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MUSIELAK - ORLICZ ALGEBRAS

Henryk Hudzik

There are characterized Musielak-Orlicz spaces which are Banach algebras under pointwise multiplication of functions. It is an extension of the results of [1]. Let (T, Σ, μ) be a space of positive σ -finite measure and let $\Phi: T \times \mathbb{R} \rightarrow [0, +\infty]$ be a Musielak-Orlicz function, i.e. $\Phi(t, \cdot)$ is convex, even, vanishing and continuous at 0 and not identically equal to 0 for μ -a.a. $t \in T$ and $\Phi(\cdot, u)$ is a Σ -measurable function for any $u \geq 0$. Let L^Φ be the corresponding Musielak-Orlicz space, i.e. L^Φ consists of all equivalence classes of Σ -measurable functions $f: T \rightarrow \mathbb{R}$ for which there exists $\lambda > 0$ such that $M_\Phi(\lambda f) = \int_T \Phi(t, \lambda f(t)) d\mu < +\infty$. With respect to the Luxemburg norm $\|f\|_\Phi$, defined by

$$\|f\|_\Phi = \inf \{ \lambda > 0 : M_\Phi(\lambda^{-1} f) \leq 1 \},$$

L^Φ is a Banach function space with the Fatou property (see [3-6]).

Henceforth, T_a and \mathbb{N} denote the non-atomic and purely atomic part of T , respectively, i.e. the atoms will be identified with positive integers. For $n \in \mathbb{N}$ we write $\Phi_n(\cdot)$ instead of $\Phi(n, \cdot)$. L^∞ denotes the space of μ -essentially bounded functions on T with the norm defined by $\|f\|_\infty = \text{ess sup}_{t \in T} |f(t)|$ for any $f \in L^\infty$.

LEMMA. $L^\Phi \subset L^\infty$ if and only if there exists $\alpha \in (0, +\infty)$ such that

- (i) $\Phi(t, \alpha) = +\infty$ for μ -a.a. $t \in T_a$, and
- (ii) $\Phi_n(\alpha) \mu(\{n\}) \geq 1$ for all $n \in \mathbb{N}$.

Moreover, the inequality $\|f\|_\infty \leq \alpha \|f\|_\Phi$ holds for any $f \in L^\Phi$ when conditions (i) and (ii) are fulfilled.

Proof. Sufficiency. Assume that conditions (i) and (ii) hold. If $f \in L^\Phi$, then $M_\Phi(f/r \|f\|_\Phi) \leq 1$ and so $|f(t)|/r \|f\|_\Phi \leq \alpha$ for any $r > 1$ and for μ -a.a. $t \in T$. Hence it follows that $\|f\|_\infty \leq \alpha \|f\|_\Phi$.

Necessity. Assume that $L^\Phi \subset L^\infty$. Then $L^\Phi(T_a) \subset L^\infty(T_a)$, i.e. $L^\Phi(T_a) \subset L^{\Phi_0}(T_a)$, where Φ_0 is the Orlicz function defined by $\Phi_0(u)$

$=0$ for $0 \leq u \leq 1$ and $\Phi_0(u) = +\infty$ for $u > 1$. However, this inclusion is possible if there exist $k > 0$, a set T_0 of measure 0 and a non-negative μ -summable over T_a function h such that

$$\Phi_0(u) \leq \Phi(t, ku) + h(t)$$

for all $t \in T_a \setminus T_0$ and $u \geq 0$ (see [5]). Hence it follows that $\Phi(t, 2k) = +\infty$ for μ -a.a. $t \in T_a$. So, condition (i) holds with $\alpha = 2k$.

Now, we shall prove the necessity of condition (ii). Assume that this condition does not hold. There is a sequence (n_k) of positive integers such that $\Phi_{n_k}(2^k) \mu(\{n_k\}) \leq 2^{-k}$ for $k=1, 2, \dots$. Defining $f = \sum_{k=1}^{\infty} 2^k e_{n_k}$, where e_{n_k} is the n_k th basic sequence in l^1 , we have

$$M_{\Phi}(f) = \sum_{k=1}^{\infty} \Phi_{n_k}(2^k) \mu(\{n_k\}) \leq \sum_{k=1}^{\infty} 2^{-k} = 1,$$

i.e. $f \in L^{\Phi} \setminus L^{\infty}$. This ends the proof.

DEFINITION. A Banach function space $(X, \|\cdot\|)$ is called a Banach quasi-algebra (algebra) if it is an algebra, i.e. $f \cdot g \in X$ whenever $f, g \in X$ and if there is $K > 0$ such that $\|f \cdot g\| \leq K \|f\| \|g\|$ ($\|f \cdot g\| \leq \|f\| \|g\|$) for all $f, g \in X$.

THEOREM 1. The following conditions are equivalent:

- (i) L^{Φ} is an algebra:
- (ii) $L^{\Phi} \subset L^{\infty}$:
- (iii) L^{Φ} is a Banach quasi-algebra:
- (iv) There is $\alpha \in (0, +\infty)$ such that:
 - (a) $\Phi(t, \alpha) = +\infty$ for μ -a.a. $t \in T_a$, and
 - (b) $\Phi_n(\alpha) \mu(\{n\}) \geq 1$ for all $n \in \mathbb{N}$.

Proof. (i) \Rightarrow (ii). If L^{Φ} is an algebra, then for any $f \in L^{\Phi}$ we may define on L^{Φ} the operator Π_f by $\Pi_f g = f \cdot g$ for any $g \in L^{\Phi}$. It is obvious that Π_f is an orthomorphism (see [7] and [8]), i.e. $\inf(|\Pi_f g|, |h|) = 0$ whenever $\inf(|g|, |h|) = 0$ in L^{Φ} (obviously L^{Φ} is a lattice under the natural order relation $f \leq g$ if and only if $f(t) \leq g(t)$ for μ -a.a. $t \in T$). However, it follows by [7], Th.8 that $f \in L^{\infty}$. So, $L^{\Phi} \subset L^{\infty}$. The implication (ii) \Rightarrow (iii) follows by LEMMA. Indeed, if $f, g \in L^{\Phi}$, then for any $r > 1$,

$$M_{\Phi}\left(\frac{f \cdot g}{r \alpha \|f\|_{\Phi} \|g\|_{\Phi}}\right) \leq M_{\Phi}\left(\frac{f \cdot g}{r \|f\|_{\Phi} \|g\|_{\infty}}\right) \leq M_{\Phi}\left(\frac{f}{r \|f\|_{\Phi}}\right) \leq 1,$$

i.e. $\|f \cdot g\|_{\Phi} \leq \alpha \|f\|_{\Phi} \|g\|_{\Phi}$. The implication (iii) \Rightarrow (i) is obvious and the equivalence (ii) \Leftrightarrow (iv) follows by LEMMA. The proof is finished.

THEOREM 2. L^{Φ} is a Banach algebra if and only if there exists $\alpha \in (0, 1]$ such that:

- (i) $\Phi(t, \alpha) = +\infty$ for μ -a.a. $t \in T_a$, and
- (ii) $\Phi_n(\alpha) \mu(\{n\}) \geq 1$ for any $n \in \mathbb{N}$.

Proof. The sufficiency follows by LEMMA .. Now, we shall prove the necessity. Denote $\alpha = \sup [u \geq 0: \Phi(t, u) < +\infty \text{ for } \mu\text{-a.a. } t \in T]$. Assuming that $\alpha > 1$ and defining $\beta = \alpha^{2/3}$, we have $\Phi(t, \beta) < +\infty$ for μ -a.a. $t \in T$. Assume that $\mu(T_a) > 0$ and C is a subset of T_a of positive and finite measure. Define

$$C_n = \{t \in C: \Phi(t, \beta) \leq n\}, n=1, 2, \dots$$

This sequence is ascending and $\mu(\bigcup_n C_n) = \mu(C)$. So, there is an index k such that $0 < \mu(C_k) < +\infty$. Defining $f = \beta \chi_{C_k}$, we have

$$M_{\Phi}(f) = \int_{C_k} \Phi(t, \beta) dt \leq k \mu(C_k) < +\infty.$$

There exists a set $D \subset C_k$, $D \in \Sigma$, such that $M_{\Phi}(f \chi_D) \leq 1$. However,

$$M_{\Phi}((f \chi_D)^2) = \int_D \Phi(t, \alpha^{4/3}) d\mu = +\infty.$$

Hence it follows that $\|f \chi_D\|_{\Phi} \leq 1$ and $\|(f \chi_D)^2\|_{\Phi} > 1$, i.e. L^{Φ} is not a Banach algebra.

For the proof of necessity of condition (ii), assume that L^{Φ} is a Banach algebra. Every element $f = \chi_{\{n\}}$ belongs to L^{Φ} ($n=1, 2, \dots$). We have $\|f\|_{\Phi} \leq \|f\|_{\Phi}^2$, i.e. $\|f\|_{\Phi} \geq 1$ and so $M_{\Phi}(f) = \Phi_n(1) \mu(\{n\}) \geq 1$ for $n = 1, 2, \dots$. The proof is finished.

COROLLARIES: (i). Let μ be a purely atomic measure and $\Phi = (\Phi_n)$, where $\Phi_n(u) = u^{p_n}$, where $1 \leq p_n < +\infty$ for any $|u| \geq 0$ and $n \in \mathbb{N}$. Then L^{Φ} is a Banach quasi-algebra if $\inf_n \mu(\{n\}) > 0$. L^{Φ} is a Banach algebra if and only if $\inf_n \mu(\{n\}) \geq 1$.

(ii). We may define for any Musielak-Orlicz function a subspace E^{Φ} of L^{Φ} by

$$E^{\Phi} = \{f \in L^{\Phi}: M_{\Phi}(\lambda f) < +\infty \text{ for any } \lambda > 0\}.$$

It is clear that the condition $\Phi(t, \alpha) = +\infty$ for μ -a.a. $t \in T$, where $0 < \alpha < +\infty$, implies that $E^{\Phi} = \{0\}$. So, in the case of a non-atomic measure, no non-trivial space E is an algebra under pointwise multiplication of functions.

(iii). It is well known that any Banach quasi-algebra can be re-normed to be a Banach algebra (see [9]). For Musielak-Orlicz spaces the following is true: if a Musielak-Orlicz space L^{Φ} is a Banach quasi-algebra, then there is a Musielak-Orlicz function Φ_1 equivalent to Φ (i.e. $L^{\Phi} = L^{\Phi_1}$) such that L^{Φ} equipped with the norm $\|\cdot\|_{\Phi_1}$ is

a Banach algebra. For this purpose it suffices to put $\Phi_1(t, u) = \Phi(t, \alpha u)$, where α is a positive constant satisfying conditions (i) and in THEOREM 1. It is evident that $\|\cdot\|_{\Phi_1} = \alpha \|\cdot\|_{\Phi}$. So, $\|f \cdot g\|_{\Phi_1} = \alpha \|f \cdot g\|_{\Phi} \leq \alpha^2 \|f\|_{\Phi} \|g\|_{\Phi} = \|f\|_{\Phi_1} \|g\|_{\Phi_1}$ for all $f, g \in L^{\Phi_1}$.

REMARKS. It is obvious that L^{Φ} is an algebra if and only if $f^2 \in L^{\Phi}$ whenever $f \in L^{\Phi}$. It is equivalent to $L^{\Phi} \subset L^{\Psi}$, which it is equivalent to $\Psi \rightarrow \Phi$, where $\Psi(t, u) = \Phi(t, u^2)$ for all $u \geq 0$ and μ -a.a. $t \in T$. The relation $\Psi \rightarrow \Phi$ is characterized for example in [5]. In the case of a non-decreasing but non-convex Musielak-Orlicz function Φ it is possible that $\Psi \rightarrow \Phi$ also for a non-atomic measure. For example, the function $\Phi(u) = \log(1 + |u|)$ satisfies the inequality $\Phi(u^2) \leq 3 \Phi(u)$ for all $u \geq 0$. Orlicz algebras generated by non-convex Orlicz functions has considered N.J. Kalton in [2].

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