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PARACOMPACTNESS IN BOX PRODUCTS

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For fifteen years there has been an active search for classes of spaces for which paracompactness occurs in box products (see surveys [3] and [13]). However, very little has been accomplished in the case of uncountably many factors - the case considered here.

In [2], van Douwen proved that for a very special class of paracompact P-spaces, the ω_μ-metrizable spaces, box products are paracompact if there are not too many factors. Prior to the present paper, this was the only "positive" result using nothing other than ZFC. Our first theorem is of a similar nature, but new even for the countable factors case.

1.3). If X_α is a paracompact scattered P-space for each $\alpha \in \omega_1$, then $< \omega_1 - \square_\alpha X_\alpha$ is paracompact.

Miller extended part of Kunen's nabla lemma from box products of countably many spaces to the countable support box product, $< \omega_1 - \square_\alpha X_\alpha$, of ω_1 many spaces [10]. In our section 2 we present a similar extension for the other part of the nabla lemma. This extension is applied in section 3 to prove

3.1). [CH] If X_α is a paracompact locally compact scattered space for each $\alpha \in \omega_1$, then $< \omega_1 - \square_\alpha X_\alpha$ is paracompact.

Our section 4, adds to the list of non-paracompact box products of compact spaces with an example whose proof is quite different from all the rest. We show

4.3). The box product of countably many compact orderable spaces need not be $\underline{c}$-Lindelöf.

In the final section we remark on recent efforts to understand the full box product topology of uncountably many compact first countable spaces.

This paper is in final form and no version of it will be submitted for publication elsewhere.

PRELIMINARIES: All spaces are regular Hausdorff. All cardinals are regular von Neumann ordinals. $\underline{\underline{c}} = 2^{\omega}$. CH denotes the continuum hypothesis and MA denotes Martin's axiom. If $\varkappa$ is a cardinal and if X is a set, then $[X]^{<\varkappa}$ denotes the set of all subsets Y of X whose cardinality $|Y|$ is less than $\varkappa$.

Suppose that $\varkappa$ is a cardinal, and for each $\alpha \in \varkappa$ X_α is a space. An open box in $\Pi_\alpha X_\alpha$ is a set of the form $B = \Pi_\alpha B_\alpha$ where B_α is an open subset of X_α for each α . The support of the open box B is the set $\mathrm{spt}(B) = \{\alpha \in \varkappa : B_\alpha \neq X_\alpha\}$. If λ is also a cardinal, then $< \lambda - \Box_\alpha X_\alpha$ denotes the set $\Pi_\alpha X_\alpha$ with the topology having as base the set of all open boxes B satisfying $\mathrm{spt}(B) \in [\varkappa]^{<\lambda}$. When $\varkappa < \lambda$, we use $\Box_\alpha X_\alpha$ instead of $< \lambda - \Box_\alpha X_\alpha$ and call it the full box product.

§1. A paracompact box product in ZFC

Recall that a space is called scattered provided each of its non-empty subspaces contains an isolated point. For a useful characterization of "scattered", recall the Cantor-Bendixen decomposition of a space X :

Let X' consist of the set of non-isolated points of X, and let $X^{(0)} = X$. Define, inductively, for each non-zero ordinal α, $X^{(\alpha)} = \bigcap_{\beta < \alpha} (X^{(\beta)})'$. It is known that X is scattered iff $\exists \alpha\ X^{(\alpha)} = \phi$.

1.1. DEFINITION. Suppose that X is a scattered space and $A \subseteq X$. Define the rank of A by

$$\mathrm{rk}(A) = \inf\{\alpha : A \cap X^{(\alpha)} = \phi\} .$$

Define the top of A by

$$t_p(A) = \begin{cases} A \cap X^{(\alpha)}, & \text{if } \mathrm{rk}(A) = \alpha + 1 \\ \phi, & \text{if } \mathrm{rk}(A) \text{ is a limit ordinal.} \end{cases}$$

We will say A is capped whenever $|t_p(A)| = 1$.

1.2. LEMMA [12]. Suppose that X is a paracompact scattered space. Then each open covering of X is refined by a pairwise-disjoint open capped refinement.//

The following conventions, for a family $\mathcal{U}$ of sets, are in use: For a set X, $\mathcal{U}|X = \{U \cap X : U \in \mathcal{U}\}$. A family $\mathcal{V}$ of sets refines $\mathcal{U}$ provided $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ and, for each $V \in \mathcal{V}$ there is a $U_V \in \mathcal{U}$ with $V \subseteq U_V$.

Recall that a P-space is a space in which each G_δ-set is open.

1.3. THEOREM. *Suppose that for each $\alpha \in \omega_1$, X_α is a paracompact scattered P-space. Then $< \omega - \Box_\alpha X_\alpha$ is paracompact.*

PROOF. We may assume each X_α is infinite, and we let $\square$ denote $< \omega_1 - \square_\alpha X_\alpha$. According to 1.2, $\square$ possesses a base $\mathcal{B}$ consisting of countable boxes B such that B_α is a clopen capped set of X_α for each $\alpha \in \operatorname{spt}(B)$. Obviously, we may assume, without loss of generality, that $\operatorname{spt}(B) \in \omega_1$ for each $B \in \mathcal{B}$, define

$$T_p(B) = \{x \in \Pi_\alpha X_\alpha : \{x_\alpha\} = t_p(B_\alpha)\ \forall\, \alpha \in \operatorname{spt}(B)\}.$$

Suppose that $\mathcal{G} \subseteq \mathcal{B}$ covers $\square$. We will show $\mathcal{G}$ is refined by a pairwise-disjoint subset of $\mathcal{B}$. Given a family $\mathcal{W} \subseteq \mathcal{B}$ define

$$\mathcal{W}^{\#} = \{W \in \mathcal{W}: \exists G \in \mathcal{G},\ \operatorname{spt}(G) \leq \operatorname{spt}(W) \text{ and } T_p(W) \subseteq G\}, \text{ and}$$

$$\mathcal{W}^{*} = \{W \in \mathcal{W}: \exists G \in \mathcal{G},\ W \subseteq G\}.$$

According to 1.2 X_0 is covered by a pairwise-disjoint open capped family $\mathcal{P}$. Set

$$\mathcal{U}(0) = \{B \in \mathcal{B}: B_0 \in \mathcal{P} \text{ and } \operatorname{spt}(B) = 0\}.$$

Obviously $\mathcal{U}(0)$ is a pairwise-disjoint open covering of $\square$. We now define, inductively for each non-zero ordinal $\nu \leq \omega_1$, a family $\mathcal{U}(\nu)$ subject to the restrictions (1) through (8) below:

(1). $\mathcal{U}(\nu)$ is a pairwise-disjoint subset of $\mathcal{B}$ covering $\square$.

(2). if $V \in \mathcal{U}(\nu)$, then $\operatorname{spt}(V) \leq \sup\{\mu + 1: \mu < \nu\}$.

(3). if ν is a limit ordinal and if $V \in \mathcal{U}(\nu)$, then

$$V = \bigcap\{U: \exists \mu < \nu,\ U \in \mathcal{U}(\mu),\ V \subseteq U\}.$$

For $\nu = \mu + 1$,

(4). $\mathcal{U}(\nu)$ refines $\mathcal{U}(\mu)$.

(5). $\mathcal{U}(\nu) \cap U(\mu) = \mathcal{U}(\nu)^{*}$.

(6). if $U \in \mathcal{U}(\mu)^{\#}$, then $\operatorname{spt}(V) = \operatorname{spt}(U)\ \forall\, V \in \mathcal{U}(\nu)|U$.

(7). if $U \in \mathcal{U}(\mu)^{\#}$, then $\exists V \in \mathcal{U}(\nu)^{*}$, $T_p(U) \subseteq V$.

(8). if $U \in \mathcal{U}(\mu)\backslash\mathcal{U}(\mu)^{\#}$, then $\operatorname{spt}(V) = \nu + 1\ \forall\, V \in \mathcal{U}(\nu)|\mathcal{U}$.

Suppose $\xi \leq \omega_1$ and $\forall \nu < \xi$ we have constructed $\mathcal{U}(\nu)$ subject to the conditions (1)-(8). We define $\mathcal{U}(\xi)$.

<u>Case</u> 1. ξ is a successor ordinal, say $\xi = \mu + 1$. In order to satisfy (1) and (4), we define for each $U \in \mathcal{U}(\mu)$, a pairwise-disjoint family $\mathcal{V}_U \subseteq \mathcal{B}$ covering U, and require $\mathcal{U}(\xi)|U = \mathcal{V}_U|U$. This is clear when $U \in \mathcal{U}(\mu)^{*}$ for we just let $\mathcal{V}_U = \{U\}$. So (5) will be fulfilled.

If $U \in \mathcal{U}(\mu)^{\#}|\mathcal{U}(\mu)^{*}$, choose one $G \in \mathcal{G}$ such that $spt(G) \leq spt(U)$ and $G \cap T_p(U) \neq \phi$. For each $\alpha \in spt(U)$, we apply 1.2 to find a pairwise-disjoint clopen capped refinement $\mathcal{F}_\alpha$ of $\{U_\alpha \cap G_\alpha, U_\alpha \setminus G_\alpha\}$. Now let

$$\mathcal{V}_U = \{V \in \mathcal{B}: V_\alpha \in \mathcal{F}_\alpha \text{ if } \alpha \in spt(U) \text{ and } spt(V) = spt(U)\}.$$

In order to see that (6) and (7) hold, suppose $y \in T_p(U)$. Then for each $\alpha \in spt(U)$ $\exists F_\alpha \in \mathcal{F}_\alpha$ such that $y_\alpha \in F_\alpha \subseteq U_\alpha \cap G_\alpha$.

If $U \in \mathcal{U}(\mu)\setminus\mathcal{U}(\mu)^{\#}$, apply 1.2 again to find a pairwise-disjoint open capped covering $\mathcal{P}_\alpha$ of X_α for each α satisfying $spt(U) \leq \alpha \leq \xi$. Define

$$\mathcal{V}_U = \{V \in \mathcal{B}: V_\alpha \in \mathcal{P}_\alpha \text{ if } spt(U) \leq \alpha \leq \xi, \text{ and } V_\alpha = U_\alpha \text{ otherwise}\}.$$

It is easy to see (1)-(8) are satisfied.

<u>Case</u> 2. ξ is a limit ordinal. From (1), (3), and (4) we observe that if $\mathcal{V} = \{\cap M: \cap M \neq \phi$ and $\mathfrak{M}$ is a maximal chain of $(\bigcup_{\nu<\xi} \mathcal{U}(\nu), \subseteq)\}$, then $V = \cap\{U: \exists \nu < \xi, U \in \mathcal{U}(\nu), V \subseteq U\}$ for each $V \in \mathcal{V}$. (1), (3), and (4) also imply $\mathcal{V}$ is a pairwise-disjoint cover of $\square$. If $\mathcal{V} \subseteq \mathcal{B}$, then we are done for then $\mathcal{U}(\xi) = \mathcal{V}$ easily satisfies (1)-(8). So we show $\mathcal{V} \subseteq \mathcal{B}$.

Suppose that $V \in \mathcal{V}$, and for each $\nu < \xi$ we have $U(\nu) \in \mathcal{U}(\nu)$ so that $V = \bigcap_{\nu<\xi} U(\nu) \in \mathcal{V}$. Then we have the inequality

(9). $$V = \bigcap_{\nu<\xi} \Pi_\alpha U(\nu)_\alpha = \Pi_\alpha(\bigcap_{\nu<\xi} U(\nu)_\alpha).$$

For each $\alpha \in \omega_1$, put $V_\alpha = \bigcap_{\nu<\xi} U(\nu)_\alpha$. Then $V \in \mathcal{B}$ if $|spt(V)| < \omega_1$ and if V_α is a clopen capped set of X_α for each $\alpha \in spt(V_\alpha)$.

Fix $\alpha \in \omega_1$. Then (2) and (3) show $U(\nu)_\alpha \subseteq U(\mu)_\alpha$ whenever $\mu < \nu < \xi$. So $\mu < \nu$ implies $rk(U(\nu)) \leq rk(U(\mu))$. Since every decreasing sequence of ordinals is finite, there is a $\nu_\alpha < \xi$ such that

$$rk(U(\nu)_\alpha) = rk(U(\nu_\alpha)_\alpha) \ \forall \nu, \nu_\alpha < \nu < \xi.$$

In particular, we have

(10). $$rk(V_\alpha) = rk(U(\nu_\alpha)_\alpha).$$

So if $V_\alpha \neq X_\alpha$, then V_α is a closed capped set of X_α. As X_α is a P-space, we see that (9) implies V_α is open if

(11). $\{U(\nu): \nu < \xi\}$ is countable.

Let α vary in ω_1. Note that (2) applied to the $U(\nu)$ shows $spt(V)$ is an ordinal and $spt(V) \le \xi$. Thus, $\xi < \omega_1$ implies (11) holds and hence $V \in \mathcal{B}$. So we assume $\xi = \omega_1$.

Choose an $x \in \Box$ such that $\{x_\alpha\} = t_p(V_\alpha)\ \forall\, \alpha \in spt(V)$. Then there is a $G \in \mathcal{G}$ such that $x \in G$. Let

$$\lambda = (\sup\{\nu_\alpha : \alpha \in spt(G)\}) + 1 .$$

Then (10) implies $\{x_\alpha\} = t_p(U(\lambda)_\alpha)$ whenever $\alpha < spt(G)$. So $U(\lambda) \in \mathcal{U}(\lambda)^{\#}$. From (6), $spt(U(\lambda + 1)) = spt(U(\lambda))$. Since $V \subseteq U(\lambda + 1)$, we use (10) to show $t_p(U(\lambda + 1)_\alpha) = \{x\}\ \forall\, \alpha \in spt(G)$. So (7) implies $U(\lambda + 1) \in \mathcal{U}(\lambda + 1)^*$. Therefore, $U(\nu) = U(\lambda + 1)$ whenever $\lambda < \nu < \omega_1$. Thus (11) holds for $\xi = \omega_1$. This completes our construction of $\mathcal{U}(\nu)\ \forall\, \nu \le \omega_1$.

According to (3) and (5), (11) implies that for each $U \in \mathcal{U}(\omega_1)$ there is a $\nu \in \omega_1$ such that $U \in \mathcal{U}(\nu)^*$. From (1), $\mathcal{U}(\omega_1)$ is the desired refinement of $\mathcal{G}$. //

The theorem 1.3 is new even for the countably many factors case that we presented at the 1983 Annual Winter Meetings of the American Mathematical Society. Independently, Rudin and Watson proved that the Tychonov product of countably many paracompact scattered spaces in paracompact [11]. The proof, given above, of 1.3 combines elements of both our earlier result and the theorem in [11].

Question 1: Suppose that X_α is a paracompact scattered space for each $\alpha \in \omega_1$. Is $< \omega_1 - \Box_\alpha X_\alpha$ paracompact?

Without the P-space assumption, or assuming locally compact factors (see section 3), nothing (even consistently) is known about Question 1.

§2. The countable support nabla lemma.

A most useful tool in studying box products with locally compact factors is the so-called nabla product ∇, which is not really a product.

2.1. DEFINITION. Suppose that κ is an infinite cardinal, $\mathcal{J}$ is an ideal on κ, and X_α is a set for each $\alpha \in \kappa$. On $\Pi_\alpha X_\alpha$ define a relation $\sim_{\mathcal{J}}$ by

$$x \sim_{\mathcal{J}} y \text{ iff } \{\alpha \in \kappa : x(\alpha) \ne y(\alpha)\} \in \mathcal{J} .$$

Let $\nabla_{\mathcal{J},\alpha} X_\alpha$ denote the set of equivalence classes of $\sim_{\mathcal{J}}$, and $q: \Pi_\alpha X_\alpha \to \nabla_{\mathcal{J},\alpha} X_\alpha$

the quotient map. If $\Pi_\alpha X_\alpha$ has a topology, then we give $\nabla_{\mathcal{J},\alpha} X_\alpha$ the quotient topology. For most of this paper $\mathcal{J}$ will consist of all finite subsets of $\varkappa$ and $\Pi_\alpha X_\alpha$ will be the space $< \omega_1 - \Box_\alpha X_\alpha$. In this case $\nabla_\alpha X_\alpha$ shall denote $\nabla_{\mathcal{J},\alpha} X_\alpha$.

2.2. <u>LEMMA</u> [8]. $\nabla_\alpha X_\alpha$ is a P-space and q is an open continuous map. //

2.3. <u>LEMMA</u>. Suppose that $\varkappa$ is a cardinal, and for each $\alpha \in \varkappa$ X_α is a σ-compact locally compact space. Suppose $x \in < \omega_1 - \Box_\alpha X_\alpha$ and $\mathcal{G}$ is an open cover of $q^{-1}q(x)$. Then there is an open box U, with countable support, neighborhood of x such that $q^{-1}q(U)$ is covered by a countable subfamily of $\mathcal{G}$.

PROOF. Let $\Box$ denote $< \omega_1 - \Box_{\alpha \in \varkappa} X_\alpha$. As each X_α is a σ-compact and locally compact, we may write $X_\alpha = \bigcup_{n \in \omega} X(\alpha,n)$, where $X(\alpha,n)$ is open in X_α and $X(\alpha,n)^-$ is a compact subset of $X(\alpha,n+1)$ for each $n \in \omega$. Let $F \in {}^\varkappa\omega$ be such that $x_\alpha \in X(\alpha,f(\alpha))$ $\forall\, \alpha \in \varkappa$. Let $\mathcal{B}$ be the set of all open boxes $B = \Pi_\alpha B_\alpha$ such that $|spt(B)| \leq \omega$. Without loss of generality, we may assume $\mathcal{G} \subseteq \mathcal{B}$.

There are $G(0) \in \mathcal{G}$ and $B(0) \in \mathcal{B}$ such that for each $\alpha \in spt(G(0))$,

$$x_\alpha \in B(0)_\alpha \subseteq G(0)_\alpha \cap X(\alpha,f(\alpha)) .$$

Put $S(0) = \phi$, $\mathcal{G}(0) = \{G(0)\}$, and let $\{\beta(0,i)\colon i \in \omega\}$ be an indexing of $spt(B(0))$. We construct, inductively, for each positive $n \in \omega$, a finite family $\mathcal{G}(n) \subseteq \mathcal{G}$, a neighborhood $B(n) \in \mathcal{B}$ of x, an indexing $\{\beta(n,i)\colon i \in \omega\}$ of $spt(B(n))$, and a finite set $S(n) \subseteq \varkappa$, all subject to the restrictions (1), (2), and (3) below:

(1). $x \in B(n) \subseteq B(n-1)$.

(2). $S(n) = \{\beta(m,i)\colon m,i < n\}$.

(3). $(\Pi_{\alpha \in S(n)} X(\alpha,f(\alpha)+n)) \times (\Pi_{\alpha \in \varkappa \setminus S(n)} B(n)_\alpha) \subseteq \bigcup \mathcal{G}(n)$.

Given the construction $\forall n < k \in \omega$. We construct $\mathcal{G}(k)$, $B(k)$, the set $\{\beta(k,i)\colon i \in \omega\}$, and $S(k)$ as follows:

Put $S(k) = \{\beta(m,i)\colon m,i < k\}$. Since

$$(\Pi_{\alpha \in S(k)} X(\alpha,f(\alpha)+k)^-) \times \Pi_{\alpha \in \varkappa \setminus S(k)} \{x_\alpha\}$$

is a compact subset of $\Box$, it is covered by a finite family $\mathcal{G}(k) \subseteq \mathcal{G}$. Let $x \in H \in \mathcal{G}(k)$ and define $B(k)$ by

$$B(k)_\alpha = B(k-1)_\alpha \cap \begin{cases} H_\alpha, & \text{if } \alpha \in S(k) \\ \cap\{G_\alpha : G \in \mathcal{G}(k)\}, & \text{if } \alpha \in \kappa \setminus S(k). \end{cases}$$

Arbitrarily index $spt(B(k))$ by $\{\beta(k,i): i \in \omega\}$. It is easy to see that (1), (2), and (3) hold, so we consider our construction complete for each $n \in \omega$.

Let $S = \bigcup_{n \in \omega} S(n)$. Define $U \in \mathcal{B}$ by

$$U_\alpha = \begin{cases} B(n)_\alpha, & \text{if } n \text{ is the first integer such that } \alpha \in S(n) \\ X_\alpha, & \text{if } \alpha \notin S \end{cases}$$

Then $U \in \mathcal{B}$, and $spt(U) \subseteq S$. Obviously, (1) implies $x \in U$.

To complete our proof, we show that for each finite set $A \subseteq \kappa$,

(4). $(\Pi_{\alpha \in A} X_\alpha) \times (\Pi_{\alpha \in \kappa \setminus A} U_\alpha) \subseteq \{G: \exists n, G \in \mathcal{G}(n)\}$.

So suppose A is a finite subset of κ and $y \in (\Pi_{\alpha \in A} X_\alpha) \times (\Pi_{\alpha \in \kappa \setminus A} U_\alpha)$.

(5). $A \cap spt(U) \subseteq S(k)$, and

(6). $y_\alpha \in X(\alpha,k)\ \forall \alpha \in A$

are true. We claim

(7). $y \in (\Pi_{\alpha \in S(k)} X(\alpha, f(\alpha) + k)) \times (\Pi_{\alpha \in \kappa \setminus S(k)} B(k)_\alpha)$

is also true. In fact, if we apply (3) to (7), we see that $y \in \bigcup \mathcal{G}(k)$, and, as y is arbitrary, this proves (4). In order to prove (7) we consider three cases.

Case 1. $\alpha \notin S$: From (2), $spt(B(k)) \subseteq S$. Hence, $y_\alpha \in X_\alpha = B(k)_\alpha$.

Case 2. $\alpha \in S \setminus S(k)$: From (5), $\alpha \notin U_\alpha$. Find the first n such that $\alpha \in S(n)$. Then $U_\alpha = B(n)_\alpha$. Since $\alpha \in S(n) \setminus S(k)$, (2) implies $k < n$. From (1), $y_\alpha \in B(n)_\alpha \subseteq B(k)_\alpha$.

Case 3. $\alpha \in S(k)$: From (6), $y_\alpha \in X(\alpha,k) \subseteq X(\alpha, f(\alpha)+k)$. //

The following was established for compact factors by Kunen [5] with $\kappa = \omega$ and Miller [9] with $\kappa = \omega_1$, and for σ-compact locally compact factors by van Douwen [3] with $\kappa = \omega$.

2.4. COROLLARY. Suppose that X_α is a σ-compact locally compact space for each $\alpha \in \kappa$. Then (1). $q: < \omega_1 - \square_\alpha X_\alpha \to \nabla_\alpha X_\alpha$ is a closed map, and

(2). $q^{-1}q(x)$ is Lindelof for each $x \in \Pi_\alpha X_\alpha$.

Proof: (2) is obvious. For (1) use the standard characterization [4] of closed maps: for each x and each neighborhood G of $q^{-1}q(x)$, there is a neighborhood V of $q(x)$ such that $q^{-1}(V) \subseteq G$. In 2.3 take $\mathcal{G} = \{G\}$ and $V = q(U)$. //

2.5. THE NABLA LEMMA. Suppose that Y_α is a paracompact locally compact space for each $\alpha \in \omega_1$. Then the following are equivalent:

(1). $<\omega_1 - \square_\alpha Y_\alpha$ is paracompact.

(2). $<\omega_1 - \square_\alpha X_\alpha$ is paracompact whenever X_α is a closed σ-compact locally compact subspace of Y_α for each $\alpha \in \omega_1$.

(3). $\nabla_\alpha X_\alpha$ is paracompact whenever X_α is a closed σ-compact locally compact subspace of Y_α for each $\alpha \in \omega_1$.

PROOF. (1) $\Rightarrow$ (2): This is clear since $<\omega_1 - \square_\alpha X_\alpha$ is homeomorphic to the closed subset $\Pi_\alpha X_\alpha$ of $<\omega_1 - \square_\alpha Y_\alpha$.

(2) $\Leftrightarrow$ (3): This is an immediate consequence of 2.4 and E. Michael's theorem [8, Corollary 1] on image and pre-image preservation of paracompactness.

(2) $\Rightarrow$ (1): As a paracompact locally compact space is the topological sum of paracompact spaces, we can write each $Y_\alpha = \Sigma \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha$ is a family of σ-compact locally compact spaces. Suppose $\mathcal{G}$ is a basic open covering of $\square = <\omega_1 - \square_\alpha Y_\alpha$. Given $x \in F = \Pi_\alpha F_\alpha$ with $x_\alpha \in F_\alpha \in \mathcal{F}_\alpha$ for each $\alpha \in \varkappa$, we apply 2.4(2) to choose a countable family $\mathcal{G}(x) \subseteq \mathcal{G}$ such that

(i). $q^{-1}q(x) \cap F \subseteq \cup\mathcal{G}(x)$.

Putting $\sigma = \cup\{\mathrm{spt}(G): G \in \mathcal{G}(x)\}$, we see (i) implies $F(x) \subseteq \cup\mathcal{G}(x)$, where $F(x) = (\Pi_{\alpha \in \sigma} F_\alpha) \times (\Pi_{\alpha \in \omega_1 \setminus \sigma} X_\alpha)$.

Now give each set $\mathcal{F}_\alpha$ the discrete topology, and for each $x \in \square$, let $R(x) = \Pi_\alpha R(x)_\alpha$, where

$$R(x)_\alpha = \begin{cases} \{F(x)_\alpha\}, & \text{if } \alpha \in \mathrm{spt}(F(x)) \\ \mathcal{F}_\alpha, & \text{if } \alpha \notin \mathrm{spt}(F(x)) . \end{cases}$$

Then $\mathcal{R} = \{R(x): x \in X\}$ is an open covering of $<\omega_1 - \square_{\alpha \in \omega_1} \mathcal{F}_\alpha$. From the proof of 1.3, we may find a pairwise-disjoint open refinement $\mathcal{S}$ of $\mathcal{R}$ such that if $S \in \mathcal{S}$ and if $\alpha \in \mathrm{spt}(S)$, then $|S_\alpha| = 1$. For each $S \in \mathcal{S}$ choose one $x_S \in \square$

such that $S \subseteq R(x_S)$, and let $U(S) = \Pi_\alpha U(S)_\alpha$ be defined by

$$U(S)_\alpha = \begin{cases} F_\alpha, & \text{if } \alpha \in \operatorname{spt}(S) \text{ and } S = \{F_\alpha\} \\ X_\alpha, & \text{if } \alpha \notin \operatorname{spt}(S). \end{cases}$$

Clearly $\{G \cap U(S): G \in \mathcal{G}(x_S),\ S \in \mathcal{S}\}$ is a σ-locally finite refinement of $\mathcal{G}$. //

In reality 2.5 is true if we replace $<\omega_1 - \square_{\alpha \in \omega_1} Y_\alpha$ by $<\omega_1 - \square_{\alpha \in \varkappa} Y_\alpha$ for any infinite $\varkappa$. However, in case $\varkappa > \omega_1$ the replacement is useless, because if X_α contains at least two points $\forall \alpha \in \omega_2$, then $<\omega_1 - \square_{\alpha \in \omega_2} X_\alpha$ is not normal (see [1] and [2]).

§3. Consistently paracompact.

In this section we will extend to $<\omega_1 - \square_{\alpha \in \omega_1} X_\alpha$. a theorem of Kunen [5] known for $\square_{\alpha \in \omega} X_\alpha$.

3.1. THEOREM. [CH] If X_α is a paracompact locally compact scattered space for each $\alpha \in \omega_1$, then $- <\omega_1 - \square_\alpha X_\alpha$ is paracompact.

Before proving 3.1 we need several definitions and lemmas. Recall that the Lindelof degree, $L(X)$, of a space X is the least infinite cardinal $\varkappa$ such that each open covering of X has a subcovering of cardinality at most $\varkappa$. The G_δ-topology of a space (X,τ) is the topology τ' on X whose base is the set of all G_δ-sets of (X,τ). When X denotes (X,τ), then we let X_δ denote (X,τ'). Of course, X_δ is always a P-space.

3.2. LEMMA [7]. If X is a scattered space, then $L(X) = L(X_\delta)$. //

The conclusion for the next lemma was obtained in [5] for countably many compact scattered factors case.

3.3. LEMMA. If X_α is a Lindelof scattered space for each $\alpha \in \omega_1$, then $L(<\omega_1 - \square_\alpha X_\alpha) \leq \underline{c}$.

PROOF. Obviously $L(<\omega_1 - \square_\alpha X_\alpha) \leq L(<\omega_1 - \square_\alpha (X_\alpha)_\delta)$. According to 3.2 each $(X_\alpha)_\delta$ is a Lindelof scattered P-space. So, without loss of generality, we may assume each X_α is a P-space. Now the proof is similar to the proof of 1.3. Just add a new condition

$(1\frac{1}{2})$. $|\mathcal{U}(\nu)| \le \underline{\underline{c}}$

to the recursion hypothesis. In case 1, observe that for each $\alpha \in \text{spt}(U)$ we may take $|\mathcal{F}_\alpha| \le \omega$. Since $\omega \cdot \underline{\underline{c}} = \underline{\underline{c}}$, $(1\frac{1}{2})$ holds. In case 2, we have by recursion hypothesis $|\mathcal{U}(\xi)| \le \underline{\underline{c}}^\omega = \underline{\underline{c}}$ whenever $\xi < \omega_1$. Obviously, the last paragraph of 1.3 shows that the pairwise-disjoint refinement $\mathcal{U}(\omega_1)$ has cardinality at most $\omega_1 \cdot \underline{\underline{c}} = \underline{\underline{c}}$. //

PROOF of 3.1. According to 2.5, we need only prove that $\nabla_\alpha X_\alpha$ is paracompact whenever X_α is a σ-compact locally compact scattered space for each $\alpha \in \kappa$. From 3.3, $L(<\omega_1 - \square_\alpha X_\alpha) \le \underline{\underline{c}} = \omega_1$. From 2.2 $\nabla_\alpha X_\alpha$ is a P-space with $L(\nabla_\alpha X_\alpha) \le \omega_1$, because q is continuous. It is well known and easy to prove (see [11]), that a P-space is paracompact if its Lindelof degree is at most ω_1. So $\nabla_\alpha X_\alpha$ is paracompact. //

Question 2: Suppose that ${}^\omega\omega$ is given the order topology. Is it consistent with $\omega_1 < c$ that $<\omega_1 - \square^{\omega_1}({}^\omega\omega)$ is normal?

3.4 REMARKS. Miller [9] proved that CH implies $<\omega_1 - \square_{\alpha \in \omega_1} X_\alpha$ is paracompact whenever each X_α is a compact metrizable space. However, a stronger result can be found using the methods of [13]: The axiom $\underline{\underline{d}} = \omega_1$ implies $<\omega_1 - \square_{\alpha \in \omega_1} X_\alpha$ is paracompact whenever each X_α is a σ-compact locally compact space of weight at most ω_1.

Question 3. Suppose that $\Pi^{\omega_1}[0,1]$ is given the usual Tychonov product topology. Is it consistent with $\omega_1 < \underline{\underline{d}}$ that $<\omega_1 - \square^{\omega_1}(\Pi^{\omega_1}[0,1])$ is paracompact?

One might hope that forcing with the Cohen partial order FN $(\kappa,2)$ (see [6]) might yield, for $cf(\kappa) > \omega$, a positive solution to our question 3. Indeed, each basic open set of $<\omega_1 - \square^{\omega_1}(\Pi^{\omega_1}[0,1])$ can be considered as defined at some stage $\lambda \in \kappa$. However, the criterion devised by Roitman [10], for proving the paracompactness of the nabla product, do not appear to be satisfied.

§4. A non-paracompact box product

Kunen, applying a result due to Ar'hangelski, has observed an important relationship between paracompactness and the Lindelof degree.

4.1 LEMMA [5]. Suppose that for each $n \in \omega$, X_n is a compact space. If $\square_n X_n$ is paracompact, then $L(\square_n X_n) \le \underline{\underline{c}}$. //

Recall that a space X is called <u>orderable</u> provided its topology is the interval topology induced by a linear ordering of the set X. Let

$$X^* = \{x \in X\colon x \text{ has an immediate successor}\} \cup \{\sup X\}, \text{ and}$$

$$X_* = \{x \in X\colon x \text{ has an immediate predecessor}\} \cup \{\inf X\}.$$

All previous examples of non-paracompact box products of compact spaces (see [13]) use the same technique: Find a compact space K such that its G_δ-modification K_δ (see section 3) is not normal, then observe that K_δ embeds as a closed subspace of $\square^\omega K$. However, if K is a compact orderable space, then X_δ is paracompact and $L(X_\delta) \le \underline{\underline{c}}$ for each finite power X of K [14]. Therefore, the following represents a new method for obtaining non-paracompact box products.

4.2. EXAMPLE. <u>There is a compact orderable space T such that $\square^\omega T$ is not paracompact</u>.

PROOF: Let $K = {}^{\underline{\underline{c}}}2$ be ordered lexicographically; i.e., by

$$f < g \text{ iff } f(\alpha) < g(\alpha), \text{ where } \alpha = \inf\{\zeta \in \underline{\underline{c}}\colon f(\zeta) \ne g(\zeta)\}.$$

Note that $K^* \cap K_* = \phi$. It is well-known that there is a family $\mathcal{A} = \{A_\alpha\colon \alpha \in \underline{\underline{c}}\}$ of subsets of ω satisfying

(1). for each distinct pair $\mathcal{A}_0, \mathcal{A}_1$ of finite subsets of $\mathcal{A}$ we have $\bigcap \mathcal{A}_0 \setminus \bigcup \mathcal{A}_1 \ne \phi$.

($\mathcal{A}$ is called an independent family of sets [4] and [6]). For each $f \in K$, define $\hat{f} \in {}^\omega K$ by

$$\hat{f}(n)(\alpha) = \begin{cases} f(\alpha) & \text{if } n \in A_\alpha \\ 1 - f(\alpha) & \text{if } n \notin A_\alpha \end{cases}$$

for each $\alpha \in \underline{\underline{c}}$.

<u>CLAIM</u> 1. Suppose that $f,g \in K$, $f < g$, and $\alpha = \inf\{\zeta \in \underline{\underline{c}}\colon f(\zeta) = g(\zeta)\}$.

Then the following are true:

Proof of claim 1.

(2). $\hat{f}(n) < \hat{g}(n)$ if $n \in A_\alpha$, and $\hat{g}(n) < \hat{f}(n)$ if $n \notin A_\alpha$.

(3). $F \cap \Pi_n[\inf\{\hat{f}(n),\hat{g}(n)\}, \sup\{\hat{f}(n),\hat{g}(n)\}] = \{\hat{f},\hat{g}\}$.

The statement (2) is a consequence of the obvious $\hat{f}(n)(\zeta) = \hat{g}(n)(\zeta)$ if $\zeta < \alpha$. To see (3) suppose $h \in K\backslash\{f,g\}$, $\beta = \inf\{\zeta: f(\zeta) \neq h(\zeta)\}$, and $\gamma = \inf\{\zeta: g(z) \neq h(\zeta)\}$. We consider two cases:

Case 1. $h < f$ (or similarly, $g < h$): If $h < f$, then (1) finds an $n \in A_\alpha \cap A_\beta \cap A_\gamma$. For this n (2) implies $\hat{h}(n) < \hat{f}(n) < \hat{g}(n)$. Therefore, $\hat{h}$ is not between $\hat{f}$ and $\hat{g}$.

Case 2. $f < h < g$: Since $\alpha = \beta = \gamma$ is impossible, either $\alpha \neq \beta$ or $\alpha \neq \gamma$. If $\alpha \neq \gamma$, then $f < h < g$ implies $\alpha < \beta$. Now (1) finds an $n \in A_\alpha \cap A_\beta \backslash A_\gamma$. For this n, (2) implies $\hat{f}(n) < \hat{g}(n) < \hat{h}(n)$. So $\hat{h} \notin [\hat{f}(n), \hat{g}(n)]$. The argument is similar when $\alpha \neq \gamma$. Thus, case 2, and hence the claim is proved.

Define $T = ((K\backslash(K^* \cup K_*) \times \{0,1\}) \cup ((K^* \cup K_*) \times \{0\})$, let T be lexicographically ordered (so $T = T^* \cup T_*$ and $T^* \cap T_* = \phi$), and give T the interval topology. For each $f \in K$ define $f^{\#} \in {}^{\omega}T$ by $f^{\#}(n) = \langle \hat{f}(n), 0\rangle$. Then we have

(4). for each $f,g \in K$ and each $n \in \omega$, $f^{\#}(n) < g^{\#}(n)$ iff $\hat{f}(n) < \hat{g}(n)$.

Define $C = \{f^{\#} : f \in K\backslash(K^* \cup K_*)\}$.

CLAIM 2. C is closed and discrete in $\square^{\omega}T$.

Proof of claim 2. We suppose, by way of contradiction $x \in \square^{\omega}T$ is a limit point of C. Define an open box neighborhood $U_0 = \Pi_n U_0(n)$ of x by

$$U_0(n) = \begin{cases} [\inf T,\ x(n)] & \text{if } x(n) \in T^* \\ [x(n),\ \sup T] & \text{if } x(n) \in T_*\ . \end{cases}$$

Since U_0 is an open neighborhood of x, there is a $c_0 \in U_0 \cap C$. Now construct, inductively, for each positive $i \in \omega$, an open box neighborhood $U_i = \Pi_n U_i(n)$ of x and a point $c_i \in {}^{\omega}T$ subject to the three conditions:

$$\text{(5).}\quad U_i(n) = \begin{cases} [x(n),\ \sup T] & \text{if } x(n) \in T_* \\ [\inf T,\ x(n)] & \text{if } x(n) \in T^*\backslash\{c_j(n): 0 < j < i\}\ . \\ (c_{j-1},\ x(n)] & \text{if } x(n) = c_j(n) \text{ and } 0 < j < i\ . \end{cases}$$

(6). $c_i \in U_i \cap C$

(7). if $c \in U_i \cap C$, if $m \in \omega$, and if $c(m) = x(m)$, then there is a $j \leq i$ such that $c_j = c$.

Now (5) can be reached because $c(n) \in T^* \ \forall n \in \omega \ \forall c \in C$. According to (2), if $c, c' \in C$ and if $c(m) = c'(m)$, then $c = c'$. So (7) can be achieved. So we assume U_i and c_i are constructed $\forall i \in \omega$.

Define $V = \Pi_n V(n) \subseteq \square^{\omega} T$ by

$$V(n) = \begin{cases} [x(n), \sup T] & \text{if } x(n) \in T_* \\ [\inf T, x(n)] & \text{if } x(n) \in T^* \setminus \{c_i(n) : 0 < i \in \omega\} \\ \bigcap_{j \le i} (c_j(n), x(n)] & \text{if } x(n) = c_{i+1}(n) \text{ and } i \in \omega . \end{cases}$$

Then V is an open neighborhood of x and $V \subseteq \bigcap_{i \in \omega} U_i$. We consider two cases:

Case 3. $\exists c \in C \cap V$ and $m \in \omega$ with $c(m) = x(m)$: From (7) there is a $k \in \omega$ with $c_k = c$. Since $c \in U_k$, (5) shows $k = 0$. Since $c \in V$, $V = U_0$. Let $d_1 = c_0$ and define $d_0 \in {}^{\omega}T$ by

$$d_0(n) = \begin{cases} \sup T & \text{if } x(n) \in T_* \\ \inf T & \text{if } x(n) \in T^* . \end{cases}$$

Since x is a limit point of C, we may define inductively $d_i \in C \cap V$ for $i \in \{2,3,4\}$ by

$$(8). \quad d_i(n) \in \begin{cases} [x(n), d_{i-1}(n)) & \text{if } x(n) \in T_* \\ (d_{i-1}(n), x(n)] & \text{if } x(n) \in T^* \setminus \{d_{i-1}(n)\} \\ (d_{i-2}(n), x(n)] & \text{if } x(n) = d_{i-1} . \end{cases}$$

Now (2) implies $d_1(n)$, $d_2(n)$, $d_3(n)$, and $d_4(n)$ are pairwise-distinct $\forall n \in \omega$. Since $c_0 \neq d_i \in V \ \forall i > 1$, $x(n) \neq d_i(n) \ \forall n \in \omega \ \forall i > 1$. So we have

(9). $x(n) \in T_*$ implies $d_4(n) < d_3(n) < d_2(n) < d_1(n)$.

(10). $x(n) \in T_* \setminus \{d_1(n)\}$ implies $d_1(n) < d_2(n) < d_3(n) < d_4(n)$.

(11). $x(n) = d_1(n)$ implies $d_2(n) < d_3(n) < d_4(n) < d_1(n)$.

Therefore, $d_3 \in \Pi_n[\inf\{d_2(n), d_4(n)\}, \sup\{d_2(n), d_4(n)\}]$ which contradicts (3).

Case 4. $c \in C \cap V$ implies $x(n) \neq c(n) \ \forall n \in \omega$. Choose $d_2 \in C \cap V$ randomly, and define d_i for $i \in \{3,4\}$ as in (8). The argument of the previous paragraph (neglecting d_1 of course) works here to achieve a contradiction. Thus, case 4, and hence claim 2, is true.

According to claim 2, it is sufficient to show $|C| = 2^{\underline{c}}$. But this follows since (4) shows $|C| = |K\setminus(K^* \cup K_*)|$. However, $K^* \cup K_*$ is exactly $\{f \in {}^{\underline{\underline{c}}}2 : f$ is constant on a tail$\}$. Therefore, $|K\setminus(K^* \cup K_*)| = 2^{\underline{\underline{c}}}$. So $2^{\underline{\underline{c}}} \leq |L(\square^\omega T)| \leq |\square^\omega T| = 2^{\underline{\underline{c}}}$. //

4.3. <u>COROLLARY</u>: There is a compact orderable space T such that $\square^\omega T$ is not paracompact.

PROOF. Apply 4.1 to 4.2. //

4.4. <u>COROLLARY</u> $[2^{\underline{\underline{c}}} = \underline{\underline{c}}]$: There is a compact orderable space T such that $\square^\omega T$ is not normal.

PROOF. In the example 4.2, $|K^* \cup K_*| = 2^{\underline{\underline{c}}}$ and $K^* \cup K_*$ is dense in K. Therefore, $\square^\omega T$ has a dense set of cardinal $c^\omega = c$. Since $\square^\omega T$ has a closed discrete subset of cardinal $2^{\underline{\underline{c}}}$, the F. B. Jones lemma (see [4]) shows $\square^\omega T$ is not normal. //

<u>Question</u> 3: Suppose X is a compact space. Does X_δ normal imply $L(X_\delta) \leq \underline{\underline{c}}$? [Observe that $L(\square^\omega X) = L(\square^\omega(X_\delta))$].

4.5. <u>Remarks</u>: Independently, Kunen has shown that the axiom in 4.4 is unnecessary. Specifically, he proved $\square^\omega T$ is not normal when T is the lexicographic ordered product ${}^{\omega_1+1}2$.

§5. <u>On the full box product</u>

At present, we know of no answers even consistent with ZFC to the question, "Is $\square^\varkappa \omega+1$ paracompact?" for any uncountable cardinal $\varkappa$. An affirmative answer for some full box product $\square_\alpha X_\alpha$ of compact spaces would seem to require an intermediate object like the nabla products $\nabla_{\mathcal{J},\alpha} X_\alpha$ of section 2. By "intermediate" we mean that $\nabla_\mathcal{J}$ can be shown paracompact under various hypothesis on the X_α, and that $\nabla_\mathcal{J}$ paracompact ought to imply $\square$ is paracompact.

If the ideal $\mathcal{J}$ contains an uncountable set, then the point-inverses of the map $q:\nabla_\mathcal{J} \to \square$ are closed and contain closed homeomorphs of an uncountable box product of the spaces X_α. Therefore, $\mathcal{J}$ should contain no uncountable sets.

Even in the case $\mathcal{J}$ contains an infinite set, there are difficulties. For $\mathcal{J} = [\omega_1]^{\leq\omega}$, van Douwen has offered in [3] the following reasons why $\nabla_\mathcal{J}$ is useless for our purposes: 1). $\nabla_\mathcal{J}$ is rarely a P-space; 2). The map q is rarely closed and its point-inverses are not Lindelof (so Michael's theorem can not be applied as we did in 2.5); 3). $[2^{\omega_1} = \omega_2]$ There is a compact space K such that $\nabla_{\mathcal{J}}^{\omega_1} K$ is paracompact but $\square^{\omega_1} K$ is not normal.

It is clear that $\mathcal{J} = [\kappa]^{<\omega}$ is likely our only hope - this is what we have studied recently. Although the topology on $\nabla_{\mathcal{J},\alpha} X_\alpha$ is finer than that considered in section 2, we risk confusion by using the same symbol $\nabla_\alpha X_\alpha$ to denote it. Again $\nabla_\alpha X_\alpha$ is rarely a P-space and so paracompactness is not easily proved; however, we have shown (proofs will appear elsewhere)

5.1. PROPOSITION. [MA] If $\kappa < c$ and if X_α is a compact first countable space for each $\alpha \in \omega_1$, then $\nabla_\alpha X_\alpha$ is paracompact. //

The quotient map q is probably not closed in this case as well, and certainly point-inverses are not Lindelof. However, these results are not needed to show that ∇ paracompact implies $\square$ is paracompact. What is needed is the following statement:

(#). If $\mathcal{G}$ is an open covering of $\square$, then for each $x \in \square$ there is an open neighborhood of H_x of $q(x)$ in ∇ and an open locally-finite family $\mathcal{S}_x$ covering $q(x)^{-1}$ in $\square$ such that $\mathcal{S}_x$ refines $\mathcal{G}|q^{-1}(H_x)$.

I cannot yet prove #, but I do have an approximation to it.

5.2. PROPOSITION. Suppose that for each $\alpha \in \kappa$ X_α is a compact space. Then $q(x)^{-1}$ is paracompact for each $x \in \square_\alpha X_\alpha$. //

Our proof of 5.2 shows various kinds of disjoint closed sets of $\square$ can be separated.

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