mathrm{N}$, such that $|A_m| \geq \frac{1}{2}|A|$. This simple trick reduces the proof to the case, where A is given by (15), which we shall do in the sequel. Hence, we finally assume that

$$\text{(16)} \qquad \sum_{k=m'}^{m} 2^{kN} \,.\, \frac{\varphi(x, 2^{-k})}{3 + \{|\log \theta(2^{-k})|/\log 2\}} \geq 4^{N+1} \quad \text{for every} \quad x \in A\,.$$

Let $\{K_n\}$ be selected by the optimistic procedure. By Lemma 2.1 it suffices to prove the existence of a constant $c > 0$, only depending on the dimension, and a finite subclass $\{K_{n_i}\}$ of $\{K_n\}$, such that

$$\text{(17)} \qquad \sum_{i=1}^{p} |K_{n_i}| \geq c|A|\,.$$

We shall assume that $\{K_n\}$ has been enumerated according to nonincreasing measure.

If $|K_1| \geq c$, then (17) becomes trivial, because $A \subseteq]0, 1[^N$. Thus assume in the following that $|K_n| < c$ for every $n \in \mathbf{N}$.

Let $\mathcal{N}_p$, $p \in \mathbf{N}$, be the dyadic division of $[0, 1]^N$ into meshes of edge-length 2^{-p}, continued to the whole of $\mathbf{R}^N$.

Each K_n is a cube by definition, so define $k_n \in \mathbf{N}$ by

$$2^{-k_n-1} \leqq l(K_n) < 2^{-k_n}. \tag{18}$$

For later purposes we note that it follows from (18) that

$$\frac{1}{k_n} > \frac{N \log 2}{|\log |K_n||}. \tag{19}$$

Choose 2^N meshes from $\mathcal{N}_{k_n}$, the union of which is a cube denoted by $Q^n_{k_n}$, such that

$$K_n \subset Q^n_{k_n} \quad \text{and} \quad \|c(K_n) - c(Q^n_{k_n})\|_\infty \leqq 2^{-k_n-1}.$$

Then

$$|K_n| \geqq 2^{-(k_n+1)N} = 2^{-2N}|Q^n_{k_n}| \tag{20}$$

and

$$K_n \subset B(x, 2^{1-k_n}) \quad \text{for every} \quad x \in Q^n_{k_n}.$$

Then choose 2^N meshes from $\mathcal{N}_{k_1-1}$, the union of which is a cube $Q^n_{k_n-1}$, such that

$$Q^n_{k_n} \subset Q^n_{k_n-1} \quad \text{and} \quad \|c(K_n) - c(Q^n_{k_n-1})\|_\infty \leqq 2^{-(k_n-1)-1}.$$

Then

$$|K_n| \geqq 2^{-2N} . 2^{-2N}|Q^n_{k_n-1}|$$

and

$$K_n \subset B(x, 2^{1-k_n} + 2^{-k_n}) = B(x, 3 . 2^{-k_n}) \quad \text{for} \quad x \in Q^n_{k_n-1}.$$

It is very important for the proof that $K_n \cap B(x, 2^{-k_n}) \neq \emptyset$ implies $x \in Q^n_{k_n-1}$ and that $K_n \cap B(x, 2^{-(k_n-1)}) \neq \emptyset$ implies $x \in 2Q^n_{k_n-1}$.

By proceeding this way we define a finite sequence

$$(K_n \subset)\, Q^n_{k_n} \subset Q^n_{k_n-1} \subset \ldots \subset Q^n_2 \subset Q^n_1$$

associated with each K_n, such that

$$\|c(K_n) - c(Q^n_k)\|_\infty \leqq 2^{-k-1}, \quad k = 1, 2, \ldots, k_n,$$

and whence

$$K_n \subset B(x, 3 . 2^{-k-1}) \quad \text{for} \quad k = 1, 2, \ldots, k_n - 1 \quad \text{and} \quad x \in Q^n_{k_n},$$

where each cube Q^n_k is composed of 2^N meshes from $\mathcal{N}_k$. We have furthermore, that $K_n \cap B(x, 2^{-k}) \neq \emptyset$ implies $x \in 2Q^n_k$. Note that Q^n_k no longer can be assumed to be contained in $[0, 1]^N$.

In order to obtain a *screening effect* we define $R_k^n = 2Q_k^n$ as the cube consisting of 4^N meshes from $\mathcal{N}_k$ with Q_k^n as the central subcube.

The meshes in R_k^n from $\mathcal{N}_k$ are denoted by $J_{k,j}^n$, where $j = 1, \ldots, 4^N$, and where the enumeration with respect to j follows a fixed pattern for all R_k^n. The only restriction is that the centre $c(K_n)$ must belong to the meshes of the same index j as k varies, which will almost fix the rest of the enumeration, if a fixed pattern is followed.

We shall especially consider the meshes

$$\{Q \in \mathcal{N}_{m+1} \mid A \cap \operatorname{int} Q \neq \emptyset\} = \{T_s \mid s = 1, \ldots, L\}, \tag{21}$$

where $m \in \mathbf{N}$ is the same constant as in the upper bound of the sum (16). In each mesh T_s we choose a *control point* $x_s \in A \cap \operatorname{int} T_s$, and the set of all control points is denoted by

$$P = \{x_s \mid s = 1, \ldots, L\}.$$

According to Lemma 2.2 we have two alternatives for $k \in \{1, 2, \ldots, m+1\}$ and $x_s \in P$. Either

a) there exists an $n \in \mathbf{N}$, such that

$$K_n \cap B(x_s, 2^{-k}) \neq \emptyset \quad \text{and} \quad |K_n| \geqq \tfrac{1}{2}\{\tfrac{1}{2}\,\theta(2^{-k})\}^N 2^{-Nk}, \tag{22}$$

b) or one can find $n_1, n_2 \in \mathbf{N}$, $n_1 \neq n_2$, such that for $i = 1, 2$,

$$K_{n_i} \cap B(x_s, 2^{-k}) \neq \emptyset \quad \text{and} \quad |K_{n_i}| \geqq \tfrac{1}{2}\,\varphi(x_s, 2^{-k}), \tag{23}$$

where $B(x_s, 2^{-k-m_0(2^{-k})})$ is disjoint from at least one of the sets K_{n_1} and K_{n_2}.

According to (4), the constant $m_0(2^{-k})$ is given by

$$2 + \frac{|\log \theta(2^{-k})|}{\log 2} < m_0(2^{-k}) \leqq 3 + \frac{|\log \theta(2^{-k})|}{\log 2}. \tag{24}$$

For later use we denote the selected sets K_n above, for a given k and a given x_s, by S_{s1}^k and S_{s2}^k. Note that $S_{s1}^k = S_{s2}^k$ in case a), while $S_{s1}^k \cap S_{s2}^k = \emptyset$ in case b).

For every $x_s \in P$ we can find a finite number of pairs (S_{s1}^k, S_{s2}^k), $k = 1, \ldots, m+1$, of compact sets from our chosen sequence $\{K_n\}$. As $P = \{x_s \mid s = 1, \ldots, L\}$ is finite, we have altogether chosen a finite number of sets

$$\mathcal{S}'' = \{S_{si}^k \mid k = 1, \ldots, m+1;\ s = 1, \ldots, L;\ i = 1, 2\}, \tag{25}$$

where the same set may occur several times. In some sense the system (25) describes the geometry. We shall add some extra sets to $\mathcal{S}''$. Let $x_s \in P$, and let $k \in \{1, \ldots, m+1\}$ be an index, such that we are in alternative b). Then only a finite number of the disjoint sets $\{K_n\}$ can satisfy

$$K_n \cap B(x_s, 2^{-k}) \neq \emptyset \quad \text{and} \quad |K_n| \geqq \tfrac{1}{2}\,\varphi(x_s, 2^{-k}),$$

cf. (23). Add every such K_n, which have not already been included in $\mathcal{S}''$. In this way we obtain our finite system $\mathcal{S}' = \{K_{n_1}, K_{n_2}, \ldots, K_{n_p}\}$.

For each $K_n \in \mathscr{S}''$ we have defined a set $R^n_{k_n}$, which we for short shall denote by γ_n. If we define

$$\Gamma = [0, 1]^N \cap \bigcup \{\gamma_n \mid K_n \in \mathscr{S}''\},$$

we obtain by (20) for $K_n \in \mathscr{S}''$,

$$|K_n| \geqq 2^{-2N}|Q^n_{k_n}| = 2^{-3N}|R^n_{k_n}| = 2^{-3N}|\gamma_n|,$$

so

$$\sum_{i=1}^{p} |K_{n_i}| \geqq |\bigcup \{K_n \mid K_n \in \mathscr{S}''\}| \geqq 8^{-N}|\Gamma|.$$

In the last part of the proof we prove that also

$$(26) \qquad \sum_{i=1}^{p} |K_{n_i}| \geqq |A \setminus \Gamma|.$$

Assume for the time being that (26) holds. Then

$$|A| \leqq |A \setminus \Gamma| + |\Gamma| \leqq (1 + 8^N) \sum_{i=1}^{p} |K_{n_i}|,$$

and (17) follows with $c = (1 + 8^N)^{-1}$, so the theorem follows from Lemma 2.1. Hence only (26) remains to be proved.

Now $A \setminus \Gamma$ is disjoint from every $K_n \in \mathscr{S}''$, so the idea is instead to consider the sets from $\mathscr{S}' \supseteq \mathscr{S}''$ as a *pool of gauges* which can be divided and given to selected meshes as we please, as long as we at most use each subgauge once. If this can be done is such a way that the sum of all gauges over each particular subcube $Q \in \mathscr{N}_{m+1}$, which is contained in $[0, 1]^N \setminus \Gamma$ and which contains at least one point from A in its interior, exceeds $|Q|$, then (26) follows.

Let $D = \max\{k_1, \ldots, k_p, m + 1\}$. Then the closure of $[0, 1]^N \setminus \Gamma$ is composed of meshes from $\mathscr{N}_D$,

$$\mathrm{cl}([0, 1]^N \setminus \Gamma) = \bigcap_{j=1}^{q'} Q^j_D, \quad Q^j_D \in \mathscr{N}_D.$$

Assume that the Q^j_D have been enumerated, such that for some $q \leqq q'$ and some nullset Ω,

$$(27) \qquad A \setminus \Gamma \subseteq \Omega \cup \bigcup_{j=1}^{q} Q^j_D \quad \text{and} \quad A \cap \mathrm{int}\, Q^j_D \neq \emptyset \quad \text{for} \quad j = 1, \ldots, q.$$

To every Q^j_D in (20) we can find T_s defined by (21), such that $Q^j_D \subseteq T_s$. For the control point $x_s \in T_s$ we have

$$\max\{\|x_s - y\|_\infty \mid y \in Q^j_D\} \leqq l(T_s) = 2^{-m-1}.$$

We have finally come to the description of the mass distribution process, which is fairly difficult here, because we shall connect two different mass distribution processes.

For each $K_n \in \mathscr{S}'$ we have defined 4^N strings $\mathscr{S}_j^n$, given by

$$\mathscr{S}_j^n = \{J_{k,j}^n \mid k = 1, \ldots, k_n\}, \quad j = 1, \ldots, 4^N .$$

We shall use $\frac{1}{2}|K_n|$ in two different ways as a gauge on the total of all these strings. In both cases we give each *string* the subgauge $\frac{1}{2}\, 4^{-N}|K_n|$, but the distribution on the *levels* in the string will be different in the two cases.

We first describe the most difficult one of these two mass distribution processes. Let $x_s \in P$ be a control points, and let $I_s \subseteq \{1, 2, \ldots, m+1\}$ be the set of the indices k, for which we are in alternative b). If K_n has been attached to some index from I_s, let k^* be the smallest of these indices. We shall in this distribution process consider such sets $K_n \in \mathscr{S}'$. When k^* has been found, define the *distribution number* m^* associated with K_n by (24), i.e. $m^* \in \mathbf{N}$ and

$$2 + \frac{|\log \theta(2^{-k^*})|}{\log 2} < m^* \leqq 3 + \frac{|\log \theta(2^{-k^*})|}{\log 2}$$

In this way we select a subsystem $\{K_{n_1^*}, \ldots, K_{n_r^*}\}$ with corresponding level numbers $\{k_1^*, \ldots, k_r^*\}$ and distribution numbers $\{m_1^*, \ldots, m_r^*\}$, where

$$\{n_1^*, \ldots, n_r^*\} \subseteq \{n_1, \ldots, n_p\} \quad \text{and} \quad n_1^* < \ldots < n_r^* .$$

Let us look at the 4^N strings $\mathscr{S}_j^{n_1^*}$, $j = 1, \ldots, 4^N$, associated with the largest of these cubes $K_{n_1^*}$. Each of these have received the gauge $\frac{1}{2} \,.\, 4^N|K_{n_i^*}|$. Divide this gauge into m_1^* subgauges, each of the size $(2m_r^*)^{-1}\, 4^{-N}|K_{n_1^*}|$, and give them to

$$J_{k_1^*j}^{n_1^*},\ J_{k_1^*+1,j}^{n_1^*},\ \ldots,\ J_{k_1^*+m_1^*-1,j}^{n_1^*} ,$$

while we still let $\{J_{k_1^*+m_1^*,j}^{n_1^*}, \ldots, J_{n_1^*,j}^{n_1^*}\}$ be associated with $K_{n_1^*}$, though these sets have not yet recieved a gauge.

Then turn to the 4^N strings $\mathscr{S}_j^{n_2^*}$, $j = 1, \ldots, 4^N$, associated with $K_{n_2^*}$. In this case the subgauges have the size $(2m_2^*)^{-1}\, 4^{-N}|K_{n_2^*}|$.

If a string $\mathscr{S}_j^{n_2^*}$ does not contain elements from any of the strings $\mathscr{S}_i^{n_1^*}$, $i = 1, \ldots, 4^N$, the mass distribution process is carried out as above.

If, however, some $J_{k,j}^{n_2^*} = J_{k,i}^{n_1^*}$, where $k \in \{k_1^*, \ldots, k_1^* + m_1^* - 1\}$, we let the largest of the subgauges $(2m_1^*)^{-1}\, 4^{-N}|K_{n_1^*}|$ and $(2m_2^*)^{-1}\, 4^{-N}|K_{n_2^*}|$ win. If e.g. $(m_2^*)^{-1} \,.\, 4^{-N}|K_{n_2^*}| > (2m_1^*)^{-1}\, 4^{-N}|K_{n_1^*}|$, we let $(2m_2^*)^{-1}\, 4^{-N}|K_{n_2^*}|$ replace $(2m_1^*)^{-1}\, 4^{-N}|K_{n_1^*}|$ on $J_{k,j}^{n_2^*} = J_{k,i}^{n_1^*}$, and we let the displaced subgauge $(2m_1^*)^{-1}\, 4^{-N}|K_{n_1^*}|$ represent the first vacant element $J_{k_1^*+m_1,i}^{n_1^*}$ in the string associated with $K_{n_i^*}$ from the string number i.

If instead $(2m_2^*)^{-1}\, 4^{-N}|K_{n_2^*}| \leqq (2m_2^*)^{-1}\, 4^{-N}|K_{n_1^*}|$, we jump over all elements in the string $\mathscr{S}_j^{n_2^*}$, which already have received a subgauge from $K_{n_1^*}$ and continue to fill in the vacant places in the string, as long as we have got subgauges.

In this way we proceed, until all subgauges from $K_{n_2^*}$ have been placed, and possibly some of the subgauges from $K_{n_1^*}$ have been pushed further down their strings.

It is obvious that it would be an overwhelming task to index this process, as an addition of a subgauge from a K_{n_i*} may cause an avalanche of shifts in the gauges from the sets K_n, which already have been through this process. Instead we note that each member of a string, which has been participating in this process, at least receives a subgauge of a size, which it deserves. We can finish the proof, if *all* members of *all* strings participate in this process.

Continue successively on $K_{n_3*}, \ldots, K_{n_r*}$ as described above, and we have finished the description of the first mass distribution process.

At this step of the proof we note that condition b) ensures that if some K_{n_i*} occurs in more than m_i^* levels, then the second member of the pair in (23) is big enough to take over the mass distribution process, *seen from the control point* $x_s \in P$. We may also have added some extra sets in the definition of $\mathscr{S}'$, but this will only improve on the situation. As long as the members of the pairs (S_{s1}^k, S_{s2}^k) are distinct, no "holes" can occur in the distribution process, because at least one member of the pair is replaced by a new set from $\mathscr{S}''$ within $m_0(2^{-k})$ steps. Thus *seen from any* $x_s \in P$ we at least get the right gauge at each level, whenever we only have distinct members in the pairs (S_{s1}^k, S_{s2}^k).

The second possibility occurs when $S_{s1}^k = S_{s2}^k$ for some $x_s \in P$. When θ is constant, this can be ruled out, and we get the simple argument from [2]. In general we must deal with this case also, and it is for this reason that we have saved $\frac{1}{2}|K_n|$ for another distribution process. Here we shall use the logarithmic distribution process as described in [6]. The present situation is however simpler, because all the elements are cubes, and because the targets, namely the 4^N strings attached to each K_n, are given already, so we need not here refer to [6].

Let $K_n \in \mathscr{S}'$. Each associated string $\mathscr{I}_j^n$, $j = 1, \ldots, 4^N$, is given the gauge $\frac{1}{2} . 4^{-N}|K_n|$ as before. By (18) we are given k_n levels, so each element of each string receives the gauge $(2k_n)^{-1}\, 4^{-N}|K_n|$.

Finally, consider $x_s \in P$. For $k \in \{1, 2, \ldots, m + 1\}$ we have found (S_{s1}^k, S_{s2}^k), such that

$$S_{si}^k \cap B(x_s, 2^{-k}) \neq \emptyset\,, \quad i = 1, 2\,,$$

so T_s receives gauges from all levels by the two procedures described above. We calculate the contributions according to whether an index k gives rise to alternative a) or alternative b). Furthermore, we shall only work with *densities*, so we shall divide the gauge by $|J_{k,j}^n| = 2^{-kN}$, which will give us the density from level k.

Assume that $k \in \{1, 2, \ldots, m + 1\}$ gives rise to alternative a), cf. (22). Then T_s receives from level k at least the gauge density

$$\{|J_{k,j}^n|\, 2k_n\, 4^N\}^{-1}\, |K_n| \geqq (2k_n)^{-1}\, 4^{-N}\, 2^{kN}\, \tfrac{1}{2}\{\tfrac{1}{2}\, \theta(2^{-k})\}^N\, 2^{-kN} =$$
$$= (4k_n)^{-1}\, 8^{-N}\, \theta(2^{-k})^N\,.$$

From (12) we get for $r = 2^{-k}$,

$$\theta(2^{-k})^N \geqq C\, 2^{Nk} |\log(2^{-k}\, \theta(2^{-k}))|\, \varphi(x, 2^{-k})\,,$$

from which we deduce, using (19),

$$\frac{8^{-N}\,\theta(2^{-k})^N}{4k_n} \geqq \frac{N\log 2\,.\,8^{-N}C\,.\,2^{Nk}|\log(2^{-k}\,\theta(2^{-k}))|}{4|\log|K_n||}\,.\,\varphi(x, 2^{-k})\,.$$

From (22) we get

$$1 > |K_n| \geqq 2^{-(N+1)}\,\{2^{-k}\,\theta(2^{-k})\}^N\,,$$

whence

$$|\log|K_n|| \leqq (N+1)\log 2 + N|\log(2^{-k}\,\theta(2^{-k}))|\,.$$

Hence the contribution to the density from level k is at least

$$\frac{N\log 2}{4}\,\frac{8^{-N}C\,.\,2^{Nk}|\log(2^{-k}\,\theta(2^{-k}))|}{(N+1)\log 2 + N|\log(2^{-k}\,\theta(2^{-k}))|}\,.\,\varphi(x, 2^{-k}) =$$

$$= \left\{\frac{C}{4}\log 2\,.\,8^{-N}\right\}.\left\{\frac{N|\log(2^{-k}\,\theta(2^{-k}))|}{(N+1)\log 2 + N|\log(2^{-k}\,\theta(2^{-k}))|}\right\}.\,2^{Nk}\,\varphi(x, 2^{-k})\,.$$

If furthermore $k \geqq m'$, it follows from (13) and (14) that this contribution is at least

$$\frac{4^{-(N+1)}\,2^{Nk}}{3 + \{|\log\theta(2^{-k})|/\log 2\}}\,.\,\varphi(x, 2^{-k})\,, \tag{28}$$

which is an estimate of the gauge density coming from level k in alternative a).

If k belongs to alternative b), cf. (23), then T_s receives at least the gauge density

$$\{2m_0(2^{-k})\,4^N|J^n_{k,j}|\}^{-1}\,|K_n| \geqq 2^{Nk}\{2m_0(2^{-k})\,4^{-N}\}^{-1}\,.\,\tfrac{1}{2}\,\varphi(x, 2^{-k}) \geqq$$

$$\geqq \frac{4^{-(N+1)}\,2^{Nk}\,\varphi(x, 2^{-k})}{3 + \{|\log\theta(2^{-k})|/\log 2\}}\,,$$

where we have used (24). This contribution is therefore at least of the same amount as (28), so adding the density contributions from all levels $k = m', \ldots, m$, we get at least the following density over T_s,

$$4^{-(N+1)}\sum_{k=m'}^{m}\frac{2^{Nk}\,\varphi(x, 2^{-k})}{3 + \{|\log\theta(2^{-k})|/\log 2\}} \geqq 4^{-(N+1)}\,4^{N+1} = 1\,,$$

where we in the estimate have used (16). Since this mass distribution can be carried out simultaneously over all the relevant T_s (and even with excess mass), we conclude that

$$\sum_{i=1}^{p}|K_{n_i}| \geqq |A \setminus \Gamma|\,,$$

so we have proved (26) and hence the theorem. □

When $\theta \in \Theta^N$ is a constant, (8) is only fulfilled, when $\varphi(x, r) \leqq c_x r^N/|\log r|$, so Theorem 3.1 does not contain the neighbouring theorem from [2]. It is however easy to obtain the following corollary, which trivially contains this earlier result.

Corollary 3.1. *Assume that $\theta \in \Theta^N$ satisfies the slightly stronger condition for $R > 0$,*

$$\int_0^R \frac{\{\theta(r)\}^N}{|\log(r\,\theta(r))|} \cdot \frac{1}{|\log \theta(r)|} \frac{dr}{r} = +\infty . \tag{29}$$

If $\varphi \in \Phi^N$ fulfils

$$\int_0^R \frac{\varphi(x, r)}{|\log \theta(r)|} \cdot \frac{dr}{r^{N+1}} = +\infty$$

for every $R \in \mathbf{R}_+$ and almost every $x \in \mathbf{R}^N$, then $\mathscr{V}^N_{nbh}[\varphi; \theta]$ has the Vitali property.

Proof. We only replace $\varphi(x, r)$ by

$$\varphi_1(x, r) = \min\left\{\varphi(x, r), \frac{\{r\,\theta(r)\}^N}{|\log(r\,\theta(r))|}\right\}.$$

Then φ_1 satisfies the assumptions of Theorem 3.1, so $\mathscr{V}^N_{nbh}[\varphi_1; \theta]$ has the Vitali property. As $\varphi_1(x, r) \leqq \varphi(x, r)$, the same is true for $\mathscr{V}^N_{nbh}[\varphi; \theta]$. $\square$

Obviously every constant $\theta \in \Theta^N$ satisfies (29), so the neighbouring theorem from [2] is contained in Corollary 3.1 above. Note also that $\theta \in \Theta^N$ given by (2) satisfies (29). If

$$\theta_\alpha(r) = \{\log|\log r| \,.\, (\log\log|\log r|)^\alpha\}^{-1/N}$$

for $r > 0$ sufficiently small and suitable otherwise, then $\theta_\alpha \in \Theta^N$ for $\alpha \leqq 1$, while (29) only holds for $\alpha \leqq 0$.

Vitali theorems like Theorem 3.1 above immediately gives a weak differentiation theorem, when the result is combined with the main result in [10]. Also they are well fitted to give a description of some classes of nullsets. For these applications the reader is refered to e.g. [2] and [10].

References

[1] BANACH S., *Sur le théorème de M. Vitali*, Fund. Math. 5 (1924), 130—136.

[2] JØRSBOE O., MEJLBRO L. and TOPSØE F., *Some Vitali theorems for Lebesgue measure*, Math. Scand. 48 (1981), 259—285.

[3] LEBESGUE H., *Sur l'intégration des fonctions discontinues.* Ann. École Norm. 27 (1910), 361—450.

[4] MEJLBRO L., *Vitali type theorems for a class of measures in* $\mathbf{R}^N$. Math. Scand. 51 (1982), 289—309.

[5] MEJLBRO L., *Generalized Vitali systems of uniform type*, Acta Univ. Carolinae — Math. et Phys. 28 (1987), 83—93.

[6] MEJLBRO L., *Pointwise Vitali systems of constant tightness*, Conf. Topology and Measure V (Binz, 1987), Greifswald, 1—16 (to appear).

[7] MEJLBRO L., *Pointwise Vitali systems of variable tightness*, Terzo Convegno di Analisi Reale e Teoria della Misura, Capri 12—16 settembre 1988 (preprint).

[8] MEJLBRO L. and TOPSØE F., *A precise Vitali theorem for Lebesgue measure*, Math. Ann. 230 (1977), 183—193.

[9] TOPSØE F., *Packing and coverings with balls in finite dimensional normed spaces*. In "Measure Theory", Lecture Notes in Mathematics 541, Springer 1976, 197—199.

[10] TOPSØE F., *Extending classical criteria for differentiation theorems*, Mathematika 32, (1985), 68—74.