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This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Introduction

The purpose of this paper is to give a variant of the ergodic theorem for functions with values in a vector lattice. Let $(\Omega, \mathcal{S}, P)$ be a probability measure space and $T:\Omega\rightarrow\Omega$ be a $\mathcal{S}$-measurable mapping. T is called measure preserving if $P(T^{-1}(A)) = P(A)$ for any $A \in \mathcal{S}$.

Theorem 1.1:

Let $T:\Omega\rightarrow\Omega$ be a measure preserving mapping of a probability measure space $(\Omega, \mathcal{S}, P)$. For any integrable function $f:\Omega\rightarrow\mathbb{R}$ there exists an integrable function $g:\Omega\rightarrow\mathbb{R}$ such that :

(i) $\lim\limits_{k\to\infty} \frac{1}{k}\sum\limits_{i=0}^{k-1} f(T^i(\omega)) = g(\omega)$ almost everywhere

(ii) $\lim\limits_{k\to\infty} \int\limits_{\Omega} \left| g(\omega) - \frac{1}{k}\sum\limits_{i=0}^{k-1} f(T^i(\omega))\right| dP(\omega) = 0$.

The parts (i) and (ii) belong to G.D.Birkhoff and J. von Neumann respectively. See [5] pp. 30-38.

It is convenient to describe the limit function g using the conditional mean value. Let $(\Omega, \mathcal{S}, P)$ be a probability measure space and $\mathcal{S}_0$ be a σ-subalgebra of $\mathcal{S}$. For any integrable $\mathcal{S}$-measurable function $f:\Omega\rightarrow\mathbb{R}$ there exists an integrable $\mathcal{S}_0$-measurable function $g:\Omega\rightarrow\mathbb{R}$ such that :

$$\int\limits_A f(\omega)\,dP(\omega) = \int\limits_A g(\omega)\,dP(\omega) \quad \text{for any } A \in \mathcal{S}_0 .$$

The function f determines the function g uniquely almost everywhere . All these facts may be found in [1] pp. 193-194. The function g is called a conditional mean value of f with respect to $\mathcal{S}_0$. We shall write $g = E(f \mid \mathcal{S}_0)$.

In Theorem 1.1 it may be put $g = E(f \mid \mathcal{S}_0)$, where $\mathcal{S}_0$ is the σ-algebra of all almost T-invariant sets. (A set $A \in \mathcal{S}$ is called almost T-invariant if the symmetric difference $A \triangle T^{-1}(A)$ is a zero set.)

This paper uses the results of [3] concerning the mean value and the conditional mean value for vector lattice valued functions, see also [4] . A special case of the presented ergodic theorem has been proved by E. Hrachovina, see [2] .

2. Vector lattices

A real vector space V is called a vector lattice if it has a partial ordering $\leq$ such that $(V, \leq)$ is a lattice and :

$\forall x, y, z \in V: \ x \leq y \Rightarrow x + z \leq y + z$

$\forall x, y \in V: \forall \lambda \geq 0: \ x \leq y \Rightarrow \lambda x \leq \lambda y$.

Lattice operations are denoted by symbols $\vee$ and $\wedge$.

If $a \in V$ then the symbol $|a|$ denotes the element $a \vee (-a)$.

A vector lattice V is called σ-complete if every upper bounded sequence $\{a_n\} \subset V$ has a least upper bound which is denoted by the symbol $\bigvee_{n=1}^{\infty} a_n$ (or equivalently, every lower bounded sequence $\{a_n\}$ has a greatest lower bound which is denoted by $\bigwedge_{n=1}^{\infty} a_n$) .

Definition 2.1:

Let V be a σ-complete vector lattice. A sequence $\{a_n\} \subset V$ is called decreasing to 0 if :

$\forall n: \ 0 \leq a_{n+1} \leq a_n$ and $\bigwedge_{n=1}^{\infty} a_n = 0$.

We write $a_n \searrow 0 \ (n \to \infty)$ in this case.

A sequence $\{x_n\} \subset V$ is called converging to $x \in V$ if there exists a sequence $\{a_n\} \subset V$ decreasing to 0 such that $|x_n - x| \leq a_n$ for all n . We write $x_n \to x \ (n \to \infty)$ in this case.

Proposition 2.2:

Let V be a σ-complete vector lattice.

(i) A sequence $\{x_n\} \subset V$ converges to $x \in V$ if and only if $\{x_n\}$ is bounded and $x = \bigwedge_{n=1}^{\infty} \bigvee_{m=n}^{\infty} x_m = \bigvee_{n=1}^{\infty} \bigwedge_{m=n}^{\infty} x_m$

(ii) $a_n \searrow 0, \ b_n \searrow 0 \Rightarrow (a_n + b_n) \searrow 0$

(iii) $a_n \searrow 0, \ \lambda \geq 0 \Rightarrow \lambda a_n \searrow 0$

(iv) $x_n \to x, \ y_n \to y \Rightarrow (x_n + y_n) \to (x + y)$

(v) $x_n \to x \Rightarrow \lambda x_n \to \lambda x$.

The following lemma will be important in the proof of the main result of this paper.

Lemma 2.3:

Let V be a σ-complete vector lattice and $\{a_n\} \subset V$, $\{b_{n,k}\} \subset V$ be sequences such that :

$\forall\ n, k:\ b_{n,k} \geq 0$

$\forall\ n:\ b_{n,k} \to 0\ \ (k \to \infty)$

$a_n \searrow 0\ \ (n \to \infty)$.

Put $c_k = \bigwedge_{n=1}^{\infty} (a_n + b_{n,k})$. Then $\forall\, k$: $c_k \geq 0$ and $c_k \to 0\ (k \to \infty)$.

Proof:

The inequality $c_k \geq 0$ for all k is obvious. The sequence $\{c_k\}$ is bounded because $0 \leq c_k \leq a_1 + b_{1,k}$ for all k and $b_{1,k} \to 0\ (k \to \infty)$. It means that the element $\bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} c_j$ exists. Since $c_k \geq 0$ for all k, by Proposition 2.2 it suffices to prove that $\bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} c_j = 0$. We have :

$$\bigvee_{j=k}^{\infty} c_j = \bigvee_{j=k}^{\infty} \bigwedge_{n=1}^{\infty} (a_n + b_{n,j}) \leq \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) \quad \text{and}$$

$$\bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} c_j \leq \bigwedge_{k=1}^{\infty} \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) = \bigwedge_{n=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) =$$

$$= \bigwedge_{n=1}^{\infty} (a_n + \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} b_{n,j}) = \bigwedge_{n=1}^{\infty} (a_n + 0) = \bigwedge_{n=1}^{\infty} a_n = 0 .$$

Except for assumptions of lemma we used the obvious facts :

$$\bigvee_{j=k}^{\infty} \bigwedge_{n=1}^{\infty} (a_n + b_{n,j}) \leq \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) \quad \text{and}$$

$$\bigwedge_{k=1}^{\infty} \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) = \bigwedge_{n=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) .$$

3. Integral and conditional mean value of vector lattice valued functions

In this section we give a summary of results of author's paper [3] .

Let $(\Omega, \mathcal{S}, P)$ be a probability measure space and V be a σ-complete vector lattice. The symbol $F(\Omega, V)$ denotes the set of all functions $f: \Omega \longrightarrow V$. Obviously, $F(\Omega, V)$ is a σ-complete vector lattice under natural operations and ordering.

Two functions $f, g \in F(\Omega, V)$ are called equivalent if there exists a set $A \in \mathcal{S}$ such that $P(A) = 0$ and $\forall \omega \in \Omega - A: f(\omega) = g(\omega)$. The set of all equivalence classes is denoted by $\mathcal{F}(\Omega, \mathcal{S}, P, V)$ and it is a σ-complete vector lattice under natural operations and ordering. A function $f \in F(\Omega, V)$ is called simple if $f(\omega) = a_i$ for $\omega \in A_i$, where $\{A_i\}$ is a finite measurable partition of Ω and $a_i \in V$. We put

$$\int_\Omega f(\omega)\, dP(\omega) = \sum_{i=1}^{n} P(A_i) \cdot a_i \quad \text{in this case.}$$

A class $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$ is called simple if it contains some simple function f.
We put $E(\varphi) = \int_\Omega \varphi\, dP = \int_\Omega f(\omega)\, dP(\omega)$ in this case.

The set of all simple functions is denoted by $L_0^\infty(\Omega, \mathcal{S}, P, V)$ and the set of all simple classes is denoted by $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$.

Let $\{f_n\} \subset F(\Omega, V)$ and $f \in F(\Omega, V)$. We say that a sequence $\{f_n\}$ converges to the function f uniformly almost everywhere if there exist $A \in \mathcal{S}$, $\{a_n\} \subset V$ such that:
$P(A) = 0$
$\forall \omega \in \Omega - A: \forall n: |f_n(\omega) - f(\omega)| \leq a_n$
$a_n \to 0 \ (n \to \infty)$.
Obviously, the condition $a_n \to 0$ may be replaced by a stronger one $a_n \searrow 0$. We write $f_n \to f$ u. a. e. $(n \to \infty)$ in this case.
Let $\{\varphi_n\} \subset \mathcal{F}(\Omega, \mathcal{S}, P, V)$ and $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$. We say that the sequence $\{\varphi_n\}$ converges to the class φ uniformly almost everywhere if $f_n \to f$ u. a. e. for some $f_n \in \varphi_n$ and $f \in \varphi$. We write $\varphi_n \to \varphi$ u. a. e. $(n \to \infty)$ in this case.

Let $\mathfrak{M}$ be a system of all vector subspaces of $\mathcal{F}(\Omega, \mathcal{S}, P, V)$ which contain $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$ and are closed with respect to the convergence which was described above. Obviously, $\mathfrak{M}$ has the minimal element with respect to inclusion. This vector space is denoted by $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$.

Theorem 3.1:
(i) $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ is a vector sublattice of $\mathcal{F}(\Omega, \mathcal{S}, P, V)$, which is closed with respect to u. a. e. convergence.
(ii) There exists a unique nonnegative linear extension $\overline{E}$ of E onto $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$, which is continuous in the following sense $\varphi_n \to \varphi$ u. a. e. $\Rightarrow$ $\overline{E}(\varphi_n) \to \overline{E}(\varphi)$.

Remark : We shall write $E(\varphi)$ or $\int_\Omega \varphi\, dP$ for $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ instead of $\bar{E}(\varphi)$.

In a similar way it may be constructed a conditional mean value operator. Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, $\mathcal{S}_0$ be a σ-subalgebra of $\mathcal{S}$ and $E(.\,|\mathcal{S}_0)$ be a conditional mean value operator for real functions. Take $\varphi \in \mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$; φ is an equivalence class of some simple function f of the form $\sum_{i=1}^{n} \chi_{A_i} a_i$. Denote by ψ the equivalence class of the function

$\sum_{i=1}^{n} E(\chi_{A_i} | \mathcal{S}_0) \cdot a_i$. In this case $\psi \in \mathcal{L}^\infty(\Omega, \mathcal{S}_0, P, V)$. Putting $E(\varphi|\mathcal{S}_0) = \psi$

we obtain a linear nonnegative operator $E(.\,|\mathcal{S}_0) : \mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V) \to \mathcal{L}^\infty(\Omega, \mathcal{S}_0, P, V)$.

Theorem 3.2:

(i) There exists a unique nonnegative linear extension $\bar{E}(\,.\,|\mathcal{S}_0) : \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V) \to \mathcal{L}^\infty(\Omega, \mathcal{S}_0, P, V)$ of $E(\,.\,|\mathcal{S}_0)$.

(ii) The operator $\bar{E}(\,.\,|\mathcal{S}_0)$ is continuous in the following sense : $\varphi_n \longrightarrow \varphi$ u. a. e. $\Rightarrow$ $\bar{E}(\varphi_n|\mathcal{S}_0) \longrightarrow \bar{E}(\varphi|\mathcal{S}_0)$ u. a. e.

Remark : We shall write $E(\varphi|\mathcal{S}_0)$ instead of $\bar{E}(\varphi|\mathcal{S}_0)$.

We shall also use the pointwise convergence. Let $\{f_n\} \subset F(\Omega, V)$ and $f \in F(\Omega, V)$. We say that the sequence $\{f_n\}$ converges to f almost everywhere if there exists a set $A \in \mathcal{S}$ such that $P(A) = 0$ and $\forall\, \omega \in \Omega - A$: $f_n(\omega) \longrightarrow f(\omega)$ ($n \to \infty$). We write $f_n \to f$ a. e. in this case.

If $\{\varphi_n\} \subset \mathcal{F}(\Omega, \mathcal{S}, P, V)$ and $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$ then the notation $\varphi_n \to \varphi$ a. e. $(n \to \infty)$ means that $f_n \to f$ a. e. $(n \to \infty)$ for some $f_n \in \varphi_n$ and $f \in \varphi$.

4. The ergodic theorem

We have defined all objects which give us possibility to formulate vector lattice variant of the ergodic theorem.

Theorem 4.1:

Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, $T: \Omega \to \Omega$ be a measure preserving mapping and V be a σ-complete vector lattice. For any $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$:

(i) $\varphi \circ T \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$

(ii) $\frac{1}{k} \sum_{i=0}^{k-1} \varphi \circ T^i \longrightarrow E(\varphi|\mathcal{S}_0)$ a. e. when $k \to \infty$

(iii) $\int_{\Omega} | E(\varphi | \mathcal{S}_0) - \frac{1}{k} \sum_{i=0}^{k-1} \varphi \circ T^i | \, dP \to 0$ when $k \to \infty$

where $\mathcal{S}_0$ is the σ-subalgebra of all almost T-invariant sets $A \in \mathcal{S}$.

Proof:

(i) The set of all $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ such that $\varphi \circ T \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ is a linear subspace of $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ which contains $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$ and is closed with respect to u. a. e. convergence. So, it must coincide with $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$.

(ii) Let $\mathcal{M}$ be a set of all $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ such that

$$\frac{1}{k} \sum_{i=0}^{k-1} \varphi \circ T^i \longrightarrow E(\varphi | \mathcal{S}_0) \quad \text{a. e. when} \quad k \to \infty .$$

Obviously, $\mathcal{M}$ is a vector subspace of $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$. The inclusion $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V) \subset \mathcal{M}$ follows easily from the ergodic theorem for real functions. If we show that $\mathcal{M}$ is closed with respect to u. a. e. convergence we prove the equality $\mathcal{M} = \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$.
Let $\{\varphi_n\} \subset \mathcal{M}$ be a sequence which converges to $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$ uniformly almost everywhere. Obviously $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$. For any n we have :

$$\frac{1}{k} \sum_{i=0}^{k-1} \varphi_n \circ T^i \to E(\varphi_n | \mathcal{S}_0) \text{ almost everywhere when } k \to \infty .$$

Theorem 3.2 implies $E(\varphi_n | \mathcal{S}_0) \to E(\varphi | \mathcal{S}_0)$ u. a. e.
Let f_n, f, g_n, g be representants of the equivalence classes φ_n , φ , $E(\varphi_n | \mathcal{S}_0)$, $E(\varphi | \mathcal{S}_0)$ respectively. Since $f_n \to f$, $g_n \to g$ u. a. e. there are $A_1, A_2 \in \mathcal{S}$ and $\{d_n\}, \{e_n\} \subset V$ such that :
$P(A_1) = P(A_2) = 0$

$$\forall \omega \in \Omega - A_1, \ \forall n: |f_n(\omega) - f(\omega)| \le d_n$$
$$\forall \omega \in \Omega - A_2, \ \forall n: |g_n(\omega) - g(\omega)| \le e_n$$

and $d_n \searrow 0$, $e_n \searrow 0$ when $n \to \infty$.

Because $\frac{1}{k} \sum_{i=0}^{k-1} f_n(T^i(\omega)) \to g_n(\omega)$ a. e. when $k \to \infty$ for any n,

there are sets $B_n \in \mathcal{S}$ such that :

$$\forall n \ \forall \omega \in \Omega - B_n: \ \frac{1}{k} \sum_{i=0}^{k-1} f_n(T^i(\omega)) \longrightarrow g_n(\omega) \quad \text{when} \quad k \to \infty \text{ and}$$

$P(B_n) = 0$. Put $B = A_1 \cup A_2 \cup \bigcup_{n=1}^{\infty} B_n$. Obviously $P(B) = 0$.
Let $A = \bigcup_{i=0}^{\infty} T^{-i}(B)$. Then $P(A) = 0$, $A_1 \cup A_2 \cup \bigcup_{n=1}^{\infty} B_n \subset A$ and $T^{-i}(A) \subset A$ for any i . Take fixed $\omega \in \Omega - A$. Then for any n :

$\forall i: |f_n(T^i(\omega)) - f(T^i(\omega))| \leq d_n$

$|g_n(\omega) - g(\omega)| \leq e_n$

$\frac{1}{k} \sum_{i=0}^{k-1} f_n(T^i(\omega)) \longrightarrow g_n(\omega)$ when $k \to \infty$.

It means that for any n and k the following inequalities hold:

$$|g(\omega) - \frac{1}{k} \sum_{i=0}^{k-1} f(T^i(\omega))| \leq |g(\omega) - g_n(\omega)| +$$
$$+ |g_n(\omega) - \frac{1}{k} \sum_{i=0}^{k-1} f_n(T^i(\omega))| + \frac{1}{k} \sum_{i=0}^{k-1} |f_n(T^i(\omega)) - f(T^i(\omega))| \leq$$
$$\leq e_n + |g_n(\omega) - \frac{1}{k} \sum_{i=0}^{k-1} f_n(T^i(\omega))| + d_n \quad .$$

Denote $a_n = e_n + d_n$, $b_{n,k} = |g_n(\omega) - \frac{1}{k} \sum_{i=0}^{k-1} f_n(T^i(\omega))|$ and $c_k = \bigwedge_{n=1}^{\infty} (a_n + b_{n,k})$. Then $|g(\omega) - \frac{1}{k} \sum_{i=0}^{k-1} f(T^i(\omega))| \leq c_k$ and

$c_k \to 0$ by Lemma 2.3 .

The proof of (ii) is complete.

(iii) Since $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ is a sublattice of $\mathcal{F}(\Omega, \mathcal{S}, P, V)$ the integrals

$\int_\Omega | E(\varphi \mid \mathcal{S}_0) - \frac{1}{k} \sum_{i=0}^{k-1} \varphi \circ T^i | \, dP$ are defined and we may repeat

the proof of (ii) (without using the representants f_n, f, g_n, g) .

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