

Dirk Werner

Extreme points in spaces of operators and vector – valued measures

In: Zdeněk Frolík (ed.): Proceedings of the 12th Winter School on Abstract Analysis, Section of Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 5. pp. [135]–143.

Persistent URL: <http://dml.cz/dmlcz/701820>

## Terms of use:

© Circolo Matematico di Palermo, 1984

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

# EXTREME POINTS IN SPACES OF OPERATORS AND VECTOR-VALUED MEASURES

Dirk Werner

## 1. Introduction.

If one wants to prove that an operator  $T$  acting between Banach spaces  $X$  and  $Y$  is an extreme operator, that is an extreme point of the unit ball of the space of bounded linear operators  $L(X, Y)$ , it suffices to check that  $T'$  (the adjoint of  $T$ ) maps extreme functionals on  $Y$  onto extreme functionals on  $X$ . An operator with this property is called a nice operator [8]. It is even enough that  $T'$  maps a weak\*-dense subset of  $\text{ex } B_Y$  into  $\text{ex } B_X$ , but this is not a substantial generalization. ( $B_Z$  denotes the unit ball of a Banach space  $Z$ ,  $\text{ex } C$  the set of extreme points of a convex set  $C$ .)

Consider some examples:

1. The identity operator  $\text{id}: X \rightarrow X$  is always a nice, hence extreme operator. Using this and the canonical isometry between  $L(X, X'')$  and  $L(X', X')$  one can easily see that the natural injection  $i_X: X \rightarrow X''$  is always extreme.
2. To see that there are non-nice extreme operators consider the injection operator from  $l^1$  into  $l^p$  ( $1 < p < \infty$ ). Moreover, no operator from some space  $L^1(\mu)$  into some space  $L^p(\nu)$  ( $p$ 's above) is nice, but  $B_{L(L^1, L^p)}$  is compact with respect to the weak\*-operator topology so that there are many extreme operators.
3. A fairly easy description of nice operators is possible in the setting of spaces of continuous functions. The main tool here is to represent operators into  $CL$  ( $L$  a compact Hausdorff space) as vector-valued functions. If  $X$  is a Banach space, we associate with an operator from  $X$  into  $CL$  a function from  $L$  into  $X'$  in the following way:  $T \mapsto T^*$ ,  $T^*(1) := \delta_1 \circ T$ . This mapping induces an isometric isomorphism between  $L(X, CL)$  and the space of weak\*-continuous functions  $C(L, X'(\text{weak}^*))$  and between the space of compact operators  $K(X, CL)$  and the space  $C(L, X')$  of norm-continuous functions [7, p.490].

---

This paper is in final form and no version of it will be submitted for publication elsewhere.

Now,  $T \in L(X, CL)$  is nice if and only if the representing function assumes only extremal values:  $T^*(L) \subseteq \text{ex } B_{X^*}$ . More can be said if  $X = CK$ .  $T \in L(CK, CL)$  is nice if and only if  $T$  has the form  $Tf = \lambda \cdot f \circ \psi$  with  $\psi: L \rightarrow K$  continuous,  $\lambda \in CL$  with modulus one. Equivalently,  $T$  is nice if and only if  $T$  is essentially multiplicative ( $Tf \cdot Tg = T(\lambda \cdot fg)$ ).

As we pointed out before, in general it is not necessary for an extreme operator to be nice. In the case of operators between  $CK$ -spaces, however, the situation is much more involved. Several authors have treated the question if an extreme operator  $T$  from  $CK$  into  $CL$  is necessarily nice, cf. [3], [4], [8], [9], [11]. Although a variety of properties of  $K$ ,  $L$  or  $T$  is known that ensure that  $T$  is nice (e.g.  $K$  metrizable, real scalars; or  $L$  extremally disconnected, no matter if the scalars are real or complex), the general answer is no since Sharir has constructed counterexamples both for the real and complex case [12], [13].

All these results may be translated into results concerning measure-valued functions, thanks to the isometry  $T \mapsto T^*$ . In this paper we shall consider operators from  $C(K, E)$  to  $C(L)$ , where  $E$  is a Banach space. By Singer's theorem, the dual of  $C(K, E)$  may be thought of as the space  $M(K, E')$  of  $E'$ -valued regular Borel measures of finite variation, equipped with the total variation norm, cf. [15], [16] or [10]. Thus, we shall investigate functions on  $L$  the values of which are vector measures. The most far-reaching results can be obtained for compact operators, represented by elements of  $C(L, M(K, E'))$ . As it turns out, the proofs work for the space  $C(L, M(K, Z))$ , too, when  $Z$  is not a dual Banach space. Throughout,  $X, Y, Z, E$  denote Banach spaces,  $K, L$  denote compact Hausdorff spaces. The results apply to real as well as complex Banach spaces.

## 2. Characterization of extreme points.

**Proposition 1:** Let  $T$  be an extreme point of the unit ball of  $K(C(K, E), CL)$ . Then, for  $1 \in L$ ,  $T^*(1)$  is a point measure with norm one:  $T^*(1) = p \otimes \delta_k$ ,  $p \in E'$  with norm one.

The case  $E = \text{scalars}$  has been settled in [8]. Proposition 1 is a consequence of the following result.

**Proposition 1\*:** Let  $f$  be an extreme point of the unit ball of  $C(L, M(K, Z))$ . Then, for  $l \in L$ ,  $f(l)$  is a point measure with norm one.

**Proof:** First of all  $\|f(l)\| = 1$  since  $f(l) \pm (1 - \|f(l)\|)m$  lie in the unit ball of  $M(K, Z)$  whenever  $\|m\| \leq 1$ . Next, consider the function  $F: L \rightarrow M(K)$ ,  $F(l) := |f(l)|$ . Since  $\| |m_1| - |m_2| \| \leq \|m_1 - m_2\|$  for vector measures  $m_i$  of bounded variation,  $F$  is a norm-continuous function which has probability measures as values. We claim:

$F \in \text{ex} \{g \mid g \in C(L, MK), \text{ all } g(l) \text{ are probability measures}\}$ .

Once this claim is established, we finish the proof as follows.  $F$  defines an operator  $S: CK \rightarrow CL$  by  $(S\psi)(l) := \int \psi d(F(l))$ , of course  $S^* = F$ . If  $F$  is extreme,  $S$ , too, is extreme and positive. By a result due to Phelps [9]  $S$  is nice, that means  $F(l)$  is a point measure. It follows that  $f(l)$  is a point measure.

To prove the claim let  $u: L \rightarrow M(K)$  be a continuous function such that  $F(l) \pm u(l) \geq 0$  and  $(F(l) \pm u(l))(K) = 1$  for all  $l \in L$ . In this case  $u(l)$  is absolutely continuous with respect to  $F(l)$  so that there is a Borel function  $h_l$  with  $u(l) = h_l \cdot F(l)$ . We may (and shall) assume that  $h_l$  is real-valued and  $-1 \leq h_l(k) \leq 1$  for all  $k \in K$ . Let  $m(l) := h_l \cdot f(l) \in M(K, Z)$ . Then (a)  $l \mapsto m(l)$  is continuous, and (b)  $\|f(l) \pm m(l)\| \leq 1$  for all  $l \in L$ . We conclude that  $m = 0$  and consequently  $u = 0$ .

It remains to prove (a) and (b).

$$(a) \quad \|m(l) - m(l')\| = \|h_l \cdot f(l) - h_{l'} \cdot f(l')\| \\ \leq \|h_l(f(l) - f(l'))\| + \|(h_l - h_{l'})f(l')\|$$

$$\text{The first term} = \int |h_l| d|f(l) - f(l')| \quad [6, p. 173] \\ \leq \|f(l) - f(l')\|.$$

$$\text{The second term} = \int |h_l - h_{l'}| dF(l') \quad [6, p. 173] \\ = \int_A \dots + \int_B \dots$$

with  $A := \{k \mid k \in K, h_l(k) \geq h_{l'}(k)\}$ ,  $B := K \setminus A$ .

$$\text{Now } \int_A \dots = \int_A h_l dF(l') - \int_A h_{l'} dF(l') \\ = \int_A h_l dF(l) - \int_A h_{l'} dF(l') + \int_A h_l d(F(l') - F(l)) \\ \leq |u(l) - u(l')|(A) + |F(l) - F(l')|(A).$$

$B$  may be treated analogously so that

$$\|m(l) - m(l')\| \leq \|f(l) - f(l')\| + \|u(l) - u(l')\| + \|F(l) - F(l')\|$$

Therefore  $m(\cdot)$  is continuous.

$$(b) \quad \|f(l) \pm m(l)\| = \|(\mathbb{1} \pm h_l) \cdot f(l)\| \\ = \int |\mathbb{1} \pm h_l| dF(l) \quad [6, p. 173] \\ = \int dF(l) \pm \int h_l dF(l) \\ = (F(l) \pm u(l))(K) = 1.$$

In order to tackle the problem if an extreme operator from  $C(K, E)$  into  $CL$  is nice we need to know the extreme functionals on  $C(K, E)$ . They were first described by Singer [14], but the proofs in the literature are quite complicated, cf. also [17]. Here we shall present a very simple argument.

**Theorem 2:** A vector measure  $m \in M(K, Z)$  is an extreme point of the unit ball if and only if  $m = z \otimes \delta_k$  for some  $z \in \text{ex } B_Z$ ,  $k \in K$ .

**Proof:** Assume  $m$  is extreme. If it were not a point measure, the total variation measure  $|m|$  wouldn't be either. Therefore  $0 < |m|(A) =: a < 1$  for some Borel set  $A$ . Then we can represent  $m$  as a non-trivial convex combination of norm-one measures:  
 $m = a (a^{-1} \cdot \mathbb{1}_A m) + (1-a) ((1-a)^{-1} \cdot \mathbb{1}_{A^c} m)$ . We infer that  $m$  is of the form  $z \otimes \delta_k$ , since  $m$  is extreme we have  $z \in \text{ex } B_Z$ .

On the other hand, if  $m$  is as stated, we shall show that it is extreme. Indeed,  $m$  is extreme in the subspace  $Z_0 := \{x \otimes \delta_k \mid x \in Z\}$ . But  $Z_0$  is a very well complemented subspace, namely, it is the range of a projection  $P$  satisfying  $\|n\| = \|Pn\| + \|n - Pn\|$  for all  $n \in M(K, Z)$ , a so-called  $L$ -projection. Here,  $Pn := n(\{k\}) \otimes \delta_k$  defines the required projection. It follows easily from the defining norm-condition that  $\text{ex } B_{Z_0} \subseteq \text{ex } B_{M(K, Z)}$ , and  $m$  is extreme.

Specializing to  $Z = E'$  we get Singer's theorem on extreme functionals. We may state it in the form:  $Q: C(K, E) \rightarrow E$ ,  $Qf := f(k)$  ( $k \in K$  fixed, but arbitrary) is a nice operator. Looking at the above argument we see that the proof used the facts that  $Q'$  is an isometric embedding and that the polar of  $\text{Ker } Q$  (that is  $Z_0$ ) is the range of an  $L$ -projection. A space with this property is called an  $M$ -ideal [1],[2]. So we have actually shown:

**Proposition 3:** Suppose  $Q: X \rightarrow Y$  is a quotient map such that  $\text{Ker } Q$  is an  $M$ -ideal. Then  $Q$  is a nice operator. Moreover,  $p \in \text{ex } B_Y$ , iff  $Q'p \in \text{ex } B_X$ .

Note that Proposition 3 includes [5, Theorem 1(b)].

It is known that an  $M$ -ideal  $J$  of a Banach space  $X$  satisfies the following intersection property  $IP(n)$  for each  $n \in \mathbb{N}$ :

Whenever  $U_1, \dots, U_n$  are open balls in  $X$  with non-void intersection  $\neq \emptyset$  such that  $J \cap U_i \neq \emptyset$  for all  $i$ , then  $J \cap U \neq \emptyset$ .

Conversely, if a closed subspace  $J$  of  $X$  satisfies  $IP(3)$ , then  $J$

is an M-ideal [1, Th. 5.9], [2, Th. 2.17]. Alfsen and Effros [1, p.125] exhibit an example of a subspace of some 3-dimensional real Banach space which fails to be an M-ideal but satisfies IP(2). It is, however, an open problem, whether such an example exists in a complex Banach space [20]. Here it is interesting to note that Proposition 3 holds if  $\text{Ker } Q$  satisfies the slightly weaker condition of having IP(2). This is shown in [18] by means of a direct application of the intersection property.

Proposition 3 can be used to prove the following extension theorem for extreme operators.

Proposition 4: Suppose  $Q: X \rightarrow Y$  is a quotient map and that  $\text{Ker } Q$  is an M-ideal of  $X$ . If  $S: Y \rightarrow \text{CL}$  is an extreme operator, then  $T := S \circ Q: X \rightarrow \text{CL}$  is an extreme operator, and  $T^*(1)$  is extreme iff  $S^*(1)$  is extreme. (Analogously for compact operators.)

Proof: We have only to show that  $T$  is extreme, the other statements are proved in Proposition 3. Let  $\|T \pm U\| \leq 1$ . Then we have  $\|Q'(S^*(1)) \pm U^*(1)\| \leq 1$  for all  $1 \in L$ . By assumption on  $Q$   $Q'$  is an isometric isomorphism from  $Y'$  onto  $W := (\text{Ker } Q)^{\perp} \subseteq X'$ , and there is a decomposition  $X' = W \oplus W^{\perp}$  with some closed subspace  $W^{\perp} \subseteq X'$ . Using this we may write  $U^*(1) = w(1) + w^{\perp}(1) \in W \oplus W^{\perp}$ , and therefore  $1 \geq \|Q'(S^*(1)) \pm w(1)\| + \|w^{\perp}(1)\|$ . Since  $S$  is extreme, a theorem due to Sharir [11] yields  $1 = \|S^*(1)\| = \|Q'(S^*(1))\|$  on a dense subset  $H$  of  $L$ . Hence  $w^{\perp}(1) = 0$  on  $H$ , equivalently  $U^*(H) \subseteq W$ . But  $U^*$  is weak\*-continuous, and  $W$  is weak\*-closed so that  $U^*(L) \subseteq W$ . Thus, for  $1 \in L$  there exists  $v(1) \in Y'$  such that  $U^*(1) = Q'(v(1))$ . Now  $v$  is seen to be weak\*-continuous, hence  $v = 0$  and  $U = 0$  because  $\|Q'(S^*(1)) \pm Q'(v(1))\| = \|S^*(1) \pm v(1)\| \leq 1$ . The (easier) proof for compact operators is established in the same way.

Corollary 5: Let  $S: E \rightarrow \text{CL}$  be an extreme operator (resp. extreme compact operator). Then for every compact Hausdorff space  $K$  there exists an extreme operator (resp. extreme compact operator)  $T: C(K, E) \rightarrow \text{CL}$  with  $T^*(1)$  extreme iff  $S^*(1)$  extreme.

Proof: Choose any  $k \in K$  and consider  $Q: C(K, E) \rightarrow E$ ,  $Qf := f(k)$ . Proposition 4 gives the result, the M-ideal property has been observed in the proof of Theorem 2. Cf. also [2, prop. 10.1].

In [3] it is shown that for every compact Hausdorff space  $L$  there exists a Banach space  $E$  and an extreme operator  $S: E \rightarrow CL$  such that  $S^*(1)$  is extreme if and only if  $1$  is an isolated point of  $L$ . If  $L$  is the unit interval,  $E$  may be chosen to be 4-dimensional. The following theorem shows that the space  $C(K, E)$  perfectly reflects the properties of  $CK$  and  $E$  as far as the characterization of compact extreme operators into  $CL$  is concerned; for merely bounded operators we shall need additional assumptions.

Theorem 6: The following statements are equivalent:

- (a) Every extreme point of the unit ball of  $K(E, CL)$  is nice.
- (b) Every extreme point of the unit ball of  $K(C(K, E), CL)$  is nice, where  $K$  is an arbitrary compact Hausdorff space.

Proof: That (b) implies (a) is the contents of Corollary 5. Assume (a) and let  $T: C(K, E) \rightarrow CL$  be an extreme point of the unit ball of compact operators. By Proposition 1 we have  $T^*(1) = \rho(1) \otimes \delta_{\psi(1)}$ . Since  $T^*$  is norm-continuous,  $\psi$  is locally constant: If  $1 \in U$ , then  $\|T^*(1) - T^*(1')\| < 1$  for  $1'$  in some neighbourhood  $U$  of  $1$ . We conclude  $\psi(1') = \psi(1)$  for all  $1' \in U$ .  $L$  is compact, so there are finitely many pairwise disjoint clopen sets  $L_1, \dots, L_n$  covering  $L$  and  $k_1, \dots, k_n \in K$  with  $\psi(1) = k_i$  for  $1 \in L_i$ . Now  $\rho(\cdot)$  is seen to be norm-continuous, hence it represents a compact operator from  $E$  to  $CL$ . By assumption (a) we have only to show that it is extreme. Let  $u: L \rightarrow E'$  be a norm-continuous function such that  $\|\rho(1) \pm u(1)\| \leq 1$  for all  $1 \in L$ . Define  $w(1) := \sum \chi_{L_i}(1) \cdot u(1) \otimes \delta_{k_i}$ . Then  $w$  is norm-continuous,  $\|T^*(1) \pm w(1)\| \leq 1$  for all  $1$ , and hence  $w = 0$ . We infer  $u = 0$ .

A consequence of the above proof is:

Corollary 7: If  $T$  is an extreme point of the unit ball of  $K(C(K, CL))$ , then  $T$  is a finite rank operator.

Proof: We have  $T^*(1) = \rho(1) \cdot \delta_{\psi(1)}$ ,  $\rho = T^*1 \in CL$ . As pointed out above, there are pairwise disjoint clopen sets  $L_1, \dots, L_n$  covering  $L$  and  $k_1, \dots, k_n \in K$  with  $\psi(1) = k_i$  for  $1 \in L_i$ . It follows

$$(Tf)(1) = T^*(1)(f) = \rho(1) f(k_i) \quad \text{for } 1 \in L_i, f \in CK.$$

Hence  $Tf = \sum f(k_i) \chi_{L_i} \cdot T1$   
and  $\text{range}(T) \subseteq \text{span} \{ \chi_{L_i} \cdot T1 \mid i = 1, \dots, n \}$ .

Theorem 8. Suppose that every extreme operator from  $E$  to  $CL$  is nice. Let  $T: C(K, E) \rightarrow CL$  be an extreme operator such that  $T^*(l)$  is a non-zero point measure for all  $l \in L$ . Then  $T$  is nice.

Proof:  $T^*$  is of the form  $T^*(l) = \rho(l) \otimes \delta_{\psi(l)}$ ,  $\rho(l) \in E' \setminus \{0\}$ . The function  $l \mapsto \rho(l)$  is seen to be weak\*-continuous, hence represents a bounded linear operator  $S: E \rightarrow CL$ . In order to show that  $T$  is nice it is sufficient to show that  $S$  is nice by Theorem 2. For this it is enough to prove that  $S$  is extreme according to our present assumptions. In fact, let  $u$  be a weak\*-continuous function from  $L$  into  $E'$  such that for all  $l \in L$   $\|\rho(l) \pm u(l)\| \leq 1$ . Then  $\|T^*(l) \pm u(l) \otimes \delta_{\psi(l)}\| \leq 1$  for all  $l$ . But  $l \mapsto u(l) \otimes \delta_{\psi(l)}$  is weak\*-continuous since  $\psi: L \rightarrow K$  and the map  $(B_{E'}, \text{weak}^*) \times K \rightarrow (B_{C(K, E)}, \text{weak}^*)$ ,  $(g, k) \mapsto g \otimes \delta_k$  are continuous. Therefore  $u = 0$ , and  $S$  is extreme.

(To prove that  $\psi$  is continuous fix  $l_0 \in L$ . We shall show that  $\delta_{\psi(\cdot)}$  is continuous at  $l_0$ . Choose  $x \in E$  so that  $\rho(l_0)(x) = 1$ . Then for  $g \in CL$   $T^*(l)(g \otimes x) = \rho(l)(x) g(\psi(l))$ , and the weak\*-continuity of  $T^*$  and  $\rho$  together with  $\rho(l_0)(x) \neq 0$  imply the weak\*-continuity of  $\delta_{\psi(\cdot)}$ . Since  $l_0$  is arbitrary we are done.)

It can be shown that the conditions in Theorem 8 concerning  $T$  are fulfilled whenever  $K$  is dispersed and  $E$  is finite dimensional, cf. [18].

To finish with, let us rephrase Theorem 6 in terms of vector-valued functions.

Theorem 6\*: The following statements are equivalent:

- (a) Every extreme function in  $C(L, Z)$  assumes only extreme values.
- (b) Every extreme function in  $C(L, M(K, Z))$  assumes only extreme values, where  $K$  is an arbitrary compact Hausdorff space,

In [19] it is proved that an abstract  $L$ -space  $Z$  satisfies the above conditions.

## REFERENCES

1. ALFSEN E. M. and EFFROS E. G. "Structure in real Banach spaces I", Ann. of Math. 96 (1972), 98-128
2. BEHREND S. "M-Structure and the Banach-Stone Theorem", Lecture Notes in Mathematics 736, Berlin-Heidelberg-New York: Springer 1979
3. BLUMENTHAL R. M., LINDENSTRAUSS J. and PHELPS R. R. "Extreme operators into  $C(K)$ ", Pacific J. Math. 15 (1965), 747-756
4. CORSON H. H. and LINDENSTRAUSS J. "Continuous selections with nonmetrizable range", Trans. Amer. Math. Soc. 121 (1966), 492-504
5. CUNNINGHAM F. and ROY N. M. "Extreme functionals on an upper semicontinuous function space", Proc. Amer. Math. Soc. 42 (1974), 461-465
6. DINULEANU N. "Vector Measures", International Series of Monographs in Pure and Applied Mathematics 95, New York: Pergamon Press 1967
7. DUNFORD N. and SCHWARTZ J. "Linear Operators. I: General Theory", Pure and Applied Math. 7, New York: Interscience 1958
8. MORRIS P. D. and PHELPS R. R. "Theorems of Krein-Milman type for certain convex sets of operators", Trans. Amer. Math. Soc. 150 (1970), 183-200
9. PHELPS R. R. "Extreme positive operators and homomorphisms", Trans. Amer. Math. Soc. 108 (1963), 265-274
10. SCHACHERMAYER W. "Sur un théorème de Grothendieck", Séminaire Pierre Lelong (Analyse), 14e année, 1974/75, Lecture Notes in Mathematics 524, 193-212, Berlin-Heidelberg-New York: Springer 1976
11. SHARIR M. "Characterization and properties of extreme operators into  $C(Y)$ ", Isr. J. Math. 12 (1972), 174-183
12. SHARIR M. "A counterexample on extreme operators", Isr. J. Math. 24 (1976), 320-328
13. SHARIR M. "A non-nice extreme operator", Isr. J. Math. 26 (1977), 306-312
14. SINGER I. "Sur la meilleure approximation des fonctions abstraites continues à valeurs dans un espace de Banach", Rev. Math. Pures Appl. 2 (1957), 245-262
15. SINGER I. "Les duals des certains espaces de Banach de champs de vecteurs I", Bull. Sci. Math. 82 (1958), 29-40
16. SINGER I. "Sur les applications linéaires intégrales des espaces de fonctions continues I", Rev. Math. Pures Appl. 4 (1959), 391-401
17. STRÖBELE W. J. "On the representation of the extreme functionals on  $C_0(T, X)$ ", J. Appl. Th. 10 (1974), 64-68
18. WERNER D. "Extremale Operatoren zwischen Räumen stetiger Funktionen"

Diplomarbeit, FU Berlin 1982

19. WERNER D. "Extreme points in function spaces", Proc. Amer. Math. Soc. 89 (1983), 598-600
20. YOST D. "Semi-M-ideals in complex Banach spaces", Rev. Math. Pure Appl., to appear

MATHEMATISCHES INSTITUT, FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 2-6.  
D-1000 BERLIN 33