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On Generalizations of Dyadic Spaces

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The class of dyadic compact spaces (the continuous images of generalized Cantor discontinua) which is a natural generalization of the class of compact metric spaces and which is the smallest class containing the class of compact metric spaces closed with respect to cartesian products and continuous images was introduced by P. S. Alexandroff [1] in 1936. This class has a lot of nice properties and is the subject of many papers.

In the 70-es a new aproach concerning the theory of dyadic spaces appeared. A. V. Arhangelskii in [2] introduced the class of dantian spaces and the class of thick spaces and put the question "is each dantian space dyadic?" It was proved later by L. B. Sapiro [9] that the answer is no. In 1970 S. Mrówka [7] introduced the class of polyadic spaces; the continuous images of the products of the one point compactifications of the discrete spaces, and in 1985 M. G. Bell [4] defined the class of centered spaces which generalized the class of polyadic spaces. In paper [6] W. Kulpa and M. Turzański introduced the class of weakly dyadic spaces. The common feature of these generalizations is that many theorems which were originally proved for the class of dyadic spaces can be proved for them too.

Another generalization of the class of metric compact spaces is the class of Corson-compact spaces.

The aim of this paper is to present some connections between diferent generalizations of compact metric spaces.

The first part of our paper will present some conditions of weakly dyadic spaces and some examples which show that there exists a space which is weakly dyadic but not centered (this space will be also Corson-compact space). It will be shown that the space of closed subsets of the space D^{ω_2} is not weakly dyadic (a generalization of a dyadic theorem of Sapiro and a centered theorem of Bell).

The second part will show that each Corson-compact space is weakly dyadic and from this will follow immediately that the weight of Corson-compact space is equal to the density.

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On the other hand each dantian space has calibre $\aleph_1$ and Corson-compact space with calibre $\aleph_1$ is separable, hence each dantian Corson-compact space is metrizable.

In the third part of our paper an example of centered and dantian space of weight ω_1 which is not polyadic will be constructed. The first who showed that there exists a space which is dantian and not dyadic was Sapiro [9] who proved that exp D^{ω_2} is not dyadic, but is dantian as was shown by Arhangelskii [2]. Sapiro [10] later gave an example of space of weight ω_1 which is dantian but not dyadic. We do not know if this space is centered or not. Examples of dantian spaces of weight ω_1 which are not dyadic play the important role because it was proved by Sirota [8] that exp D^{ω_1} is homeomorphic to D^{ω_1}.

The dantian spaces have the Suslin property as was shown by Arhangelskii. R. Engelking proved that if polyadic space has the Suslin property then is dyadic. From this it follows that each dantian and polyadic space is dyadic.

No example of weakly dyadic and dantian space which is not centered is known. Also, no example of centered Corson-compact space which is not polyadic is known.

§ 1

Let T be an infinite set. Denote a Cantor cube by

$$D^T := \{p: p: T \to \{0, 1\}\}.$$

For $s \subset T$, $i: s \to \{0, 1\}$, $p \in D^T$ we shall use the following notation

$$H_s^i := \{f \in D^T: f \restriction s = i\}$$

$$G_s(p) := \{f \in D^T: f \restriction s = p \restriction s \text{ and } p^{-1}(0) \subset f^{-1}(0)\}$$

One can observe that

(I) $G_s(p) \subset H_s^i$.

Definition. A subset $X \subset D^T$ is said to be an ω-set iff for each $p \in X$ there exists and $s \subset T$ such that $|s| \leqq \omega$ and $G_s(p) \subset X$.

Definition. A space Y is said to be a weakly dyadic space if Y is a continuous image of a compact ω-set in D^T.

M. G. Bell [4] has defined for any infinite collection T of sets, a space Cen (T) by the following way:

$$\text{Cen}(T) := \{S: S \text{ is centered subcollection of } T\} \cup \{\emptyset\}.$$

If $s \in T$, then

$$s^+ = \{S \in \text{Cen}(T): s \in S\}$$

$$s^- = \{S \in \text{Cen}(T): s \notin S\}$$

Use $\{s^+: s \in T\} \cup \{s^-: s \in T\}$ as a clopen subbase for a topology on Cen (T).

The family Cen (T) as a topological space can be identified with a subspace $X \subset D^T$;

$$X := \{f: f \text{ is a characteristic function of a centered subcollection of } T\} .$$

The space X is a compact subspace of D^T. Notice that for each $p \in X$,

$$G_\emptyset(p) := \{f \in D^T: p^{-1}(0) \subset f^{-1}(0)\} \subset X .$$

Thus, the set X is an ω-set in the sense of our definition.

Definition. A space Y is said to be centered if Y is a continuous image of a space Cen (T) for some family T.

The following is a generalization of one of the most important theorems for dyadic spaces (it is still not known if the theorem is valid for centered spaces).

Theorem [6]. Each compact G_δ subset of a weakly dyadic space is a weakly dyadic space.

For completness we would like to give an example of a weakly dyadic space which is not a centered space.

Example [6]. Consider the Cantor cube $D^\mathfrak{c}$, where $\mathfrak{c} = 2^\omega$. Choose a subset $S \subset \mathfrak{c}$ such that $|\mathfrak{c} \setminus S| = \omega$. Define

$$H := \{f \in D^\mathfrak{c}: f \mid S = 0\} .$$

It is clear that the set H is homeomorphic to the Cantor cube D^ω, so the cardinality of H is equal to $\mathfrak{c}$. Thus we can denumerate points from the set H by indexes from the set S;

$$H = \{x^\alpha: \alpha \in S\} .$$

Now, let us define

$$M := \{f \in D^\mathfrak{c}: \exists \alpha \in S \text{ such that } f(\alpha) = 1 \text{ and } f \mid \mathfrak{c} \setminus \{\alpha\} = x^\alpha \mid \mathfrak{c} \setminus \{\alpha\}\} .$$

The subspace $X := H \cup M$ of the cube $D^\mathfrak{c}$ is a closed subspace and satisfies the first axiom of countability i.e., $\chi(X) = \omega$. The weight of the space X is equal to continuum, because $M \subset X$ is a discrete subspace and $|M| = \mathfrak{c}$. One can verify that the space X is weakly dyadic. On the other hand the space X cannot be centered because how it was proved in [4] if X is centered then $wX = \chi(X)$.

(II) If X is ω-set, then for each cardinal number $\mathfrak{m}$ the set $X_m := \{x \in X: |\{t \in T: x(t) \neq 0\}| \leqq \mathfrak{m}\}$ is the dense subset of X.

Proof. Let H_v^i, where v is finite be such a basic subset of the Cantor cube D^T, that $H_v^i \cap X \neq \emptyset$. Let $p \in H_v^i \cap X$. From the definition of X it follows that there exists $S \subset T$, $|S| \leqq \omega$ such that $G_s(p) \subset X$. Hence to X belongs a point q such that $q \mid S \cup v = p \mid S \cup v$ and $q(\alpha) = 0$ for $\alpha \in T \setminus (S \cup v)$ and $q \in H_v^i \cap X_\mathfrak{m}$.

(III) If X is a compact ω-set, then

(*) if $M \subset X_{\mathfrak{m}}$ and $|M| \leqq \mathfrak{m}$, then $clM \subset X_{\mathfrak{m}}$ and $wclM \leqq \mathfrak{m}$.

Proof. Each point $x \in M$ has no more than $\mathfrak{m}$ coordinates equal 1. Hence the set $S = \{t \in T\colon$ there exists $x \in M$ such that $x(t) = 1\}$ has the cardinality not greater than $\mathfrak{m}$. So $M \subset H^i_{T\setminus S}$ where $i(T \setminus S) = 0$. The set $H^i_{T\setminus S}$ is homeomorphic with $D^{T\setminus(T\setminus S)} = D^S$. Hence $clM \subset H^i_{T\setminus S} \cap X \subset X_m$ and $wclM \leqq \mathfrak{m}$.

(IV) If $\chi(x, X) \leqq \mathfrak{m}$, then $x \in X_{\mathfrak{m}}$.

Proof. Let $\mathscr{B}$ be a base in point x such that $|\mathscr{B}| \leqq \mathfrak{m}$. For each $U \in \mathscr{B}$ let us take a point $x_U \in U \cap X_{\mathfrak{m}}$. We have $|\{x_U\colon U \in \mathscr{B}\}| \leqq \mathfrak{m}$. Hence $cl\{x_U\colon U \in \mathscr{B}\} \subset X_{\mathfrak{m}}$ and $x \in cl\{x_U\colon U \in \mathscr{B}\}$.

Corollary 1. If X is separable, weakly dyadic space and satisfies the first axiom of countability i.e., $\chi(X) = \omega$, then X is metrizable.

Corollary 2. If X is hereditarily separable, weakly dyadic space, then X is metrizable.

Let us denote that conditions (II) and (III) are preserved under continuous functions. The condition (*) was used by Arhangelskii [2] for defining the class of thick spaces.

Definition. A Hausdorff space X is called thick if for each cardinal number $\mathfrak{m}$ there exists a dense subset $X_{\mathfrak{m}}$ such that

(*) if $M \subset X_{\mathfrak{m}}$ and $|M| \leqq \mathfrak{m}$, then $clM \subset X_{\mathfrak{m}}$ and $wclM \leqq \mathfrak{m}$.

From (III) it folows that each weakly dyadic space is thick.

As was proved by Arhangelski [2] continuous image, cartezian product and space of closed subsets of thick space is thick.

In the paper [11] (see also Comfort [5]) was proved

1. If X is thick, then $wX = ssX$.
2. If X is thick and $sX \leqq \chi(X)$, then $wX = \chi(X)$.
3. If X is thick and $\chi(X) \leqq sX$, then $wX = sX$.

By sX we denote density of X, by ssX heraditary density. This theorem are theorems of Esenin-Volpin type for dyadic spaces.

Now we shall prove that the space of closed subset of space $D^{\omega_2}(\exp D^{\omega_2})$ is not weakly dyadic but is thick. For this purpose we give first an example of space which satisfies the first axiom of countability, separable of weight ω_1 linearly ordered and zero-dimensional. Such a space as follows from corollary 1 is not weakly dyadic.

Example 2. Let us take an interval $[0, 1)$ and a subset S of interval $(0, 1]$ such that $|S| = \omega_1$ and S is dense in $(0, 1]$ and $1 \in S$.

Let $W = [0, 1) \times \{0\} \cup S \times \{1\}$.

In W we take a topology generated by the lexicografic order. The topology on W is generated by the base $\mathscr{B}$ consisting of the following sets

$$U = [(a, 0), (b, 0)) \cup ((a, 1), (b, 1))$$
$$V = ((a, 0), (b, 0)) \cup ((a, 1), (b, 1)] \quad \text{where} \quad a, b \in S\,.$$

The base $\mathscr{B}$ has cardinality ω_1. The space W is compact, linearly ordered, separable and zero-dimensional space of weight ω_1. Hence W is not weakly dyadic.

Lemma. If X is linearly ordered compact zero-dimensional space, then X is a retract of $\exp X$.

M. G. Bell [3] proved that.

Theorem. Let $\mathscr{S}$ be a subbase consisting of closed subsets of the space X closed with respect to finite intersection and let X map onto $\exp D^{\varkappa^+}$. If Y is a compact zero-dimensional space and $wY \leqq \varkappa$, then there exists $S \in \mathscr{S}$ such that S has a continuous function onto $\exp Y$.

The following is a generalization of a dyadic theorem of Sapiro and of a centered theorem of Bell.

Theorem 1. Exp D^{ω_2} is not weakly dyadic.

Proof. Suppose that there exists weakly dyadic space X such that $\exp D^{\omega_2}$ is a continuous image of X. By Bell's theorem there exists $S \in \mathscr{S}$ such that S has a continuous function onto $\exp W$, where W is the space from example 2. Since closed G_δ subset of weakly dyadic space is weakly dyadic, hence S is weakly dyadic and $\exp W$ too. But space W which is not weakly dyadic is a retract of $\exp W$, a contradiction.

§ 2

Definition. A compact space X is called Corson-compact if there are a set Γ and a homeomorphic embeding of X into the $\omega^+ - \Sigma$ product of R^Γ based at 0, i.e., into

$$\Sigma(R^\Gamma) = \{x \in R^\Gamma \colon |\{\gamma \in \Gamma \colon x_\gamma \neq 0\}| \leqq \omega\} .$$

Theorem 2. If X is a Corson-compact space, then X is a weakly dyadic space.

Proof. Let us consider the set C of all real numbers of the segment $[-1, 1]$ that have a tryadic expansion in which the digit 1 and -1 do not occur, i.e., the set of all numbers of the form

$$x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} \quad \text{where} \quad x_i \in \{-1, 0, 1\} \quad \text{for} \quad i = 1, 2, \ldots .$$

The formula

$$(x) = \sum_{i=1} \frac{x_i}{2^i} \quad \text{defines a continuous function}$$

from C onto interval $[-1, 1]$. Denote by $K = C \cap (-1, 1)$. Let us consider the product K^Γ and $(-1, 1)^\Gamma$. The product $(-1, 1)^\Gamma$ is homeomorphic to R^Γ.

Let $F: K^{\Gamma} \to (-1, 1)^{\Gamma}$ be given by $F(x) = \{f(x_\alpha)\}$ where $x = \{x_\alpha\}$. Since $f^{-1}(-1, 1) = K$ and f is perfect map, hence $f \mid K$ is perfect, and F as a product of perfect maps is perfect too.

Let us consider $\Sigma(K^{\Gamma})$ and $\Sigma((-1, 1)^{\Gamma})$ based at 0. Since $f^{-1}(0) = 0$ hence $F(\Sigma(K^{\Gamma})) = \Sigma((-1, 1)^{\Gamma})$. Let X be a Corson-compact space. Hence there exist a set Γ and a homeomorphic embeding of X into $\Sigma((-1, 1)^{\Gamma})$. Since F is perfect, hence $F^{-1}(X)$ is compact subspace of $\Sigma(K^{\Gamma})$. From this it follows that $F^{-1}(X)$ is the compact ω-set, and hence X is weakly dyadic.

Corollary. If X is a Corson-compact space, then $wX = sX$.

Proof. Since X is a Corson-compact space, hence X is weakly dyadic. From this it follows that there exists a compact ω-set Y in D^{Γ} such that Y is also the Corson-compact space and $Y_{\mathfrak{m}} = Y$ for each cardinal number $\mathfrak{m} \geqq \omega$ and X is continuous image of Y. From this it follows that $wX = sX$.

Remark. The space from example 1 is the Corson-compact space. Hence there exists a Corson-compact weakly dyadic space which is not centered.

§ 3

The dantian spaces have the Suslin property as was shown by Arhangelskii. R. Engelking proved that if a polyadic space has the Suslin property, then is dyadic. From this it follows that each dantian and polyadic space is dyadic.

We shall give an example of space which is dantian and centered but not polyadic.

Let X be a compact zero-dimensional space. Let $\mathscr{S}$ be a family consisting of all closed and open subsets of X.

Put $\text{Cen}(X) = \{T: T \text{ is a centered subcolection of } \mathscr{S}\} \cup \{\emptyset\}$.

Fact. If X is a compact zero-dimensional space and $sX = \mathfrak{m}$, then $s(\text{Cen}(X)) = \mathfrak{m}$.

Proof. Let S be a dense subset of X such that $|S| = sX$. For each $x \in S$ denote by $\mathscr{S}_x = \{U \in \mathscr{S}: x \in U\}$.

Let $\mathscr{R}_x = \{T \subset \mathscr{S}_x: |\mathscr{S}_x \setminus T| < \omega\}$. Let us consider the set $\mathscr{R} = \bigcap\{\mathscr{R}_x: x \in S\}$. It is easy to see that $\mathscr{R}$ is dense in $\text{Cen}(X)$ and $|\mathscr{R}| = \mathfrak{m}$.

Example 3. Let $\omega_1 + 1$ denote the set of ordinal number not greater than ω_1. In set $\omega_1 + 1$ we shall consider topology generated by order. This is the compact, Hausdorff, zero-dimensional space of weight ω_1. By Parovitchenko's theorem the space $\omega_1 + 1$ is a continuous image of ω^* (the remainder of the Čech-Stone compactification of the space ω). From this it follows that there exists a compactification of ω with remainder $\omega_1 + 1$. Let us denote this space by K. Let us consider the space $\text{Cen}(K)$. From the Fact it follows that $s(\text{Cen}(K)) = \omega$. The space which is thick and separable is dantian. Hence $\text{Cen}(K)$ is centered and dantian space of weight ω_1.

We shall prove that Cen (K) is not a polyadic space. Denote by $T = \{U \subset K\colon U$ is clopen in K and $\omega_1 \in U\}$. Let us observe that $T \in cl \bigcup \{\{n\}^+\colon n = 1, 2, \ldots\}$. Mrówka [7] proved that if X is a polyadic space and U is open in X, then for each $x \in clU$ there exists a chain $\{x_n\} \subset U$ which is convergent to x.

(*) Suppose that there exists a chain $\{x_n\} \subset \bigcap \{\{n\}^+\colon n = 1, 2, \ldots\}$ which is convergent to T.

It means that for each open neighbourhood of T there exists N_U such that for each $n > N_U$ $x_n \in U$.

Since T is the maximal centered family, hence V or $K \setminus V$ belong to T for each clopen subset of K. It is easy to see that $(K \setminus V)^+ \subset V^-$.

Hence it is sufficient to consider neighbourhoods of T of the form V^+. Since K is the compactification of a set ω, hence for each $V \subset K$ we have some subset of ω equal $\omega \cap V$.

Since each element $U \in T$ is a neighbourhood of point ω_1, hence U is infinite. Let $\{\alpha_n\}$ be a chain from $U \cap (\omega_1 + 1)$ such that $\alpha_n < \omega_1$ for each $n \in \omega$. There exists $\beta < \omega_1$ such that for each $n \in \omega$ $\alpha_n < \beta$. If $\beta < \omega_1$, then $\beta^+ < \omega_1$. Then the set $G = \{\gamma\colon \gamma \geqq \beta^+\}$ is the clopen subset of $\omega_1 + 1$. Both the sets $G \cap U \cap (\omega_1 + 1)$ and $\{\gamma\colon \gamma \leqq \beta$ and $\gamma \in U \cap (\omega_1 + 1)\}$ are infinite and clopen in $\omega_1 + 1$. Let $G = (\omega_1 + 1) \cap H$ and $\{\gamma\colon \gamma \leqq \beta$ and $\gamma \in U \cap (\omega_1 + 1)\} = (\omega_1 + 1) \cap E$ where H and E are clopen in K. Sets $H \cap \omega$ and $E \cap \omega$ are contained in $U \cap \omega$ and both are infinite. So (*) is false.

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