Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 34 (1993), No. 2, 107--111

Persistent URL: `http://dml.cz/dmlcz/702000`

On Sums of Darboux Functions

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Received 14 April 1993

Introduction

We consider only real functions defined on the real line. It is well-known that the sum of Darboux functions (i.e. functions with the intermediate value property) need not be Darboux in general. Therefore for any family $\mathscr{F}$ of real functions we can consider the following conditions:

$a(\mathscr{F})$: there exists a Darboux function g such that $f + g$ is Darboux for each $f \in \mathscr{F}$,

$c(\mathscr{F})$: there exists a Darboux function g such that $f + g$ is Darboux for no $f \in \mathscr{F}$.

It was remarked by Lindenbaum [7] that $a(\{f\})$ holds for each real function f (cf)[1], p. 106). This result was generalized by Fast [3] in the following way: $a(\mathscr{F})$ holds for each family $\mathscr{F}$ of functions with card $(\mathscr{F}) \leq 2^{\omega}$. Moreover, a $(\mathscr{F})$ holds for some families $\mathscr{F}$ of functions with card $(\mathscr{F}) = 2^{2^{\omega}}$, e.g. if $\mathscr{F}$ is ether the family of all Lebesgue measurable functions or the family of all functions with the Barire property [8] (obviously $a(\mathscr{F})$ does not hold for the family $\mathscr{F}$ of all real funcions).

Now we shall consider the opposite problem: which families $\mathscr{F}$ of functions satisfy the condition $c(\mathscr{F})$. It is well-known that $c(\{f\})$ holds iff f is non-constant [12] (cf [13] and [2]). P. Komjáth proved recently that $c(\mathscr{F})$ is satisfied for any family $\mathscr{F}$ of nowhere constant continuous functions with (card $\mathscr{F})^{+} < 2^{\omega}$ [5]. Moreover, if we assume that the additivity of the ideal of all first category sets is equal to 2^{ω} then $c(C^{*})$ holds for the family C^{*} of all nowhere constant continuous

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*) Supported by KBN Research Grant 1992-94, No 2 1144 91 01.

functions[1] [6]. It is interesting that $c(\mathcal{F})$ is false for the family $\mathcal{F}$ of all non-constant continuous functions [9]. The purpose of the present paper is the extension of the above theorems.

Notation

We use standard set theory notation. Cardinals are identified with initial ordinals, 2^ω is the cardinal of the continuum. For an ideal $I \subset P(\mathbb{R})$ we define the additivity of I as

$$\operatorname{add}(I) = \inf \{\operatorname{card}(I_0) : I_0 \subset I \text{ and } \bigcup I_0 \notin I\}.$$

All ideals considered in this paper satisfy the following conditions:
(1) $\{x\} \in I$ for each $x \in \mathbb{R}$,
(2) if G is open and $G \in I$ then $G = \emptyset$.

A function f is said to be
nowhere constant iff $\operatorname{int} f^{-1}(y) = \emptyset$ for each $y \in \mathbb{R}$,
quasi-continuous iff $f^{-1}(W) \subset \operatorname{cl} \operatorname{int} f^{-1}(W)$ for each open set W [4],
in $\mathfrak{B}_1^*$ class if in each non-void closed set F there is an open interval (a, b) with $(a, b) \cap F \neq \emptyset$ such that $f|F$ is continuous on $(a, b) \cap F$ [10].

In the same way as P. Komjáth [5] we can prove the following result.

Proposition 1 *If $\mathcal{F}$ is a family of nowhere constant, locally bounded Darboux functions with* $(\operatorname{card} \mathcal{F})^+ < 2^\omega$ *then* $c(\mathcal{F})$ *holds.*

Note that if CH is assumed then Komjáth's theorem does not determine if $c(\mathcal{F})$ holds even for countable families $\mathcal{F}$. However, it is easy to prove the following

Proposition 2 *For each countable family $\mathcal{F}$ of nowhere constant functions the condition $c(\mathcal{F})$ is fulfilled.*

Proof. Let $\mathcal{F} = \{f_n : n < \omega\}$ and let $i_n = [2n, 2n + 1]$ for each $n < \omega$. Since $f_n|I_n$ are non-constant, there exist Darboux functions $g_n : I_n \to \mathbb{R}$ such that $(f_n + g_n)|I_n$ is not Darboux (for all $n < \omega$) [12]. Now if we put

$$g(x) = \begin{cases} g_n(x) & \text{for } x \in I_n,\ n < \omega, \\ \text{linear otherwise,} & \end{cases}$$

then $f + g$ is Darboux for no $f \in \mathcal{F}$. □

[1] Note that this result is independent of **ZFC**. Indeed, during the last Winter School on Abstract Analysis in Poděbrady, prof. Todorčević informed me that recently prof. J. Steprans from Toronto constructed a model for **ZFC** $+ \neg c(C^*)$.

Proposition 3 *Assume that $\mathfrak{T}$ is an ideal and $\mathcal{F}$ is a family of locally bounded functions whith all level sets in $\mathfrak{T}$. If* $\mathrm{card}(\mathcal{F}) < \mathrm{add}(\mathfrak{T})$ *then the following condition is satisfied*
$c^*(\mathcal{F})$: *there exists a Darboux function g such that $(f+g)|I$ is Darboux for no $f \in \mathcal{F}$ and no non-degenerate interval $I \subset \mathbb{R}$.*

Proof. Let B be a countable base for the Euclidean topology in $\mathbb{R}$. Arrange all elements of $\mathcal{F} \times B \times \mathbb{R}$ in a sequence $(f_\alpha, I_\alpha, r_\alpha)_{\alpha<2^\omega}$, and select by transfinite induction a sequence $(x_\alpha)_{\alpha<2^\omega}$ such that:
(1) if $x_\alpha = x_\beta$ then $\alpha = \beta$,
(2) $f(x_\alpha) \neq -r_\alpha$ for each $f \in \mathcal{F}$ and $\alpha < 2^\omega$.
Such a choice is possible because, by hypothesis,

$$(x \in I_\alpha : f(x) = -r_\alpha \text{ for some } f \in \mathcal{F}\} \subset \bigcup_{f \in \mathcal{F}} f^{-1}(-r_\alpha) \in \mathfrak{T}$$

Since $\mathrm{card}(\mathcal{F}) < 2^\omega$, for each $x \in \mathbb{R}$ there exists $y(x)$ such that $y(x) \neq -f(x)$ for all $f \in \mathcal{F}$. Now we define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} r_\alpha & \text{for } x = x_\alpha,\ \alpha < 2^\omega, \\ y(x) & \text{otherwise.} \end{cases}$$

Since g takes every value on every non-degenerate interval, it is Darboux. Fix $f \in \mathcal{F}$ and a non-degenerate interval I. Then $(f+g)(x) \neq 0$ for each $x \in I$. Since f is locally bounded and $g|I$ is unbounded, $(f+g)(x_1) < -1$ and $(f+g)(x_2) > 1$ for some $x_1, x_2 \in I$, so $f+g$ does not have the Darboux property. □

Proposition 4 *Assume that $\mathfrak{T}$ is an ideal and $\mathcal{F}$ is a family of functions with all level sets in $\mathfrak{T}$. If* $\mathrm{card}(\mathcal{F}) = \mathrm{add}(\mathfrak{T}) = 2^\omega$ *then $c^*(\mathcal{F})$ is satisfied.*

Proof. Let $\mathcal{B}$ be a countable base for the Euclidean topology in $\mathbb{R}$. Arrange all reals in a sequence $(x_\alpha)_{\alpha<2^\omega}$ and all elements of $\mathcal{F} \times \mathcal{B} \times \mathbb{R}$ in sequence $(f_\alpha, I_\alpha, r_\alpha)_{\alpha<2^\omega}$. We select by transfintite induction sequences $(t_{\alpha,i})_{\alpha<2^\omega, i<3}$, $(y_\alpha)_{\alpha<2^\omega}$ and $(z_\alpha)_{\alpha<2^\omega}$ such that:
(1) if $t_{\alpha,i} = t_{\beta,j}$ then $\alpha = \beta$ and $i = j$,
(2) $f_\alpha(t_{\beta,0}) \neq y_\alpha - r_\beta$ for each $\alpha, \beta < 2^\omega$ and $i \in \{1, 2\}$,
(3) $y_\alpha - f_\alpha(t_{\alpha,i}) + (-1)^i \neq y_\beta - f_\beta(t_{\alpha,i})$ for $\alpha, \beta < 2^\omega$, $i \in \{1, 2\}$,
(4) $f_\alpha(x_\beta) + z_\beta \neq y_\alpha$ for each $\alpha, \beta < 2^\omega$.

Assume that for some γ there are chosen sequences $(t_{\alpha,i})_{\alpha<\gamma, i<3}$, $(y_\alpha)_{\alpha<\gamma}$ and $(z_\alpha)_{\alpha<\gamma}$ for which the conditions (1)–(4) are fulfilled.
Since $A_\gamma = \{t_{\beta,i} : \beta < \gamma,\ i < 3\} \cup \bigcup_{\beta<\gamma} f_\beta^{-1}(y_\beta - r_\gamma) \in \mathfrak{T}$, we can choose distinct points $t_{\gamma,0}, t_{\gamma,1}, t_{\gamma,2} \in I_\gamma \setminus A_\gamma$.

Since $B_\gamma = \{f_\gamma(t_{\beta,0}) + r_\beta : \beta \leqq \gamma\} \cup \{f_\gamma(x_\beta) + z_\beta : \beta < \gamma\} \cup \{y_\beta + (-1)^i - f_\beta(t_{\beta,i}) + f_\gamma(t_{\beta,i}) : \beta < \gamma, \quad i \in \{1, \quad 2\}\} \cup \{y_\beta - f_\beta(t_{\gamma,i}) + f_\gamma(t_{\gamma,i}) - (-1)^i : \beta < \gamma,\ i \in \{1, 2\}\}$ has cardinality less than 2^ω, we can choose $y_\gamma \notin B_\gamma$.

Since $C_\gamma = \{y_\beta - f_\beta(x_\gamma) : \beta < \gamma\}$ has cardinality less than 2^ω, we choose $z_\gamma \notin C_\gamma$.

Now it is easy to verify that sequences chosen in this way satisfiy the conditions (1)−(4). Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} r_\alpha & \text{for } x = t_{\alpha,0},\ \alpha < 2^\omega, \\ r_\alpha + (-1)^i - f_\alpha(x) & \text{for } x = t_{\alpha,i},\ \alpha < 2^\omega,\ i \in \{1, 2\}, \\ z_\alpha & \text{for } x = x_\alpha \neq t_{\beta,i},\ \alpha, \beta < 2^\omega,\ i < 3. \end{cases}$$

Since g takes every value on every non-degenerate interval, it is Darboux. Fix $f \in \mathbb{F}$ and a non-degenerate interval I. Then there exists $\alpha < 2^\omega$ such that $f = f_\alpha$ and $I_\alpha \subset I$. Since $t_{\alpha,1}, t_{\alpha,2} \in I_\alpha$, $(f_\alpha + g)(t_{\alpha,1}) = y_\alpha - 1 < y_\alpha < (f_\alpha + g)(t_{\alpha,2})$ and $(f_\alpha + g)(x) \neq y_\alpha$ for $x \in I_\alpha$, $(f_\alpha + g)|I$ is not Darboux. □

Now we can consider relationships between $c(\mathscr{F})$ and $c^*(\mathscr{F})$. Obviously, the following fact holds.

Fact 1 *Let $\mathscr{F}_0$, $\mathscr{F}$ be families of real functions such that for every $f \in \mathscr{F}$ there exist $f_0 \in \mathscr{F}_0$ and an interval I such that $f|I = f_0|I$. Then $c^*(\mathscr{F}_0)$ implies $c(\mathscr{F})$.*

Let $\mathfrak{I}$ denote the ideal of all first category sets. Since level sets of nowhere constant quasi-continuous or $\mathfrak{B}_1^*$ functions belong to $\mathfrak{I}$, we can apply Propositions 3 and 4 to families $\mathscr{F}$ of such functions. By Fact 1, we can omit the assumption that functions in $\mathscr{F}$ are locally bounded.

Finally, note that in the same way we can prove anologous results in more general cases, e.g. for real functions defined on dense-in-itself Polish spaces. In this case we understand the Darboux property in the Pawlak sense. Recall that a function $f: X \to Y$ has the Pawlak property iff the image of every arc in X is connected [11].

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