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Maria Cristina Pedicchio

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On n-Permutable Equivalence Relations in Regular Categories

M. C. PEDICCHIO*)

Italy

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Introduction

This note intends to be a survey on the n-commutativity for equivalence relations in regular categories where, n-commutativity means that for any relations R and S on X, it is

$$\underbrace{RSRS \ldots}_{n \text{ times}} = \underbrace{SRSR \ldots}_{n \text{ times}}$$

The most interesting cases in this sequence of conditions are those corresponding to $n = 2$ and $n = 3$; in particular the 2-commutativity (or simply commutativity) is the base to develop the theory of Mal'cev categories.

Mal'cev categories are the categorial generalization of Mal'cev varieties and have been introduced in [2], in relation with the problem of finding an appropriate context to develop "non commutative homological algebra".

As an abelian category is an exact category that is additive, similarly a Mal'cev category is an exact category with permutable equivalence relations. We showed that with this definition basic results of homological algebra remain valid.

§1 Mal'cev varieties

We first recall some classical results in the case of algebras. For more details see [8] or [14].

Let A be an algebra; by a congruence relation on A we mean an equivalence relation R on A such that it is a subalgebra of $A \times A$. We will write aRb to denote that $(a, b) \in R$.

Clearly if $(R_i)_{i \in I}$ are congruences on A, the set-theoretical intersection $\bigwedge R_i$ is

*) Dipartimento di Scienze Matematiche, Universitá degli Studi di Trieste, I-34100 Trieste, Italia.

still a congruence relation. Moreover, if we take the join $\bigvee R_i$ of the R_i as equivalence relations, i.e. the transitive closure of the set-theoretical union, then it is again a congruence. So congruences on A form a lattice, that we denote by $Congr(A)$.

Clearly $Congr(A)$ is a complete sublattice of the lattice of all equivalence relations on A; further $Congr(A)$ is an algebraic lattice. Consider now the product RS of two congruences R, S on A, where $RS = \{(a, b) \mid \exists c \text{ with } aSc \text{ and } cRb\}$.

RS is a subalgebra of $A \times A$, but not necessarily a congruence. If RS is a congruence, then $RS = R \vee S$.

We will say that the congruences on A commute if, for any R, S congruences on A, it holds $RS = SR$.

Varieties satisfying this commutativity property for all algebras, have been at once recognized as a very good framework for many purposes. For example, already in the 40's it was known (Ore) that such algebras admit a unique factorization theorem for direct decomposition of congruences.

The classical result in this area is that of A. I. Mal'cev [13].

Theorem 1.1. *All the algebras of a variety $\mathcal{V}$ have permuting congruences if and only if there is a 3-ary term p of the type of $\mathcal{V}$ such that the equations*

$$p(x, y, y) = x \text{ and } p(x, x, y) = y$$

are valid in $\mathcal{V}$.

Proof. We give this proof, because it offers the basic ideas which will be developed in the generalized n-cases.

If there is a term p such that these equations are valid in $\mathcal{V}$, then every algebra $A \in \mathcal{V}$ has a term operation p^A obeying the equations.

Suppose R and S congruences on A and let $(a, b) \in SR$ i.e., there exists $c \in A$ with aRc and cSb

$$\begin{array}{ccc} c & \xrightarrow{S} & b \\ R \uparrow & & \uparrow R \\ a & \xrightarrow{S} & p(a, c, b) \end{array}$$

Then, if we consider $d = p(a, c, b)$, we get $a = p(a, b, b)Sp(a, c, b)$ and $p(a, c, b)$ $Rp(a, a, b) = b$, so aSd and dRb, hence $aRSb$.

Conversely, let us suppose that the congruences permute for every algebra of $\mathcal{V}$ and let $A = F_3$ be the free algebra on 3 generators x, y, z. Define f as the endomorphism of A satisfying $f(x) = f(y) = x$, $f(z) = y$, and define g as the endomorphism of A satisfying $g(x) = x$ and $g(y) = g(z) = y$. Take $R = ker\ f$ and $S = ker\ g$; consequently xRy, and ySz, hence $(x, z) \in SR = RS$, so there exists an element $u \in A$ with xSu and uRz. This means that there is a 3-ary term p such that $u = p^A(x, y, z)$.

To see that p has the required properties consider

$$x = g(x) = g(u) = g(p^A(x, y, z)) = p^A(g(x), g(y), g(z)) = p^A(x, y, y).$$

Similarly it follows that $y = p^A(x, x, y)$. These equalities in $F(x, y, z)$ are equivalent to the desired conclusions. □

If the congruences on A permute, then $RS = SR$ implies that RS is a congruence, so $RS = R \vee S$. One of the main consequences of this property is that the lattice *Congr* (A) becomes modular.

Remark 1.2. It has been shown by Jónsson (1953) that, if A is an algebra with commuting congruences, then *Congr*(A) satisfies an identity that is stronger than the modular law, the so called Arguesian identity defined by the following:
consider six elements a_i, b_i $(i = 0, 1, 2)$ of a lattice and form the elements

$$c_0 = (a_1 \vee a_2) \wedge (b_1 \vee b_2)$$

and cyclically, and let

$$c' = c_0 \wedge (c_1 \vee c_2).$$

The inclusion

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leqq (a_1 \wedge (c' \vee a_2)) \vee b_1$$

is called the Arguesian identity.

Varieties with permuting congruences are also called Mal'cev varieties. Terms and operations satisfying the equations in Theorem 1.1 are called Mal'cev terms and Mal'cev operations.

Other conditions equivalent to the commutativity of congruences were found by Findley (1960) [5]; we will consider them in the general context of §2.

Examples 1.3.

1) Groups are a Mal'cev variety with $p(x, y, z) = xy^{-1}z$.
2) Rings and Modules are Mal'cev varieties.
3) Heyting algebras are a Mal'cev variety where p can be given by:

$$p(x, y, z) = ((z \to y) \to x) \bigwedge ((x \to y) \to z)$$

In this situation the lattices of congruences are distributive.

4) Boolean algebras are a Mal'cev variety.
5) Relatively complemented lattices are Mal'cev.
6) Quasi-groups are algebras $(Q, ., \backslash, /)$ with three binary operations obeying the laws:

$$y.(y\backslash x) = x = (x/y).y \text{ and}$$

$$y\backslash(y.z) = z = (z.y)/y.$$

They form a Mal'cev variety with

$$p(x, y, z) = [x/(y\backslash y)]\cdot(y\backslash z)\,.$$

7) It is easy to see that lattices do not have permuting congruences, also if they have distributive congruence lattices.

§2. Mal'cev categories

The problem of studying the categorical generalization of Mal'cev varieties was first considered by J. Meisen [15], but under quite strong conditions on the category.

Our aim is to show that it suffices to consider a regular category to maintain the validity of some algebraic results of Mal'cev varieties. We recall that a *regular category* is a category $\mathscr{E}$ such that:

(1) $\mathscr{E}$ is left exact;

(2) every effective equivalence relation (= kernel pair) has a coequalizer;

(3) regular epimorphisms are stable under pull-backs.

Toposes, categories of algebras and abelian categories are all examples of regular categories.

A *relation* R from X to Y in $\mathscr{E}$, is defined as a subobject $R \to X \times Y$. In a regular category the existence of regular images allows us to define the composite of two relations as follows: if $S \to Y \times Z$ is another relation, their composite SR is the image in $X \times Z$ of the pullback of $R \to Y \leftarrow S$. That composition is associative follows from the stability of regular epimorphisms. Then, if we write $R{:}X \to Y$ for $R \to X \times Y$, we obtain a category $Rel(\mathscr{E})$ of relations of $\mathscr{E}$, whose identities are diagonals $X \to X \times X$.

Note that $Rel(\mathscr{E})$ has the following additional structure:

(i) a *local order* preserved by composition, which has finite intersection.

(ii) an *involution* $(\,)^\circ : Rel(\mathscr{E})^\circ \to Rel(\mathscr{E})$ which is the identity on objects and which preserves the local order.

(iii) an *embedding* $\mathscr{E} \to Rel(\mathscr{E})$ which associates to every arrow $f : X \to Y$ in $\mathscr{E}$ its graph $|f| \to X \times Y$.
We will write f for $|f|$, and call such relations *maps*.

We also recall the following properties:

(a) An arrow $R{:}X \to Y$ of $Rel(\mathscr{E})$ is a map if and only if it has a right adjoint in the bicategory $Rel(\mathscr{E})$, that is $RR^\circ \leqq 1$ and $R^\circ R \geqq 1$.

(b) Composition with maps on the right distributes over interesection:

$$(R \cap S)f = Rf \cap Sf\,.$$

(c) An arrow f is a mono in $\mathscr{E}$ if and only if $f^\circ f = 1$ and a regular epi if and only if $ff^\circ = 1$.

(d) For every relation $R : X \to Y$, there exists a pair of maps f and g such that

$$R = gf^\circ, \quad f^\circ f \cap g^\circ g = 1 .$$

Such a pair is essentially unique and is called a tabulation of R.

(e) The regular image of a map $f : X \to Y$ viewed as a subobject of Y, is characterized as a map i such that

$$i^\circ i = 1, \; ii^\circ = ff^\circ .$$

(f) In $Rel(\mathscr{E})$ we can apply the following Freyd modular laws:

$$RS \cap T \leqq R(S \cap R^\circ T)$$
$$RS \cap T \leqq (R \cap TS^\circ)S .$$

It is clear that in $Rel(\mathscr{E})$ an equivalence relation (or congruence) can be simply described as a relation $R : X \to X$, such that $1_X \leqq R$, $R^\circ \leqq R$ and $RR \leqq R$.

Then, we have the following:

Proposition 2.1. *For a regular category $\mathscr{E}$, the following conditions are equivalent:*

(1) *composition of equivalence relations is commutative;*

(2) *composition of effective equivalence relations is commutative;*

(3) *every relation $R : X \to Y$ is difunctional, that is $RR^\circ R = R$;*

(4) *every reflexive relation is an equivalence;*

(5) *every reflexive relation is symmetric;*

(6) *every reflexive relation is transitive;*

(7) *the composite of two equivalence relations is an equivalence relation.*

Proof.

(1) $\Rightarrow$ (2): trivially.

(2) $\Rightarrow$ (3): if $R = gf^\circ$ is a representation of R, then
$RR^\circ R = gf^\circ fg^\circ gf^\circ = gg^\circ gf^\circ ff^\circ = gf^\circ = R$ since $f^\circ f$ and $g^\circ g$ are effective equivalence relations and maps are difunctional.

(3) $\Rightarrow$ (4): if $1 \leqq R$, the $R^\circ = 1R1^\circ \leqq RR^\circ R = R$ and $RR = R1R \leqq RR^\circ R = R$.

(4) $\Rightarrow$ (5): trivial.

(5) $\Rightarrow$ (7): $1 \leqq SR$ implies $SR = (SR)^\circ = R^\circ S^\circ = RS$, hence $SRSR = SSRR =$
$= SR$.

(7) $\Rightarrow$ (1): similar to (5) $\Rightarrow$ (6).

(4) $\Rightarrow$ (6): trivial.

(6) $\Rightarrow$ (1): $RS = 1RS1 \leqq SRSR = SR$ since $1 \leqq SR$. □

Remark 2.2.

In an algebraic context, difunctionality can be described geometrically in the following way:
$R \leqq A \times A$ is difunctional, if and only if for any "square" in $A \times A$, when three vertixes are in R, then the fourth is in R too.
This idea has been made more precise by Carboni [3]. He showed, that for pretopoi difunctional relations correspond to exact square.

Corollary 2.3. *For a regular category $\mathscr{E}$, satisfying any of the conditions of Proposition* 2.1, *the semilattice of equivalence relations on any object X is a modular lattice.*

Proof. If R and S are equivalence relations on X, then $RS = SR$ is also an equivalence relation, and $R \leqq RS$, $S \leqq RS$.
If T is another equivalence relation on X with $R \leqq T$ and $S \leqq T$, then $RS \leqq TT \leqq$ $\leqq T$, so $RS = R \vee S$.
If $R \leqq T$, Freyd modular laws imply $(R \vee S) \wedge T = RS \wedge T \leqq R(S \wedge R^\circ T) \leqq$ $\leqq R(S \wedge T^\circ T) \leqq R(S \wedge T) = R \vee (S \wedge T)$. □

All the examples of regular categories mentioned before have the following additional property:

Definition 2.4. An *exact category* is a regular category in which every equivalence relation is effective.
It is easy to see that $\mathscr{E}$ is exact if and only if $Rel(\mathscr{E})$ satisfies the following axiom: for every equivalence relation R, there is a map p such that $p^\circ p = R$ and $pp^\circ = 1$.
Now, we can give the following:

Definition 2.5. A *Mal'cev category* is an exact category such that equivalence relations commute.

Examples of Mal'cev categories 2.6.

(1) Models of a Mal'cev variety in any exact category;
(2) abelian categories;
(3) inf-semilattices viewed as ordered categories;
(4) any slice of a Mal'cev category;
(5) any functor category $\mathscr{E}^{\mathscr{C}}$ if $\mathscr{E}$ is Mal'cev;
(6) the dual of any topos.

§3. An application

Mal'cev varieties have been widely studied in relation with different problems (see for example [18]); Lambek in [12] pointed out that they should offer a good context to develop basic tools of homological algebra, thus serving as a non-additive generalization of modules.

Our aim is to show that, while proving these results for Mal'cev varieties, one never uses the fact that the category is varietal, but just the semantical condition of having permuting congruences.

This remark suggests to define a notion of Mal'cev category as given in §2, to obtain a categorical non-additive generalization of abelian eategories.
So, as

$$\text{abelian} = \text{exact} + \text{additive}$$

now, we have

$$\text{Mal'cev} = \text{exact} + RS = SR, \ \forall\, R, S \text{ congruences.}$$

We have proved that with this definition basic facts of diagram chasing, like the Snake Lemma, remain true.
We recall the main results; for more details see [2].
In modul theory, we have the following.

Lemma 3.1. *Consider the diagram*

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(1)"] & B \arrow[r, "\mu"] \arrow[d, "\beta"] \arrow[dr, phantom, "(2)"] & C \arrow[d, "\gamma"] \\ D \arrow[r, "\lambda'"'] & E \arrow[r, "\mu'"'] & F \end{tikzcd}$$

and suppose that

(*i*) *the rows are exact*;
(*ii*) *the two squares are commutative*;

then,

$$Im(1) \cong Ker(2), \ \textit{where}$$

$$Im\,(1) = \frac{Im\,\beta \cap Im\,\lambda'}{Im\,\beta\lambda}, \quad ker\,(2) = \frac{Ker\,(\gamma\mu)}{Ker\,\beta \ \vee \ ker\,\mu}. \quad \square$$

This result is extremely useful in the diagram chasing since it allows us to replace elements by squares, and a good part of homological algebra can be developed in this way.

We show that this Lemma can be also considered in a Mal'cev category. For any regular $\mathscr{E}$, we say that the following rows

(I) $$A \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} B \xrightarrow{\mu} C$$

(II) $$D \xrightarrow{\lambda} E \underset{\mu_2}{\overset{\mu_1}{\rightrightarrows}} F$$

are exact, if and only if

(I) $Im(\lambda_1, \lambda_2) = ker\ \mu$, which in the category of relations is

$$\mu^\circ\mu = \lambda_1\lambda_2^\circ, \text{ and}$$

(I) $Im\ \lambda = Ker\ (\mu_1, \mu_2)$, (where *Ker* denots the equalizer), which in the category of relations is

$$\lambda\lambda^\circ = 1 \cap \mu_2^\circ\mu\,.$$

Theorem 3.2. *Let $\mathscr{E}$ be a Malcev category and consider the following diagram:*

$$\begin{tikzcd}
A \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\alpha_2"] \arrow[dr, phantom, "(1)"] & B \arrow[r, "\mu"] \arrow[d, "\beta"] \arrow[dr, phantom, "(2)"] & C \arrow[d, shift right, "\gamma_1"'] \arrow[d, shift left, "\gamma_2"] \\
D \arrow[r, "\lambda"'] & E \arrow[r, shift left, "\mu_1"] \arrow[r, shift right, "\mu_2"'] & F
\end{tikzcd}$$

If the rows are exact, and the four squares commute, then

$$Im\ (1) \cong Ker\ (2), \textit{ where}$$

$$Im\ (1) = \frac{Im\ \beta \cap Im\ \lambda}{Im\ (\beta\lambda_1, \beta\lambda_2)}; \quad Ker\ (2) = \frac{Ker\ (\gamma_1\mu, \gamma_2\mu)}{Ker\ \mu \vee Ker\ \beta}. \quad \square$$

The use of the equational calculus of relations gives a very useful tool to prove the Theorem, and also to clarify the exact nature of the hypothesis.

It is clear that this result reduces to the classical situation of modules, by taking the second maps of each pair to be trivial.

We applied the theorem to show the validity of the Snake Lemma in Malcev categories (again see [2] for the complete details).

Similarly, many other results of varieties, can be interpreted in this general context.

§4. On the n-commutativity

The original proof of Theorem 3.2 in the case of groups (Lambek [11]), was based on a classical Isomorphism Theorem due to Goursat (1889 [7]), saying that if $R : A \to B$ is relation in Group, then

$$\frac{Dom\ R}{R^\circ R} \cong \frac{Im\ R}{RR^\circ};$$

the same argument was also used by Lambek [12] in the context of Malcev varieties.

Now, if $\mathscr{E}$ is an exact category and $R : X \to Y$ a relation in $\mathscr{E}$, we can define the domain of R as the map $i : U \to X$ such that $i^\circ i = 1$ and $ii^\circ = 1 \wedge R^\circ R$, and the image of R as the map $\gamma : V \to Y$ such that $\gamma^\circ\gamma = 1$ and $\gamma\gamma^\circ = 1 \wedge RR^\circ$. Then it is possible to show that the Goursat Isomorphism Theorem holds in $\mathscr{E}$ if and only if:

(G) for any relation $R : X \to Y$, it is $RR^\circ RR^\circ = RR^\circ$ (see [2]).

We will call this axiom (G) the Goursat axiom and an exact category verifying (G) a Goursat category.
Trivially, $\mathscr{E}$ Mal'cev implies $\mathscr{E}$ is Goursat.
It is possible to characterize axiom (G) in terms of 3-commutativity for the composition of equivalence relations, giving a result similar to Proposition 2.1.

Furthermore these Mal'cev and Goursat situations are just the first two cases of the following general setting:
Given two equivalence relations R and S on an object X of a regular category $\mathscr{E}$, we have on X the increasing sequence of relations:

$$1 \leqq R \leqq RS \leqq RSR \leqq RSRS \leqq \dots, \tag{4.0}$$

which we denote by

$$(R, S)_0 \leqq (R, S)_1 \leqq (R, S)_2 \leqq (R, S)_3 \leqq \dots .$$

For a general relation $P : X \to Y$, we use a similar notation, writing:

$$(P, P^\circ)_1, (P, P^\circ)_2, (P, P^\circ)_3, (P, P^\circ)_4, \dots$$

for the terms of the sequence:

$$P, PP^\circ, PP^\circ P, PP^\circ PP^\circ, \dots .$$

Then, the following holds:

Proposition 4.1. *For a regular category $\mathscr{E}$ and for any $n \geqq 2$, the following conditions are equivalent*:

(a) *for any relation $P : X \to Y$ in $\mathscr{E}$, we have $(P, P^\circ)_{n+1} = (P, P^\circ)_{n-1}$;*
(b) *for any reflexive relation $E : X \to X$ in $\mathscr{E}$, $(E, E^\circ)_{n-1}$ is an equivalence relation;*
(c) *for any such reflexive relation we have $(E, E^\circ)_{n-1} = (E^\circ, E)_{n-1}$;*
(c′) *for any such reflexive relation we have $(E, E^\circ)_{n-1}\,(E, E^\circ)_{n-1} = (E, E^\circ)_{n-1}$;*
(d) *for all equivalence relations $R,S : X \to X$ in $\mathscr{E}$, we have $(R, S)_n = (S, R)_n$;*
(e) *for all effective equivalence relations R and S on X, we have $(R,S)_n = (S, R)_n$;*
(f) *for all equivalence relations R, S on X we have $(R, S)_{n+1} = (R, S)_n$;*
(g) *for all such equivalence relations, we have $(R, S)_m = (R, S)_n$ for all $m \geqq n$ that is, the sequence $(R, S)_k$ is stationary at $k = n$;*
(h) *for all such R, S, the relation $(R, S)_n$ is an equivalence relation.*

Proof. See [10] for the proof, except for condition (c′) that is not given in that paper.
In this case it is trivial that (b) $\Rightarrow$ (c′).
Conversely, we show that (c′) $\Rightarrow$ (f).

If R and S are equivalence, take $E = RS$ then, since $(E, E^{\circ})_{n-1} = (R, S)_n$, we get $(R, S)_n (R, S)_n = (R, S)_n$, so (f) follows by (4.0). $\square$

We say that $\mathscr{E}$ verifies the condition C_n (n commutativity) if it satisfies one of the conditions of Proposition 4.1; here $n \geqq 2$, the proposition above being false for $n = 1$.
Of course C_n implies C_m for $m \geqq n$, and if $\mathscr{E}$ is exact, then C_2 corresponds to the case of Mal'cev categories and C_3 to the case of the Goursat categories.

Corollary 4.2. *If $\mathscr{E}$ satisfies C_n, the ordered set $\mathscr{E}q(X)$ of equivalence relations on X admits a join given by $R \bigwedge S = (R, S)_n$.*

Proof. Similar to the case $n = 2$.
Moreover, we also get:

Corollary 4.3. *If $\mathscr{E}$ satisfies C_3, the lattice $\mathscr{E}q(X)$ is modular, for any $X \in \mathscr{E}$.*

Proof (see [10]. You must apply Freyd modular laws twice. $\square$

If the regular category $\mathscr{E}$ is a variety of algebras, properties concerning n-commutativity of congruences have been widely studied (see [8], ([14]).
One of the main result in this direction shows that n-commutativity can be described in terms of operations and "Mal'cev like" conditions.

Proposition 4.4. [9]. *All the algebras in a variety $\mathscr{V}$ have n-permutable congruences if and only if, there are 3-ary terms $p_0, p_1, \ldots, p_n$ of the type of $\mathscr{V}$, such that $p_0(x, y, z) = x$, $p_n(x, y, z) = z$, $p_i(x, x, z) = p_{i+1}(x, z, z)$ $i < n$, hold in $\mathscr{V}$.*
From universal algebra we also get a very interesting list of examples:

(1) An implication algebra is an algebra $(A, \rightarrow)$ with one binary operation satisfying the laws:

$$(x \rightarrow y) \rightarrow x = x$$

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$$

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$$

Implication algebras are congruences 3-permutable but not 2-permutable [16].

(2) A right-complemented semigroups is an algebra $(A, ., *)$ with two binary operations satisfying the identities

$$x.(x * y) = y.(y * x)$$

$$(x . y)*z = y*(x * z)$$

$$x . (y * y) = x$$

Right complemented semigroups have 3-permutable congruences [16].

(3) Let k be an integer greater than 1, and let C_k denote a k-elements chain constructed as a lattice, then C_k has k-permuting congruences but does not have $(k\text{-}1)$ permuting congruences.

(4) Clearly models of algebras of type (1) and (2), in any exact category, give examples of Goursat categories.

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