Charles Stegall

More facts about conjugate Banach spaces with the Radon-Nikodym property, II.

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 32 (1991), No. 2, 47--54

Persistent URL: http://dml.cz/dmlcz/701967

More Facts about Conjugate Banach Spaces with the Radon-Nikodym Property, II

C. STEGALL*)

Austria

Received 11 March 1991

We extend the results of [*S0*] by proving: if X is an Asplund space (respectively, X is a subspace of a gsg space) and K is a Corson compact then any operator from X to $C(K)$ interpolates through a Banach space Y such that Y is both Asplund and hereditarily weakly compactly generated (respectively, Y is wcg). The techniques are much easier than those of [*S0*] and yield stronger results *e. g.* if K is a Corson compact that is the continuous image of a so called Radon-Nikodym compact then K is an Eberlein compact.

We use the same terminology as in [S8] and [S0]. Let Y be an Asplund space. In [S0] we efficiently show (motivated by [F] and [R]) that there exist increasing filters of norm closed and linear subspaces of Y and Y^*

$$\mathbf{Y} = \{Z \subseteq Y : Z \text{ is normed closed, linear, and } Z \text{ norms } Z^\#\}$$

$$\mathbf{Y}^\# = \{Z^\# \subseteq Y^* : Z^\# \text{ is normed closed, linear, and } Z^\#|Z \text{ is onto}\}\,.$$

Moreover, if $W \subseteq Y$, and $V \subseteq Y^*$ and both V and W have the same norm density then there exists $Z \in \mathbf{Y}$ of the same density as W, $W \subseteq Z$, $V \subseteq Z^\#$, and if $\mathbf{F} \subseteq \mathbf{Y}$ is an increasing filter then

$$\overline{\bigcup_{\mathbf{F}} Z}^{\text{norm}} \text{ is in } \mathbf{Y} \text{ and}$$

$$\left(\overline{\bigcup_{\mathbf{F}} Z}^{\text{norm}}\right)^\# = \overline{\bigcup_{\mathbf{F}} Z^\#}^{\text{norm}} \text{ is in } \mathbf{Y}^\#\,.$$

See [S0]. We need two applications of this construction. Suppose that Y is Asplund and weakly countably determined; equivalently, the unit ball of the dual of Y^* is a Gul'ko compact. Choose any closed linear subspace W of Y. Choose any Z_1 in $\mathbf{Y}$

*) Institut für Mathematik, Johannes Kepler Universität, A-4040 Linz, Austria.

We are grateful to the participants of the Winter School of 1991 held in Srní, ČSFR. Particularly, we thank M. Fabian for reminding us of [*S0*].

such that $d(Z_1) = d(W)$ and $W \subseteq Z_1$. Choose a subspace W_1 that is the image of a contractive projection P_1, $d(Z_1) = d(W_1)$ and $Z_1 \subseteq W_1$ (this is done as in [S4]). Continue, constructing $\{Z_n\} \subseteq \mathbf{Y}$ and subspaces $\{W_n\}$ complemented by the contractive projections $\{P_n\}$, each having the same density as W, such that

$$W \subseteq Z_1 \subseteq W_1 \subseteq \ldots \subseteq Z_n \subseteq W_n \subseteq Z_{n+1} \subseteq W_{n+1} \subseteq \ldots$$

and $P_n^*(W_n) \subseteq Z_{n+1}^*$. It follows that

$$Z = \overline{\bigcup_n Z_n} = \overline{\bigcup_n W_n}$$

is in $\mathbf{Y}$ and is also the image of a contractive projection. The discussion given in [S0] provides sufficient details to see that Y is (hereditarily) weakly compactly generated. This is an efficient way of getting at the results of [F]; in [F] are also the details of constructing equivalent Fréchet smooth norms. A second application is to suppose that K is a Corson compact and $\{f_\alpha : \alpha \in \Gamma\}$ is a point countable (each point in K belongs to at most countably many of the sets $\{|f_\alpha| > 0 : \alpha \in \Gamma\}$) family of continuous functions on K that separates the points of K. Suppose that $X = [\{f_\alpha\}] \subseteq C(K)$ is an Asplund space. Then X is (hereditarily) weakly compactly generated and K is an Eberlein compact. A property of Corson compacta is the following: given any infinite subset Λ_1 of Γ then there exists a set Λ so that $\Lambda_1 \subseteq \Lambda \subseteq \Gamma$, Λ has the cardinality as Λ_1, and there exists a multiplicative projection P on $C(K)$ such that $P(1_K) = 1_K$, $P(f_\alpha) = f_\alpha$ if $\alpha \in \Lambda$ and $P(f_\alpha) = 0$ if $\alpha \notin \Lambda$ (this can be found in [Ne]). Obviously, P leaves X invariant and defines a projection on X. As in the case that X is weakly countably determined we may intertwine the projections as above and show that X is weakly compactly generated. We deduce and generalize the main result of [S0] by using neither the results of [F] nor the results of [**Pol**]; of course, everything depends heavily on [S8]. We require the following technical Lemma, which is based on the following trivial inequalities: $|a \vee c - b \vee c| \leqq |a - b| = a \vee b - a \wedge b$.

Lemma. *If C is an equimeasurable subset of $C(K)$ and f and g are in $C(K)$ then $(C \vee f) - g$ and $\bigcup_n(C \vee 2^{-n}) - 2^{-n}$ are equimeasurable.*

Proof. Suppose that $\{g_1, \ldots, g_n\}$ is an ε net for $C|K_\varepsilon$ where K_ε is a compact subset of K. Observe that

$$|h \vee f - g_i \vee f| \leqq |h - g_i|$$

and we have that $\{g_1 \vee f, \ldots, g_n \vee f\}$ is an ε net for $(C \vee f)|K_\varepsilon$. The next part follows by translation. The final part can be shown by assuming that C is norm compact and then showing that

$$\bigcup_n (C \vee 2^{-n}) - 2^{-n}$$

is relatively norm compact. Choose $\{g_1, \ldots, g_k\}$ an $\varepsilon/4$ net for C. Choose m such that

$$\sum_{m \leqq p} 2^{-p} < \frac{\varepsilon}{8}.$$

We shall show that

$$\{(g_i \vee 2^{-j}) - 2^{-j} : i \leqq k \text{ and } j \leqq m\}$$

is an ε net for

$$\bigcup_n (C \vee 2^{-n}) - 2^{-n}.$$

If $n \leqq m, f \in C$ and g_i are such that $\|f - g_i\| < \varepsilon/4$ then

$$\|((f \vee 2^{-n}) - 2^{-n}) - ((g_i \vee 2^{-n}) - 2^{-n})\| \leqq \|f - g_i\| < \frac{\varepsilon}{4}.$$

For $q > m$ it follows that

$$\begin{aligned} &\|((f \vee 2^{-q}) - 2^{-q}) - ((f \vee 2^{-m}) - 2^{-m})\| \\ &\leqq \sum_{m \leqq p} \|((f \vee 2^{-p}) - 2^{-p}) - ((f \vee 2^{-p-1}) - 2^{-p-1})\| \\ &\leqq \sum_{m \leqq p} 2^{-p} + 2^{-p-1} + \|((f \vee 2^{-p}) - (f \vee 2^{-p-1})\| \\ &\leqq 4 \sum_{m \leqq p} 2^{-p} < \frac{\varepsilon}{2}. \end{aligned}$$

This completes the proof.

Lemma. *Let K a compact Hausdorff space and C a subset of $C(K)$ that is equimeasurable and separates the points of K (in some circles K is called a Radon-Nikodym compact). Let $\mathbf{F}$ be any subset of $C(K)$ that is point countable. There exists a subset $\mathbf{G}$ of $C(K)$ that is both equimeasurable and point countable and the algebra A_1 generated by $\mathbf{G}$ contains $\mathbf{F}$.*

Proof. We may assume that C is a convex and symmetrical subset of the unit ball with $1_K \in C$. It is easy to check that $C \,.\, C$ is also equimeasurable and, by induction. C^n is equimeasurable. It follows that

$$E = \sum_n 2^{-n} C^n$$

is equimeasurable (see [S8]). Observe that the family of all polynomials in $\mathbf{F}$ with rational coefficients is also point countable. Thus, we may assume that $\mathbf{F}$ is a dense subset of the unit ball of A_0, the smallest algebra containing $\mathbf{F}$. Partition $\mathbf{F}$ by

$$\mathbf{F} = \bigcup_n \mathbf{F}_n$$

so that

$$\mathbf{F}_n \subseteq (2^n \,.\, E) + B(0, \tfrac{1}{4})\,.$$

For each $f_{n,\alpha} \in \mathbf{F}_n$ choose $h_{n,\alpha} \in E$ so that $\|2^n h_{n,\alpha} - f_{n,\alpha}\| < \frac{1}{4}$. Define

$$u_{n,\alpha} = (2^n h_{n,\alpha} \vee \tfrac{1}{2}) - \tfrac{1}{2} \geqq 0$$

and observe that

$$2^{-n} u_{n,\alpha} = \left(h_{n,\alpha} \vee \frac{1}{2^{n+1}}\right) - \frac{1}{2^{n+1}} \in \left(E \vee \frac{1}{2^{n+1}}\right) - \frac{1}{2^{n+1}}\,.$$

Thus, $\{2^{-n} u_{n,\alpha} : n, \alpha\}$ is equimeasurable. Fix $k \in K$; if $2^{-n} u_{n,\alpha}(k) > 0$ then $2^n h_{n,\alpha}(k) > \frac{1}{2}$ which implies that $|f_{n,\alpha}(k)| > \frac{1}{4}$. Thus, $\{2^{-n} u_{n,\alpha} : n, \alpha\}$ is also point countable. Fix two points k_0 and k_1 in K so that $k_0|A_0 = a_0$ and $k_1|A_0 = a_1$ are distinct (positive, multiplicative) states of A_0 and choose a function $h \in A_0$ so that

$$-1 = h(a_0) < h(a_1) = 1 = \|h\|\,.$$

There exist n and $f_{n,\alpha} \in \mathbf{F}_n$ so that $\|f_{n,\alpha} - h\| < \frac{1}{4}$. Therefore, $\|2^n h_{n,\alpha} - h\| < \frac{1}{2}$ and it follows that

$$0 = u_{n,\alpha}(a_0) < u_{n,\alpha}(a_1)\,.$$

Let $\mathbf{G} = \{2^{-n} u_{n,\alpha} : n, \alpha\}$. Let A_2 be the algebra generated by $\mathbf{F} \cup \mathbf{G}$ and let A_1 be the algebra generated by $\mathbf{G}$. The arguments above show that the state spaces of A_2 and A_1 are the "same"; in other words, if some $h \in A_0$ separates two points in K then those two points are also separated by some $u_{n,\alpha}$, This means that $A_0 \subseteq A_1 = A_2$. The algebra A_1 is generated by the point countable and equimeasurable set $\{2^{-n} u_{n,\alpha} : n, \alpha\}$; thus, the state space of A_1 is both a Corson compact and a Radon-Nikodym compact.

We point out some trivial consequences of this Lemma. Suppose that $f : K \to T$ is a continuous function from the Radon-Nikodym compact K to the Corson compact T. Then f factors through a space S that is both a Radon-Nikodym and a Corson compact. If follows from [S0] that S is an Eberlein compact; however, the main result of [S0] can be trivially deduced from the Lemma and the remarks at the beginning of the paper. The following is both stronger and easier to prove.

Theorem. *Suppose that $U : X \to C(K)$ is an operator, X is an Asplund space and K is a Corson compact. Then the operator factors through a Banach space Z that is both Asplund and weakly k-analytic (it follows that Z is hereditarily weakly compactly generated). Hence, the algebra A generated by $U(X)$ is weakly compactly generated (equivalently, the state space of A is an Eberlein compact).*

Proof. We may assume that $\|U\| = 1$. If we let A be the subalgebra of $C(K)$ generated by $U(X)$ and the constants then the state space of A is also a Corson compact (see below). Thus, without loss of generality, we may assume that $A = C(K)$, or, in other words, that $U(X)$ separates the points of K. Now, we make the first interpolation. Let D be an equimeasurable subset of the unit ball of $C(K)$ that is closed in the simple topology, convex, symmetrical, $\bigcup_n nD$ is norm dense in $C(K)$ and $U(B_X(0,1)) \subseteq D$. There exists $E \subseteq D$ that is point countable and separates the points of K (see the Lemma above). There exists an Asplund space Y and an operator $S : Y \to C(K)$ with the following properties:

(i) $E \subseteq S(B_Y(0,1)) \subseteq B_{C(K)}(0,1)$,

(ii) $[S^{-1}(E)] = Y$ and

(iii) S^{**} is one to one.

This is in [S8]. Since $S^*(C(K))$ is norm dense in Y^* (condition (iii)) and the convex hull of $S^*(K)$ is norm dense in $S^*(C(K))$ (this is because Y^* has the Radon-Nikodym property) it is easy to check that the unit ball L of Y^* is a Corson compact. To see this, fix any $y^* \in Y^*$ and choose a sequence $\{k_n\} \subseteq K$ so that y^* is in the norm closed convex hull of $\{S^*(k_n)\}$. Then, S maps the set $\{y : S(y) \in E \text{ and } |y^*(y)| > 0\}$ one to one into the countable set $\bigcup_n \{f \in E : |f(k_n)| > 0\}$. Consider the canonical operator $J : Y \to C(L)$. This is an isometry and $J(Y)$ is the span of the point countable and point separating family $J(S^{-1}(E))$ of continuous functions. Thus, Y is weakly compactly generated and it follows that the unit ball of Y^* is an Eberlein compact; since S^* is one to one on K it follows that K is an Eberlein compact. Now, we make the second interpolation. There exists an Asplund space Z and operator $T : Z \to C(K)$ with the following properties:

(i) the image of the unit ball of Z contains D,

(ii) $[T^{-1}(D)] = Z$ and

(iii) T^{**} is one to one.

This is in [S8]. We have proved that K is an Eberlein compact and it follows that D is k-analytic in the weak topology; since T^{**} is one to one it follows that $T^{-1}(D)$ is also k-analytic in the weak topology and since $T^{-1}(D)$ spans Z it follows that Z itself is k-analytic in the weak topology (see [T] and [S1] for lots of details). In particular, Z is an Asplund space that is also wcd. We know that Z is hereditarily wcg. Since $U(B_X(0,1)) \subseteq D$ it follows that U factors canonically through Z and this is the desired result. See [Gu], [So] and [S4].

There are hosts of corollaries.

Corollary. *Suppose that X is an Asplund space and $K \subseteq X^*$ is weak* compact and is a Corson compact. Then K is an Eberlein compact. If K is norming then X is hereditarily wcg.*

Proof. Let $T : X \to C(K)$ be the canonical operator. Of course, $T(X)$ separates the points of K. By the Theorem, we know that T factors through a hereditary wcg space, which proves that K is an Eberlein compact. If K is norming, by which we mean that T is an isomorphism, it follows that X is isomorphic to a subspace of a hereditarily wcg space.

Corollary. *A Corson compact which is the image of a Radon-Nikodym compact is an Eberlein compact.*

Proof. The techniques required are the same as those above, except for one gap, which is filled by [**BRW**]. If K is a Corson compact and $C \subseteq C(K)$ is equimeasurable and separates the points of K then there exists an Asplund space X and an operator $T: X \to C(K)$ such that $C \subseteq T(X)$ [S8]; thus, T^* is a homeomorphism on K. Since T factors through a wcg space this proves that K is an Eberlein compact. Do not assume that K is a Corson compact but keep the same hypothesis on $C \subseteq C(K)$ and suppose that A_0 is any subalgebra of $C(K)$ whose state space is a Corson compact. We know that there exists another subalgebra A_1, $A_0 \subseteq A_1$, so that the state space of A_1 is both a Corson compact and a Radon-Nikodym compact; hence the state space of A_1 is an Eberlein compact. The state space of A_0 is the continuous image of the state space of A_1 and hence, is also an Eberlein compact [**BRW**] (see also [Ne] and its references).

Corollary. *Suppose that $U : X \to C(K)$ is an operator, X is a subspace of a gsg space and K is a Corson compact. Then U factors through a wcg space and the subalgebra A of $C(K)$ generated by $U(X)$ is weakly compactly generated (the state space of A is an Eberlein compact). If U is an isomorphism then the unit ball of X^* in the weak* topology is an Eberlein compact (but, in general, X is not wcg).*

Proof. The statement that X is a subspace of a gsg space is equivalent to the unit ball of X^* being the continuous image of a Radon-Nikodym compact [S8]. Thus, $U^*(K)$ is Corson compact and the continuous image of a Radon-Nikodym compact. Thus, $U^*(K)$ is an Eberlein compact and is (homeomorphic to) the state space of A. If U is an isomorphism, then X is isomorphic to a subspace of the wcg algebra A and the result follows. The example of Rosenthal (complete details are in [S8], [S0] and [Ne]) shows that, in general, X need not be wcg but it is weakly k-analytic.

Corollary. *Suppose that $U : X \to Y$ is an operator, the unit ball of X^* is Corson compact and Y is gsg. If $Z = \overline{U(X)}$ then the unit ball of Z^* is an Eberlein compact. If Y is an Asplund space then Z is (hereditarily) wcg.*

Corollary. *If we can decompose the Corson compact K so that $K = \bigcup_n K_n$ and for each n there exists a Radon-Nikodym compact L_n so that K_n is the continuous image of L_n then $C(K)$ is k-analytic in the weak topology (in other words, K is a Talagrand compact).*

Proof. Add [T] and [So] to the ingredients here. Mix well. Do not add water.

Why is the continuous image of a Corson compact also a Corson compact? Our proof of this is quite easy (there are, however, some horrendous proofs in the literature and none as easy as ours). In fact, the Lemma above essentially contains a proof. Suppose that K is a Corson compact and $\{f_\alpha\}$ is point countable, separates the points of K and $1_K \in \{f_\alpha\}$. We may replace $\{f_\alpha\}$ by all polynomials in $\{f_\alpha\}$ with rational coefficients. We may assume that $\{f_\alpha\}$ is norm dense in the unit ball of $C(K)$. Let A be a uniformly closed subalgebra of $C(K)$ containing the constants. Let B be the unit ball of A and for each α choose

$$h_\alpha \in B(f_\alpha, \tfrac{1}{4}) \cap B \text{ if } B(f_\alpha, \tfrac{1}{4}) \cap B \neq \emptyset \text{ and}$$
$$h_\alpha = 0 \text{ otherwise.}$$

Define $u_\alpha = (h_\alpha \vee \frac{1}{2}) - \frac{1}{2}$. Since A is also a lattice it follows that each u_α is in A. Suppose $a \neq a'$ are states of A. There exists $h \in A$ such that $-1 = h(a) < h(a') = = 1 = \|h\|$. There exists f_α such that $\|f_\alpha - h\| < \frac{1}{4}$ and it follows that $\|h - h_\alpha\| < \frac{1}{2}$. Therefore, $0 = u_\alpha(a) < u_\alpha(a')$ and $\{u_\alpha\}$ separates the states of A. Fix a state a of A and $k \in K$ so that $a = k|A$. Then

$$\{a : |u_\alpha(a)| > 0\} \subseteq \{\alpha : |h_\alpha(a)| > \tfrac{1}{2}\} \subseteq \{\alpha : |f_\alpha(k)| > \tfrac{1}{4}\}$$

and the latter set is countable.

We can do somewhat more with the same ideas. Suppose that K is a Corson compact and $\mathbf{U} = \{U_\alpha\}$ is any collection of open subsets of K. Let

$$\mathscr{F} = \{g \in C(K) : \|g\| = 1,\ g \geqq 0 \text{ and } \overline{\{g > 0\}} \subseteq U_\alpha \text{ for some } \alpha\}.$$

Replace B by $\mathscr{F}$, let $\{f_\alpha\}$, $\{h_\alpha\}$ and $\{u_\alpha\}$ be as above. It follows that

(i) $\{\{u_\alpha > 0\}\}$ is point countable,
(ii) $\{\{u_\alpha > 0\}\}$ is subordinate to $\mathbf{U}$,
(iii) $\bigcup\{\{u_\alpha > 0\}\} = \bigcup \mathbf{U}$,
(iv) $\{\{u_\alpha > 0\}\}$ separates the points of $\bigcup\mathbf{U}$ and
(v) each $\{u_\alpha > 0\}$ is an open F_σ.

This improves a result in [Y] that, using the terminology in [**Grl**], Corson compact spaces are hereditarily (with respect to open subsets) metalindelöf; the improvement being conditions (iv) and (v). A topological space is metalindelöf if every open cover has a point countable open refinement. In [**Grl**], there is an interesting converse; if K is a compact space and the complement of the diagonal of K^2 is metalindelöf then K is Corson compact.

References

[AL] Amir, D. and Lindenstrauss, J., The structure of weakly compact sets in Banach spaces, Ann. of Math. 88 (1968), 35—46.

[BRW] Benjamini, Y., Rudin, M. E., and Wage, M., Continuous images of weakly compact subsets of Banach spaces, Pacific Jour. Math. 70 (1977), 309—324.

[DFJP] Davis, W., Figiel, T., Johnson, W. and Pelczynski, A., Factoring weakly compact operators, Jour. Functional Analysis 17 (1974), 311—327.

[F] Fabian, M., Each weakly countably determined Asplund space admits a Fréchet differentiable norm, Bull. Austıal. Math. Soc. 36 (1987), 367—384.

[G] Grothendieck, A., *Produits tensoriels topologiques et espaces nucléaires*, Memoire AMS 16, 1955.

[Gu] Gul'ko, S. P., On the structure of spaces of continuous functions and their complete paracompactness, Russian Math. Surveys 34 (1979), 36—44.

[Gr] Gruenhage, G., A note on Gul'ko compact spaces, Proc. AMS (100), 371—376.

[Gr1] Gruenhage, G., Covering properties on $X^2 \setminus \Delta$, W-sets, and compact subsets of Σ-products, Topology and its Applications 17 (1984), 287—304.

[Ne] Negrepontis, S., Banach Spaces and Topology, in *Handbook of Set Theoretic Topology,* North Holland, Amsterdam, 1984.

[Pol] Pol, R., On pointswise and weak topology in function spaces, Warsaw University, 1984.

[R] Rode, G., Superkonvexität und schwache kompakthiet, Arch. Math. 36 (1981), 62—72.

[So] Sokolov, G. A., On some classes of compact spaces lying in Σ-products, Comm. Math. Univ. Carolinae 25 (1984), 219—231.

[S0] Stegall, C., More Facts about conjugate Banach Spaces with the Radon-Nikodym Property, Acta Universitatis Carolinae-Math. et Phys. 31 (1990), 107—117.

[S1] Stegall, C., *Applications of Descriptive Topology in Functional Analysis,* Universität Linz, 1985.

[S2] Stegall, C., The duality between Asplund spaces and spaces with the Radon-Nikodym property, Israel Jour. Math. 29 (1978), 408—412.

[S3] Stegall, C., Functions of the first Baire class with values in Banach spaces, to appear, Proc. AMS.

[S4] Stelgal, C., A Proof of the Theorem of Amir and Lindenstrauss, Israel Jour. Math. 68 (1989), 185—192.

[S5] Stegall, C., preprint 1987, incorrect.

[S6] Stegall, C., Permutations of facts about dual Banach spaces with the Radon Nikodym property, notes, 1988—89.

[S7] Stegall, C., The Radon-Nikodym property in conjugate Banach spaces, Trans. AMS 106 (1975), 213—223.

[S8] Stegall, C., The Radon-Nikodym property in conjugate Banach spaces II, Trans. AMS 264 (1981), 507—519.

[S9] Stegall, C., The Topology of Certain Spaces of Measures, preprint 1988, to appear in Topology and its Applications.

[S10] Stegall, C., Dunford-Pettis sets, in *General Topology and its Relations to Modern Analysis and Algebra VI,* Z. Frolík, editor, Heldermann, Berlin, 1988.

[T] Talagrand, M., Espaces de Banach faiblement k-analytique, Annals. Math. 110 (1979), 407—438.

[V] Vasak, L., On one generalization of weakly compactly generated Banach spaces, Studia Math. 70 (1981), 11—19.

[Y] Yakolev, N. N., On bicompacta in Σ-products and related spaces, Comment. Math. Univ. Carolinae 21 (1980), 256—283.