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Conjugation in spinor spaces Majorana and Weyl spinors

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CONJUGATION IN SPINOR SPACES
MAJORANA AND WEYL SPINORS.

by A. Crumeyrolle.

Preamble.

This subject is a relatively old subject, but according my experience there is any systematic and rigourous development of these topics, particulary in dimension different of 4. Usually these notions are derived from the Pauli-Dirac tricks and are presented in "patch-work" form. I intend to expose a complete and (I hope) satisfying solution using the modern approach in spinor theories initiated by Chevalley in the famous book "The Algebraic theory of spinors" [2] and some results of my own tracts and papers [3].

1. Conjugations and group representations.

Γ is a group, (W,ρ) a complex representation space

$$\rho : (g,x) \in \Gamma \ltimes W \longrightarrow g.x \in W,$$

W is called a Γ-module.

Definition 1.

A conjugation J is a Γ-module homomorphism :

$J(g.x) = g.J(x) \qquad , g \in \Gamma, x \in W$

$J(x_1+x_2) = J(x_1)+J(x_2), \ x_1, x_2 \in W,$

such that :

(I) $\underline{J(\alpha x) = \bar{\alpha}J(x)}$, $\alpha \in \mathbb{C}$,

(II) $\underline{J^2 = \pm Id}$.

One immediatly sees that the product of two conjugations is a linear isomorphism.

If $J^2 = -Id$, W posseses a quaternionic structure (vector space over $\mathbb{H}$) (quaternionic case).

If $J^2 = Id$, we can associate with (W,ρ) two real equivalent representations (real case).

The precedent results are well known.

2. Recalls about the spinors [2,3]

Consider first, a real vector space with $n = 2r$ dimensions - the even dimensions are the most important for geometrical and physical applications - and construct the Clifford algebra $C(Q)$ of (E,Q) provided with a non degenerated quadratic form with (p,q) signature, B is the bilinear symmetrie associated form ; we can suppose first $p \le q$.

(E',Q') is the complexified space of (E,Q).

Let $(e_1, e_2 \ldots e_n)$ an orthonormed frame of (E,Q) with $(e_i)^2 = 1$, $i \le p$ and $(e_i)^2 = -1$, $i > p$.

We construct a Witt decomposition $E' = F \oplus F'$,

$$F = (x_1 = \frac{e_1 + e_n}{2}, \ldots, x_p = \frac{e_p + e_{n-p+1}}{2},\ x_{p+1} = \frac{ie_{p+1} + e_{n-p}}{2}, \ldots, x_r = \frac{ie_r + e_{n-r+1}}{2}$$

$$F' = (y_1 = \frac{e_1 - e_n}{2}, \ldots, y_p = \frac{e_p - e_{n-p+1}}{2},\ y_{p+1} = \frac{ie_{p+1} - e_{n-p}}{2}, \ldots, y_r = \frac{ie_r - e_{n-r+1}}{2}$$

$$B(x_i, y_j) = \frac{1}{2}\delta_{ij},\ B(x_i, x_j) = B(y_i, y_j) = 0.$$

$\{x_i, y_j\}$ constitutes a "real" Witt frame.

Note : $x_i y_j + y_j x_i = \delta_{ij}$.

We put : $y_1 y_2 \ldots y_r = f$, and call f an isotropic r-vector ; classically, $C(Q')f$ is a standard spinor space, viz. a minimal left ideal in $C(Q')$. If f_1 is another isotropic r-vector, and $f_1\, C(Q')$ the corresponding right ideal, any element in $f_1\, C(Q')$ and $C(Q')f$ is called a pure spinor.

Fixing the standard spinor space, any maximal totally isotropic subspace (m.t.i.s.) characterizes, modulo a scalar factor, a pure spinor [1,3b]. For example $x_1 x_2 \ldots x_r f$ is a pure spinor.

In the complexified Clifford algebra $C(Q')$ there exists a complex-conjugation, it is a semi-linear isomorphism commuting with the main involution α and with the main anti-involution β.

3. The pure spinor of conjugation γf [3].

Consider again an isotropic r-vector f,

$f = x_{1^*} x_{2^*} \ldots x_{r^*}$, where $x_{1^*}, x_{2^*}, \ldots, x_{r^*}$ are r linearly independant isotropic vectors. According the classical Witt theorem there exists $\gamma \in \mathrm{Pin}\, Q'$ sending every x_{α^*} over $\bar{x}_{\alpha^*}$, $\alpha = 1, 2, \ldots r$,

$$\bar{x}_{\alpha^*} = \gamma x_{\alpha^*} \gamma^{-1} ;$$

then we have : $\gamma f = \bar{f}\gamma$, and γf is a pure spinor, we can suppose $\gamma \in G_o(Q')$, i.e. $N(\gamma) = 1$, where N is the spinorial norm. Neverthless we can choose several elements in $G_o(Q')$ sending also f over $\bar{f}$, if

γ' is a such one element :

$$\bar{f} = \gamma' f \gamma'^{-1}$$
$$f = g f g^{-1}, \text{ with } g = \gamma'^{-1} \gamma.$$

We give now a very important lemma :

Lemma 1 :

If g belongs to the Clifford group of C(E,Q) dim E = 2r, with neutral form Q, and if :

$$g f g^{-1} = \lambda f, \quad f \text{ isotropic r-vector,}$$

λ scalar, $\lambda \neq 0$, there exists a scalar μ such that $gf = \mu f$ and conversely ; moreover $\lambda = \mu^2 N(g)$.

We remark that $gf = \lambda fg$ is a pure spinor μf, applying β, we obtain :

$$fg^{-1} N(g) = \mu f$$
$$\lambda f = \mu N(g) gf$$
$$gf = \mu f, \lambda = \mu^2 N(g).$$

Reciprocally, if $gf = \mu f$, applying β, we get :

$$gfg^{-1} = \mu^2 N(g)f.$$

Coming back to $f = gfg^{-1}$, $g \in G_o(Q') = G'_o$, we have $gf = \pm f$. Then g belongs to the subgroup H' of G'_o with elements g such that $gf = \pm f$, finally $\gamma' = \gamma\tau$, $\tau \in H'$.

We define now $\rho = \bar{\gamma}\gamma$, and from the lemma 1, we obtain : $\rho f = \varepsilon' f$, $\varepsilon' = \pm 1$.

Lemma 2 :

The scalar ε' such that $\rho f = \bar{\gamma}\gamma f = \varepsilon' f$, with

$\bar{f} = \gamma f \gamma^{-1}$, $\gamma \in G'_o$, does not depend :

neither of the choice of $\gamma \in G'_o$ such that $\gamma f \gamma^{-1} = \bar{f}$;

neither of the factorisation of f in product of r isotropic vectors ;

neither of the choice of the isotropic r-vector f.

The proof is easy.

The lemma 2 displays some geometric meaning for ε', also we develop now several lemmas and propositions leading to the checking of ε'. We gave these results in the paper [3,a] and the tract [3,b] where the reader can find the proofs.

Lemma 3 :

If γf is a pure spinor defining $\bar{F}'$, $\gamma \in \text{Pin } Q'$, x belongs to $\bar{F}'$, if and only if $x\gamma f = 0$, $\forall x \in \bar{F}'$. If $\gamma \in \text{Pin } Q'$ and if $x\gamma f = 0$, $\forall x \in \bar{F}'$, then γf is a pure spinor defining $\bar{F}'$.

Lemma 4 :

If B is the real, symmetric, bilinear form associated to Q, and if $B(y,\overline{y}) = 0$, $\forall\, y \in F'$, then $B(\overline{y},x) = 0$, $\forall\, x \in F'$ and $\overline{F}' = F'$.

Lemma 4 :

If $B(y,\overline{y}) \neq 0$, $\forall\, y \in F'$, $y \neq 0$, there exists a Witt frame

$(x_1,x_2,\ldots x_r,\ y_1,y_2,\ldots y_r)$ such that :

$$y_i = \delta\overline{x}_i,\ \delta = \pm 1,\ i = 1,2,\ldots r,$$

$F' \cap \overline{F}' = 0$ and δ does not depend of i.

Proposition 1.

Let F' be a m.t.i.s. of $E_{\mathbb{C}} = E'$, with $\dim(F' \cap \overline{F}') = r-h$.

1^{st}). We can construct a Witt frame :

$$(x_1,x_2,\ldots,x_r,\ y_1,y_2,\ldots,y_r)$$

such that the r isotropic vectors $y_1,y_2,\ldots y_r$ generate F', $y_{h+1},\ y_{h+2},\ldots y_r$ constitute a frame of $F' \cap \overline{F}'$ and

$$y_1 = \delta\overline{x}_1,\ldots,y_h = \delta\overline{x}_h,\ \delta = \pm 1.$$

2^{nd}). If γf is a pure spinor representing $\overline{F}'$, we can choose, modulo a scalar factor :

$$\gamma = (x_1+y_1)\ (x_2+y_2)\ldots(x_h+y_h),$$

$$\gamma f = x_1x_2\ldots x_h f,\ f = y_1y_2\ldots y_r\,.$$

Lemma 5 :

If W is a complex vector subspace ($\dim W = h$) in the m.t.i.s. F', such that $\overline{W} = W$, there exists a W-frame $y_1,y_2,\ldots y_h$, with $y_1=\overline{y}_1,\ldots y_h=\overline{y}_h$.

Proposition 2.

If F' is a m.t.i.s. of $E_{\mathbb{C}} = E'$, with $\dim(F' \cap \overline{F}') = r-h$, we can construct a special Witt frame :

$$(x_1,x_2,\ldots x_r,\ y_1,y_2\ldots y_r)$$

such that $y_1,y_2,\ldots y_r$ generate F' with :

(i) $y_{h+1} = \overline{y}_{h+1},\ y_{h+2} = \overline{y}_{h+2},\ldots\ y_r = \overline{y}_r$;

(ii) $y_1 = \delta\overline{x}_1,\ \ y_2 = \delta\overline{x}_2,\ldots\ y_h = \delta\overline{x}_h,\ \ \delta = 1$ or (-1) ;

(iii) $x_{h+1} = \overline{x}_{h+1},\ldots,\ x_r = \overline{x}_r$.

Corollary 1 :

In any m.t.i.s. F' belonging to the complexified of a real space E', we can find r isotropic vectors and adding them r isotropic vectors in E', we can obtain a "real" Witt frame.

Corollary 2 :

If f is a r-isotropic vector defining the m.t.i.s. F' in E', $\overline{f}f \neq 0$ is equivalent with Q definite and $F' \cap \overline{F}' = 0$.

Finally we explain the calculus of ε' such that

$$\rho f = \varepsilon' f, \ \rho = \overline{\gamma}\gamma.$$

If $\delta = 1$, in the proposition 2 above, we consider :

$e_j = x_j + \overline{x}_j$, $e'_j = i(x_j - \overline{x}_j)$, $j = 1,2,\ldots h$.

$f_j = (x_j + y_j)$, $f'_j = (x_j - y_j)$, $j = k+1,\ldots,r$.

$(e_j)^2 = 1$, $(e'_j)^2 = 1$, $(f_j)^2 = 1$, $(f'_j)^2 = -1$.

The Sylvester decomposition contains h+r "positive" squares and r-h "negative" squares. If p is the number of the positive squares p = h+r. If $\delta = -1$, a analogous method gives h+r negative squares, and there are p = r-h positive squares. Note that (p, n-p) is the signature of Q with p positive terms, h = p-r if $2p \geq n$, and r-p if $2p \leq n$.

We take $\gamma = (x_1+y_1)(x_2+y_2)\ldots(x_h+y_h)$, $N(\gamma) = 1$,

$$\gamma f = x_1 x_2 \ldots x_h f,$$

$$\overline{\gamma}\gamma f = \overline{x}_1\overline{x}_2\ldots\overline{x}_h \, x_1 x_2 \ldots x_h f = (-1)^{\frac{h(h-1)}{2}} \, \overline{x}_1\overline{x}_2\ldots\overline{x}_h \, x_h x_{h-1}\ldots x_1 f.$$

If $\delta = 1$, $y_1 = \overline{x}_1, \ldots y_h = \overline{x}_h$, $h = p-r$;

$$\overline{\gamma}\gamma f = (-1)^{\frac{h(h-1)}{2}} f, \quad \varepsilon' = (-1)^{\frac{(p-r)(p-r-1)}{2}}$$

If $\delta = -1$, $y_1 = -\overline{x}_1, \ldots y_h = -\overline{x}_h$, $h = r-p$,

$$\varepsilon' = (-1)^{\frac{h(h+1)}{2}} = (-1)^{\frac{(p-r)(p-r-1)}{2}}$$

Thus, we obtain :

Theorem :

If γf is the pure spinor of conjugation,

$$\gamma \in \text{Pin } Q', \ \rho f = \overline{\gamma}\gamma f = \varepsilon' f,$$

$$\varepsilon' = (-1)^{\frac{(p-r)(p-r-1)}{2}}, \quad [3,a].$$

where p is the number of positive squares in the Sylvester decomposition of Q, $0 \leq p \leq 2r$.

We can choose $\gamma f = x_1 x_2 \ldots x_h f$, $h = |p-r|$.

4. The charge conjugation and the Dirac adjunction.

We put, if uf is any spinor ; as $\overline{uf\gamma} = \overline{u}\gamma f$:

$\underline{\mathscr{C}(uf = \dot{\exp}\ i\theta(\overline{u\gamma f})}$

θ is a real arbitrary coefficient, and modulo a constant scalar factor, any γ choice in Pin Q', does not modify $\mathscr{C}(uf)$.

$\mathscr{C}$ commutes with the G-action (G real Clifford group).

$\mathscr{C}$ is semi-linear, and

$$\underline{\mathscr{C}^2 = \varepsilon' \mathrm{Id}}.$$

Then $\underline{\mathscr{C}^2 = \mathrm{Id}}$, $\underline{\text{iff } p-r = 0 \text{ or } 1}$, $\underline{\text{mod } 4}$, ($\underline{\text{or } p-q = 0 \text{ or } 2 \text{ mod } 8}$).

The Dirac adjunction $\mathscr{A}$ is defined according :

$$\mathscr{A} \circ \mathscr{C} = \beta_{/S}$$

where $\beta_{/S}$ is the main anti-involution in the spinor space S.

$$\mathscr{A}(uf) = \varepsilon' \beta(\mathscr{C}(uf))$$

$$\mathscr{A} \circ \mathscr{C} = \mathscr{C} \circ \mathscr{A}$$

are immediate.

Remark 1 :

The particular case $\overline{f}f \neq 0$ is obtained when $p = 0$.

$$f\overline{f}f = \lambda f,\ \lambda \neq 0,\ \lambda \in \mathbb{R}^*.$$

Note that $f\overline{f}f \neq 0$, according anterior results means that $F' \cap \overline{F'} = 0$.

If we replace f by σf, $\sigma \in \mathbb{C}^*$, we can consider only two cases $\lambda = 1$ or $\lambda = -1$.

Because $\overline{f}f$ belongs to $C(Q')f \cap \overline{f}\, C(Q')$,

$$\overline{f}f = \alpha\gamma f,\ \alpha \in \mathbb{C}^*$$

and $\alpha\overline{\alpha} = \lambda\varepsilon'$,

λ and ε' possess the same sign.

When $\overline{f}f \neq 0$, we can define a conjugation $\mathscr{C}$ according :

$$\mathscr{C}(uf) = \sqrt{\lambda}\, \exp(i\theta)\overline{uf}\, f\ ,\ (\sqrt{\lambda} = 1 \text{ or } i).$$

$$\mathscr{C}^2 = \mathrm{Id},\ \text{if } \lambda = 1$$

$$\mathscr{C}^2 = -\mathrm{Id},\ \text{if } \lambda = -1.$$

We shall examine below if there exists another conjugations in the spinor spaces commuting with the G (or G^+)-action.

Remark 2 :

We can define a hermitian sesquilinear form $\mathscr{H}$ on the spinor space according : $\beta(\overline{uf})vf = a\ \mathscr{H}(uf,vf)\ \gamma f$, $a^2 = \varepsilon\varepsilon'$,

ε such that $\beta(f) = \varepsilon f$.

$\mathscr{H}$ owns interesting property connected with the conjugation $\mathscr{C}$:

$$\mathscr{H}(\mathscr{C}(uf), vf) = \varepsilon\, \mathscr{H}(\mathscr{C}(vf), uf)$$

and also :

$$\mathscr{H}(\mathscr{C}(uf, \mathscr{C}(vf) = \varepsilon\varepsilon'\, \overline{\mathscr{H}(uf, vf)}.$$

The reader can consult [3,a] for more details about hermitian forms and conjugation.

5. Majorana spinors.

We consider spinor spaces such that the ε' constant is 1, with a conjugation $\mathcal{C}$ defined in 4 above, then $\mathcal{C}^2 = \mathrm{Id}$. The spinor space S is the direct sum of two real spaces S_1 and S_2 such that :

$$S_1 = (I + \mathcal{C})(rS),\ S_2 = (I - \mathcal{C})(rS)$$

I is the identity and rS the realification of S. If we call ψ any spinor in S, ψ is the sum of two spinors ψ_1 and ψ_2, with :

$$\psi_1 = \frac{1+\mathcal{C}}{2}\psi, \quad \psi_2 = \frac{1-\mathcal{C}}{2}\psi .$$

The spinors ψ_1 (or ψ_2) are called Majorana spinors ; the product by i exchange these spinors and :

$$\mathcal{C}\psi_1 = \psi_1, \quad \mathcal{C}\psi_2 = -\psi_2.$$

If the dimension of S is 2^r on $\mathbb{C}$, the Majorana spinors ψ_1 (or ψ_2) generate a real 2^r-dimensional space. Because $\mathcal{C}$ commutes with the G-action S_1 and S_2 are globally invariant for this action and the representation of G in S splits in two real representations.

We obtain this situation when p-q = 0,2, (mod 8).

Remark :

If $\mathcal{C}^2 = -\mathrm{Id}$, according the 1 above, the spinor representation of G is quaternionic. It is possible writing, unically $\psi = \psi_1 + \psi_2$, with

$$\psi_1 = \frac{1+i\,\mathcal{C}}{2}\ \psi, \quad \psi_2 = \frac{1-i\,\mathcal{C}}{2}\ \psi$$

$$\mathcal{C}\psi_1 = i\psi_2, \quad \mathcal{C}\psi_2 = -i\psi_1.$$

Naturally there not exist a decomposition of S in direct sum of two complex spaces, because

$$\frac{1+i\,\mathcal{C}}{2}\ \frac{1-i\,\mathcal{C}}{2} = 1,$$

any element ψ is written :

$$\psi = \frac{1+i\,\mathcal{C}}{2}\left(\frac{1-i\,\mathcal{C}}{2}\psi\right) = \frac{1+i\,\mathcal{C}}{2}\psi' \quad (\psi_1 \text{ form}),$$

or $$\psi = \frac{1-i\,\mathcal{C}}{2}\ \frac{1+i\,\mathcal{C}}{2}\psi = \frac{1-i\,\mathcal{C}}{2}\psi'' \quad (\psi_2 \text{ form}).$$

We meet this situation if p-q = 4,6 (mod 8).

6. Weyl spinors.

Consider the product $e_1 e_2 \ldots e_n = e_N$, of the n elements (e_i) belonging to a orthonormal frame of E. If n = 2r, we know that e_N anticommutes with any $x \in E$. If we change the orthonormed frame into (e_i'), we obtain the product $\pm e_N$. The classical result $\varphi_g(x) = gxg^{-1}$, $x \in E$, $g \in G$, gives easely $\varphi(e_N) = -\mathrm{Id}_E$ and $\varphi(e_N)$ extends naturally to the main involution of the Clifford algebra.

We can sea immediately that :

$$e_N f = (-1)^r f\ e_N$$

$$e_N f\, e_N^{-1} = (-1)^r f = \alpha(f)$$

and $e_N f$ is a pure spinor defining $\alpha(f)$ (here α plays a role liking complex conjugation above).

But $\alpha(e_N) = (-1)^n e_N$

$$\alpha(e_N)e_N f = \xi f, \quad \xi = \pm 1,$$

with $(e_N)^2 = \xi$, $(n=2r)$.

$$\xi = (-1)^{r-p}.$$

According an evident analogy with the $\mathscr{C}$-definition, we could call "Weyl conjugation" :

$$\mathscr{W}(uf) = e_N\, uf,$$

obtained from the sequence of transformations :

$uf \longrightarrow \alpha(uf) \longrightarrow \alpha(uf)e_N = \alpha(u)\, e_N f = e_N\, uf$

(liking : $uf \longrightarrow \overline{uf} \longrightarrow \overline{uf}\gamma = \overline{u}\gamma f$).

However, we wish obtain $\mathscr{W}^2 = \mathrm{Id}$, also, taking account of $(e_N)^2 = (-1)^{r-p} = (-1)^h$, we introduce $\eta \in C^*$, such that

$\eta^2 = (-1)^h$ and define, "Weyl conjugation" by :

$$\underline{\mathscr{W}(uf) = \eta\, e_N\, uf.}$$

$\mathscr{W}$ commutes with the G^+-action.

Taking :

$$\varphi_1 = \frac{1+\mathscr{W}}{2}\psi, \quad \varphi_2 = \frac{1-\mathscr{W}}{2}\psi \;;$$

one writes any spinor as a direct sum of two "Weyl spinors", φ_1 and φ_2 with :

$$\mathscr{W}(\varphi_1) = \varphi_1, \quad \mathscr{W}(\varphi_2) = -\varphi_2.$$

It is easy seeing that a spinor is a Weyl and Majorana spinor in the precedent meaning iff $\mathscr{C}$ commutes with ηe_N, viz. $p = r$, mod 2, but as $p = r$ or $r+1$, mod 4, for a Majorana spinor, this condition give finally :

$$p=r, \text{ mod. } 4 \text{ or } p-q = 0, \text{ mod } 8.$$

(We recall that $n = 2r$, here).

7. Weyl-Dirac charge conjugation (Chiral conjugation) and generalizations.

e_N defines a Clifford algebra automorphism, of which the product with the usual complex conjugation gives a new complex conjugation. With the notation used in 3, we must change ε' into

$$\varepsilon'(-1)^{r-p}$$

and a conjugation (called Weyl-Dirac conjugation) is coming from there if you are using $\alpha(\overline{u})$. The Majorana condition is $\varepsilon'(-1)^{r-p} = 1$

$$p = r \text{ (or } r-1\text{), mod } 4$$

$p-q = 0, 6 \pmod 8$.

Note this conjugation commutes only with the $G^{\pm}$-action.

Ending our study in even dimension, we are looking for the conjugations commuting with the G-action or the G^+-action in spinor spaces.

Given a conjugation $\mathcal{C}$, one obtains any conjugation by product of $\mathcal{C}$ with a linear isomorphism commuting with G(or G^+).

According the classical properties of Clifford algebras in even dimension and neutral signature any linear morphism of $C(Q')f$ is a left product by a regular element of $C(Q')$; but G is generated by non isotropic vectors of E. If a linear isomorphism of $C(Q')$ f commutes with any element of G, it is necessarily a scalar, as $J^2 = \pm Id$, this scalar will be $\exp(i\theta)$, $\theta \in \mathbb{R}$, but we find again the conjugation $\mathcal{C}$ obtained in 4 above. Now, we change G into G^+, the linear isomorphism is the left product by a central element of $C^+(Q')$, because G^+ is generated by a even number of non isotropic vectors. One can obtain a conjugation J, applying $\mathcal{C}$ (defined in 4 above) followed by a product $\lambda + \mu e_N$, with $\lambda, \mu \in \mathbb{C}$:

$$* \quad \begin{cases} \lambda^2 + \mu^2(-1)^h \neq 0 \\ |\lambda|^2 + |\mu|^2(-1)^h = \pm 1 \\ \lambda\bar{\mu} + \mu\bar{\lambda} = 0 \end{cases}$$

We have obtained thus :

<u>Proposition 3</u> :

In even dimension $n = 2r$:

a) The conjugations $\mathcal{C}$, commuting with the G-action take the following form :

$$\mathcal{C}(uf) = \exp(i\theta)(\overline{u\gamma f}), \quad \theta \in \mathbb{R},$$

where γf is the pure spinor of conjugation.

If the Sylvester decomposition of Q contains p positive terms and q negative terms ($p+q = n$) there exists Majorana spinor iff $p = r, r+1$, (mod 4) (or $p-q = 0,2$, mod 8).

There exists Majorana-Weyl spinors iff $p=r$, mod 4,

(or $p-q = 0$, mod 8).

b) The conjugations J commuting with the G^+-action are the product of by the left action of $\lambda + \mu e_N$, $e_N = e_1 e_2 \dots e_n$, λ and μ satisfying to the conditions * above.

$$J^2 = \varepsilon' \, (|\lambda|^2 + |\mu|^2(-1)^h) Id$$

ε' is defined in the theorem above, $h = |r-p|$.

<u>Remarks</u> :

If h is even, $J^2 = \varepsilon' Id$, then if $h = |r-p|$ is even it is impossible to define Majorana spinors for a conjugation commuting with G^+ if $|r-p|=2$, mod.4, (or $p-q = 4$, mod 8).

These representations are always quaternionic, but they are real for the other cases.

Note that for the precedent generalized conjugations there exist Dirac-adjunctions.

8. The odd dimensions.

The real space E, dim $E = n = 2r+1$, is endowed with a quadratic form Q with (p,q) signature.

According a classical result [1,3b], if $z_o \in E$, $Q(z_o) = a \neq 0$; $C^+(Q)$ is isomorphic with the Clifford algebra $C(Q_1)$ of $(z_o)^\perp = E_1$, endowed with the quadratic form $Q_1 = -aQ$; thus $C^+(Q)$ is a central simple algebra.

Consider now the space (E',Q') complexified of (E,Q) with the Witt decomposition :

$$E' = F \oplus F' \oplus (z_o),$$

$F \oplus F'$, also is a Witt decomposition for (E_1',Q_1') complexified space of (E_1, Q_1) and we can choose $S = \{x_{i_1} x_{i_2} \ldots x_{i_h} f,\ f = y_1 y_2 \ldots y_r\}$ as a spinor space for the algebra $C'(Q_1) \simeq C^+(Q')$ and $a = \pm 1$ is permissible.

We define $\overline{f} = \gamma f \gamma^{-1}$ with $\gamma \in \mathrm{Pin}\, Q_1'$, and we take : $f_1(uf) = \overline{u}\gamma f$, and we see immediately that $\mathcal{C}_1^2 = \varepsilon' \mathrm{Id}$, ε' is calculated with (E_1,Q_1) (henceforth the coefficient exp $i\theta$ used above is choosed equal to 1).

With $\mathcal{C}_1$ we can define Majorana spinors iff $\varepsilon' = 1$, this condition gives :

$$p = \frac{p+q-1}{2} \quad \left(\text{or } \frac{p+q-1}{2} + 1\right), \text{ mod } 4, \text{ viz.}$$

$$p - q = \pm 1, \text{ mod } 8.$$

One defines a conjugation $\mathcal{C}$ in the spinor space of $C^+(Q')$, carrying $\mathcal{C}_1$ by the isomorphism j :

$$j : C(Q_1) \longrightarrow C^+(Q)$$

associating to $y \in E_1$, $z_o\, y \in C^+(Q)$, [1] ;

then one sets :

$$\mathcal{C}(j(uf) = j(\mathcal{C}_1(uf)).$$

We carry also, using j, the definition of Weyl spinors.

We can give a second method for define a conjugation in the odd dimension case.

We write : $u^+ f = (u_1^+ + z_o\, u_2^-) f \in C^+(Q')f$,

(+ and (-) denote the parity).

According a classical result $C^+(Q)$ et $C^+(-Q)$ are isomorphic algebras, also $q > 1$ is permissible.

We choose $Q(z_o) = -1$, $(z_o)^\perp$ is a vector space with (p,q-1) signature, and if $\overline{f} = \gamma f \gamma^{-1}$, there exists $\gamma \in \mathrm{Pin}\, Q'$, whose the parity is the one of $|r-p|$ (theorem above).

a) If $h = |r-p|$ is even, we take :

$$\mathscr{C}(u^+ f) = \overline{u}^+ \gamma f$$

b) If h is odd, $\gamma' = \gamma z_o$ is such that $\gamma' f \gamma'^{-1} = (-1)^r f$.

We have two subcases :

r even, we take :

$$\mathscr{C}(u^+ f) = u^+ \gamma z_o f$$

r odd, we consider in $(z_o)^\perp$ the orthonormed frame $e_1, e_2, \dots, e_{2r}$,

$\gamma'' = \gamma z_o\, e_1 e_2 \dots e_{2r}$ is such that $\gamma'' f \gamma''^{-1} = \overline{f}$, and we define :

$$\mathscr{C}(u^+ f) = \overline{u}^+ \gamma z_o\, e_1 e_2 \dots e_{2r} f.$$

$\mathscr{C}$ commutes with the G^+-action.

There exists Majorana spinors iff $p - q = \pm 1$, mod 8.

Finally, we are taking account of the isomorphism of $C^+(Q')$ onto the central simple algebra $C(Q_1')$, so that conjugations which commute with G^+ (which contains a group isomorphic with G_1, Clifford group of $C(Q_1)$) are, modulo $\exp(i\theta)$, the conjugations above (according 7).

Puting together the precedent results we obtain the final table giving the existence of G - Majorana spinors, or G^+-Majorana spinors :

(p−q) mod.8	0	1	2	3	4	5	6	7
G-Majorana spinors	exist		exist		not exist		not exist	
G^+-Majorana spinors	exist	exist	exist	not exist	not exist	not exist	exist	exist

Remark :

The classical results about the "periodicity" of Clifford algebras [3,b] are perfectly coherent with our study if we consider G-Majorana spinors when $p - q = 0, 2$ (real case) or when $p - q = 4,6$ (quaterniomic case) ; likely when $p - q = 1$, if we consider G^+-Majorana spinors.

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