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Formal Languages in Dynamical Systems

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We treat here the interrelation between formal languages and those dynamical systems that can be described by cellular automata (CA). There is a well-known injective map which identifies any CA-invariant subshift with a formal language in a natural way. However, in the special case of a symbolic dynamics, i.e. where the CA is just the shift map, one gets a stronger result: the identification map can be extended to a functor which additionally maps topological conjugacies between subshifts to generalized sequential machines between languages. The Chomsky hierarchy measuring the complexity of formal languages can be transferred via this functor to the symbolic dynamics and proves to be a conjugacy invariant. After reviewing some results of the complexity of CA-invariant subshifts, special attention is given to a new kind of invariant subshift: the trapped set, which originates from the theory of chaotic scattering and for which one can study complexity transitions. Required concepts and definitions are repeated, so that this article is essentially self-contained.

1 Introduction

This article is organized as follows: Section 2 treats cellular automata and in particular the shift map as dynamical systems on subshifts, i.e. closed and shift invariant subsets of sequence spaces (cf. [2, 4]). Besides some standard CA-invariant subshifts a new kind (the trapped set) is defined, which originates from chaotic scattering theory. To this purpose formal multiscattering systems (FMS) are introduced and two examples, the truncated double horseshoe and the truncated sawtooth are given (cf. [13, 14]). Section 3 is a short introduction in formal language theory (cf. [11, 3, 7]). The relevant objects are formal languages, which are generated by grammars and can be classified by the Chomsky hierarchy. The morphisms which will make a link to symbolic dynamics are generalized sequential machines (GSM). Then Section 4 addresses this problem of how to link the two

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categories of subshifts and languages (cf. [10, 14]). If one restricts oneself to shift dynamics then it is possible to construct an injective functor between these categories. Finally, in section 5, the complexity of dynamically generated languages is discussed. Some general results (cf. [10]) for CA and some applications in formal multiscattering systems (cf. [13, 14]) are reviewed.

2 Dynamical Systems on Subshifts

2.1 Basic definitions

We discuss here dynamical systems whose dynamical map operates on sequences. We permit sequences over some finite set A of abstract symbols with cardinality $\#A > 1$ and define the *full shift* over A as the complete set of sequences $A^{\mathbb{Z}}$.

Take the usual topology on $A^{\mathbb{Z}}$, i.e. the product topology of the discrete topology on A . A base for this topology is given by the cylinder sets

$$\text{Cyl}(a_1 \dots a_n) = \left\{ s \in A^{\mathbb{Z}}; s_i = a_{i + \lfloor \frac{n-1}{2} \rfloor + 1} \text{ for } -\lfloor \frac{n-1}{2} \rfloor \leq i \leq \lfloor \frac{n}{2} \rfloor \right\} \quad (1)$$

which are both open and closed. This topology is compact (by Tychonov) and metrisable by the *sequence metric* d :

$$d(a, b) = \sum_{i \in \mathbb{Z}} \frac{\delta_{a_i b_i}}{2^{|i|}} \quad (2)$$

where $a, b \in A^{\mathbb{Z}}$, $\delta_{xy} = 0$ if $x = y$, 1 otherwise.

Among the continuous maps operating on $A^{\mathbb{Z}}$ shift maps and cellular automata (CA) are of special interest because they are compatible with the group structure of the index set \mathbb{Z} :

Denote by σ_A the (right) *shift map* on $A^{\mathbb{Z}}$: $(\sigma(a))_i = a_{i+1}$. Usually we write just σ for σ_A . A *subshift* is a closed σ -invariant set in $A^{\mathbb{Z}}$. Σ is a *finite subshift* if it is determined by a finite set F of forbidden symbol strings, which are not permitted to appear in sequences. We call Σ *cyclic* if the periodic (w.r.t. σ) sequences are dense. A dynamical system given by a subshift $\Sigma \subset A^{\mathbb{Z}}$ and the shift map $\sigma|_{\Sigma} \rightarrow \Sigma$ is called a *symbolic dynamics*. A natural generalization leads to *cellular automata (CA)*. A (1-dimensional) CA is any continuous map τ on $A^{\mathbb{Z}}$ which commutes with the shift map. Since any CA is automatically uniformly continuous an equivalent description of a CA (cf. [5, 6]) can be given by a local function $f_{\tau}: A^{2R+1} \rightarrow A$ with the property $(\tau(s))_i = f_{\tau}(s_{i-R}, \dots, s_{i+R})$ for any $s \in A^{\mathbb{Z}}$.

The appropriate morphisms for topological dynamics are *semi-conjugacies*. We define them here just for shift maps as any continuous map Φ between subshifts Σ_A, Σ_B which intertwines between the respective shift maps σ_A, σ_B :

$$\Phi: \Sigma_A \rightarrow \Sigma_B, \quad \sigma_B \circ \Phi = \Phi \circ \sigma_A \quad (3)$$

If, moreover, Φ is a homeomorphism, it is called a *conjugacy* (between shift maps). It may be interpreted as a continuous change of variables which identifies the shift maps on \sum_A and \sum_B .

2.2 Some standard CA-invariant subshifts

As for general dynamical systems one introduces the forward (and backward) limit set, the periodic sets and the non-wandering set. Because of the commutability of a CA with the shift map they are also subshifts. Let τ be again a CA over $\hat{A}^{\mathbb{Z}}$ which need not necessarily be invertible.

Definition 2.1 • The forward limit set $\Lambda^+(\tau)$ is the intersection of all forward images:

$$\Lambda^+(\tau) = \bigcap_{n=0}^{\infty} \tau^n(\hat{A}^{\mathbb{Z}}) \quad (4)$$

- The periodic set $\Pi(\tau)$ is the topological closure of the set of all cycles:

$$\Pi(\tau) = \overline{\text{Per}(\tau, \hat{A}^{\mathbb{Z}})} \quad (5)$$

- The non-wandering set $\Omega(\tau)$:

$$\Omega(\tau) = \{s \in \hat{A}^{\mathbb{Z}}; \forall \text{ neighborhood } U \text{ of } s \exists n \in \mathbb{N} \text{ s.t. } \tau^n(U) \cap U \neq \emptyset\} \quad (6)$$

Remark 2.2 1. All the sets just introduced are τ -invariant subshifts.

2. The forward limit set is the maximal τ -invariant subshift of $\hat{A}^{\mathbb{Z}}$.
3. $\Pi(\tau) \subset \Omega(\tau) \subset \Lambda^+(\tau)$.
4. The dynamics on these invariant sets can be very complicated. We recall here the phenomenon of chaos, for which we shall assume the following (topological) definition [1]:

Suppose $\Gamma \subset \hat{A}^{\mathbb{Z}}$ is a τ -invariant subshift, then $\tau|_{\Gamma}$ is called *chaotic* if

- (i) τ depends sensitively on initial conditions

$$(\exists \delta > 0 \forall s \in \Gamma \forall \text{ neighb. } U_s \subset \Gamma \exists t \in U_s, n \in \mathbb{N} \text{ s.t. } d(\tau^n(s), \tau^n(t)) > \delta).$$

- (ii) τ has a dense orbit.
- (iii) Γ is τ -cyclic, i.e. $\Pi(\tau|_{\Gamma}) = \Gamma$.

2.3 Another CA-invariant subshift: the trapped set

We are going to introduce now a generalization of the usually studied limit sets of a CA:

Definition 2.3 Let Γ be a subshift of the compact sequence space $\hat{A}^{\mathbb{Z}}$ and $\tau: \hat{A}^{\mathbb{Z}} \rightarrow \hat{A}^{\mathbb{Z}}$ a CA. We define as trapped set $\Lambda(\tau, \Gamma)$ the Γ -maximal invariant set

$$\Sigma := \Lambda(\tau, \Gamma) = \bigcap_{i \in \mathbb{Z}} \tau^i(\Gamma) \quad (7)$$

This definition is motivated by scattering systems (cf. next subsection). The selection principle by which certain sequences in $\hat{A}^{\mathbb{Z}}$ and consequently (because of compactness) certain segments are forbidden in Σ is often called pruning.

2.3.1 Trapping in Formal Multiscattering Systems (FMS)

Definition 2.4 Let M be a metrisable space and $T: M \rightarrow M$ be a bijective map. Furthermore, let $Q \subset M$ be a compact subset and define the following images:

$$I^i := T^i(Q) \cap Q, \quad i \in \mathbb{Z} \quad (8)$$

The pair (T, Q) is called a *formal multiscattering system* (FMS) if the following conditions are satisfied:

- (i) **homeomorphism property:** The restriction $T|_Q \rightarrow T(Q)$ is a homeomorphism.
- (ii) **escape property:** $\forall n \in \mathbb{N}: T^{n+1}(I^{-n}) \setminus Q \neq \emptyset$.
- (iii) **trapping property:** $\forall n \in \mathbb{N}: T^{n+1}(I^{-n}) \cap Q \neq \emptyset$.
- (iv) **no-return property:** if $x \in Q$, $T(x) \notin Q$ then $\forall n \in \mathbb{N}: T^n(x) \notin Q$.

Remark 2.5 1. The set Q may be interpreted as the interaction set of some finite range potential. The dynamics restricted to the complement $M \setminus Q$ is then the comparison dynamics of some scattering problem.

- 2. Properties (iii)–(iv) are also valid for T^{-1} because of the bijectivity of T and the following lemma.
- 3. Properties (iii), (iv) are equivalent to $\forall n \in \mathbb{N}: I^n \neq \emptyset$ and $T(I^n) \setminus Q \neq Q$.

Lemma 2.6 *With the notation of the preceding definition we have:*

- (i) $I^{n+1} = T(I^n) \cap Q = \bigcap_{i=0}^{n+1} I^i$
- (ii) $T^{-j}(I^i) = I^{-j} \cap I^{i-j}$, for $i \geq j > 0$.

Definition 2.7 We define the trapped set of a FMS (T, Q) as its Q -maximal invariant set, i.e.

$$\Lambda := \Lambda(T, Q) := \bigcap_{i \in \mathbb{Z}} T^i(Q) \quad (9)$$

Lemma 2.8 *The trapped set Λ of a FMS is non-empty and compact.*

Proof

$\Lambda \neq \emptyset$ follows from the trapping property (iii) which implies by lemma 2.6 the finite intersection property $\bigcap_{i=-n}^n T^i(Q) = \bigcap_{i=-n}^n I^i = I^n \cap I^{-n} \neq \emptyset$. Compactness of Q yields the assertion.

Definition 2.9 (i) A FMS (T, Q) is called *expansive* if $T|_\Lambda$ is expansive in the usual sense, i.e. if $\exists \delta > 0$ s.t. $\{x \neq y \text{ implies } \exists n \in \mathbb{Z}: d(T^n x, T^n y) > \delta\}$.
(ii) A finite open cover α of $\Lambda(T, Q)$ is called a generator for the FMS (T, Q) if \forall sequences $(A_i) \in \alpha^{\mathbb{Z}}: \bigcap T^i(\bar{A}_i)$ contains no more than 1 point in Q . If $\bigcap T^i(A_i)$ contains no more than 1 point, α is called a weak generator.
As in the standard situation (cf. [16]) one has

Lemma 2.10 *The FMS is expansive iff it has a generator iff it has a weak generator.*

For an expansive FMS the existence of a symbolic dynamics is shown by practically copying the standard proof [16].

Theorem 2.11 *If (T, Q) is an expansive FMS, then there is a surjective semiconjugacy $\Phi: \Sigma \rightarrow \Lambda(T, Q)$ from a subshift Σ over some finite alphabet A .*

If there is a cover of disjoint sets one finds the following strengthening:

Corollary 2.12 *If the FMS (T, Q) has a disjoint generator, then Φ can be chosen to be a topological conjugacy. In this case the trapped set $\Lambda(T, Q)$ is totally disconnected.*

Finally we are introducing parameterized families of FMS.

Definition 2.13 *A truncated family of FMS is given by a parameterized family of FMS $(T_x, Q_x)_{x \in J}$ over some interval $J \subset \mathbb{R}^n$ as parameter set, where each Q_x is the truncation of a common set $Q \supset Q_x$ by a family of level lines given by a cut-off function $f: M \rightarrow \mathbb{R}$ and an evaluation function $e: J \rightarrow \mathbb{R}^+$:*

$$Q_x = \{u \in Q: |f(u)| \leq e(x)\} \tag{10}$$

We shall now give a simple example:

2.3.2 Example 1: The truncated double horseshoe

Consider the following variant of Smale's piecewise linear horseshoe map: As usual, we perform on the unit square $Q_1 = [0,1]^2$ first a linear horizontal contraction and a vertical expansion by positive factors $\lambda < 1$ and $\lambda^{-1} > 1$ respectively, followed by a folding, so that the folded parts fall outside Q_1 . But we choose a double folding (cf. Fig. 1), whose purpose is to make the horseshoe map $T = T^{(H)}$ restricted to $I^{-1} := Q_1 \cap T^{-1}(Q_1)$ not only linear but to remove the reflection contained in Smale's horseshoe map, i.e.

$$DT|_{I^{-1}} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{11}$$

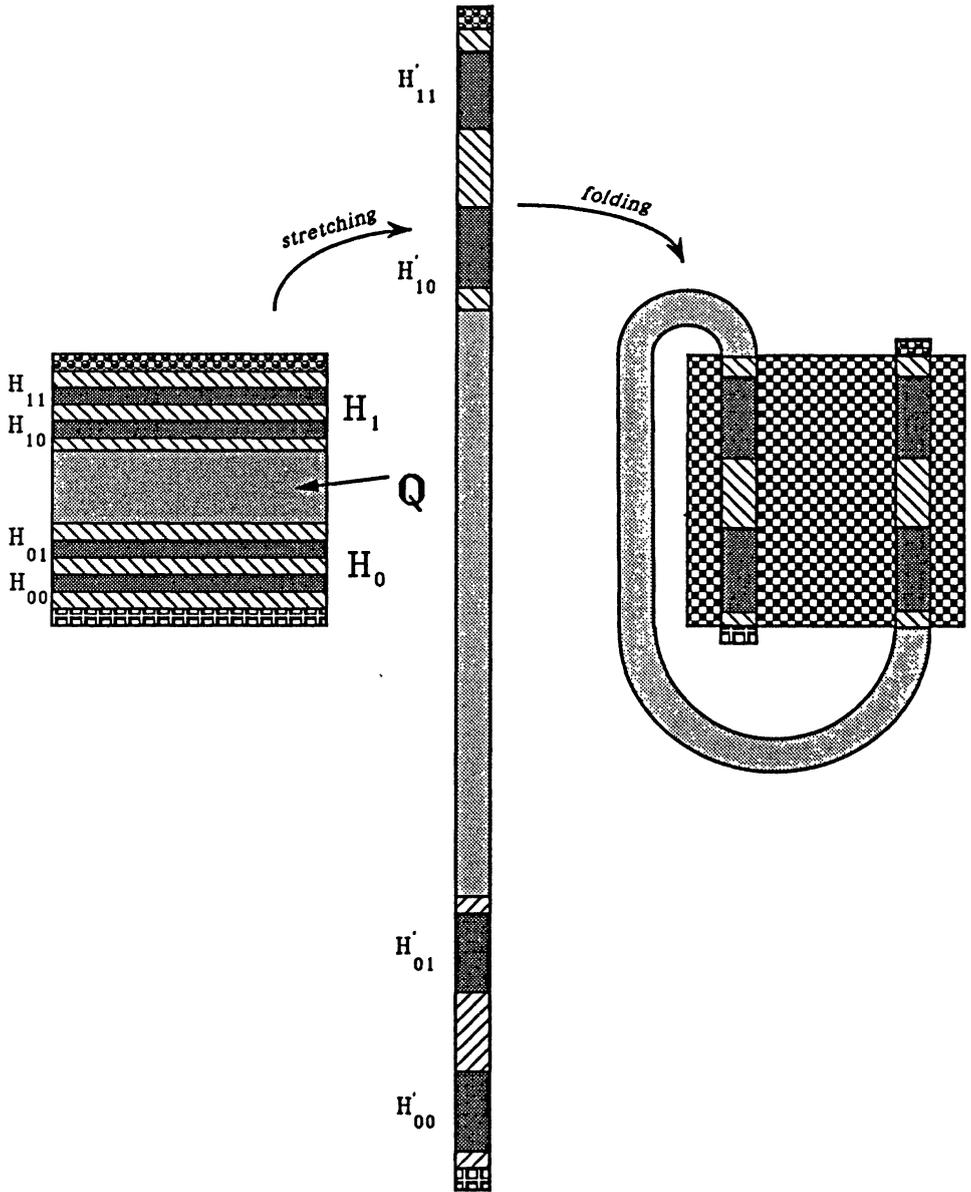


Fig. 1 The twice folded, piecewise linear horseshoe map applied to the unit square $Q = Q_1$. The symbolic dynamics chosen uses the first generation horizontal strips.

To construct a symbolic dynamics of this hyperbolic system we use the partition of I^{-1} into two disjoint horizontal strips H_0 and H_1 and define a topological conjugacy¹

$$h: \Lambda(T, Q_1) \rightarrow \{0, 1\}^{\mathbb{Z}}, \quad u \rightarrow (a_i)_{i \in \mathbb{Z}} \text{ with } T^i(x) \in H_{a_i} \quad (12)$$

We introduce the following lexical pseudo-order (not antisymmetric) on the full shift $\{0, 1\}^{\mathbb{Z}}$: Let $a, b \in \{0, 1\}^{\mathbb{Z}}$ and define the horizontal line $H_a = \bigcap_{i=0}^{\infty} H_{a_i}$, then

$$a < b: \Leftrightarrow \text{for the smallest } i_0 \geq 0 \text{ s.t. } a_{i_0} \neq b_{i_0} \text{ one has } a_{i_0} < b_{i_0} \quad (13)$$

This pseudo-order corresponds to the arrangement of the corresponding horizontal lines:

$$\pi_y(H_{(a_i)_{i \geq 0}}) \geq \pi_y(H_{(b_i)_{i \geq 0}}) \Leftrightarrow a \geq b \quad (14)$$

where we denoted by π_y , the projection to the y -component.

We now interpret Q_1 as the maximal interaction set, which will be truncated by the pruning function π_y to yield a 1-parameter family of pruned scattering systems with interaction sets (cf. Fig. 2):

$$Q_{y_{\max}} := \{u \in Q_1 : \pi_y(u) \leq y_{\max}\} \quad (15)$$

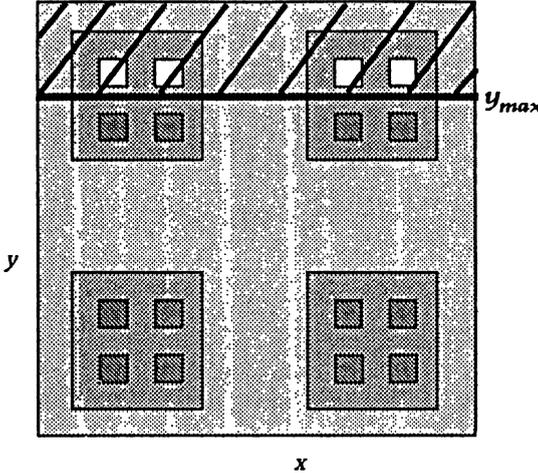


Fig. 2 The horizontally truncated double horseshoe. As the reflection has been removed, the 4 second generation rectangles lost for the chosen cut-off parameter y_{\max} correspond to the symbol combinations 0011, 0111, 1011, 1111, respectively.

In the notation of definition 2.13 the evaluation function e is the identity on $[0, 1]$ and the cut-off function f is the projection $\pi(y)$. The corresponding map $T_{y_{\max}}$ is defined by modifying the horseshoe map T only outside the box $Q_{y_{\max}}$ to make

¹ i.e. h is a homeomorphism and $h \circ T = \sigma \circ h$.

points drift to infinity without returning whereas $T_{y_{\max}}|Q_{y_{\max}} = \mathcal{T}|Q_{y_{\max}}$. This yields a 1-parameter formal multiscattering system $(T_{y_{\max}}, Q_{y_{\max}})_{y_{\max} \in (0,1)}$. Each time a test particle “jumps” out of the box $Q_{y_{\max}}$ it escapes for good. The trapped set we are interested in is the $Q_{y_{\max}}$ -maximal invariant set

$$\Lambda(T_{y_{\max}}, Q_{y_{\max}}) = \{u \in Q_1; \pi_y(T_1^i(u)) \leq y_{\max} \forall i \in \mathbb{Z}\} \quad (16)$$

It consists of all trajectories which never jump out of the box $Q_{y_{\max}}$.

Lowering y_{\max} will make it easier to get out of the box $Q_{y_{\max}}$. Formerly trapped trajectories manage to escape, so that the trapped set is bound to shrink. Actually, we are going to describe the evolution of the trapped set in the symbolic dynamics defined above, where the variation of the selection condition $y \leq y_{\max}$ will lead to pruning rules determining which symbol sequences correspond to trapped trajectories.

In this model truncation is trivially transferred to symbolic dynamics where the trapped set equivalent to $\Lambda(T_{y_{\max}}, Q_{y_{\max}})$ is the subshift

$$\Sigma_\nu := \{s \in \{0, 1\}^{\mathbb{Z}}; \forall l \in \mathbb{Z}: (s_i)_{i \geq l} \leq \nu\} \quad (17)$$

where ν is the binary expansion of y_{\max} .

2.3.3 Example 2: The truncated sawtooth

This example originates from a physical model of the scattering of a point particle in an infinite array of non-overlapping elastic scatterers which are placed at unit distance from each other along the y -axis (cf. [12, 15]). The symbolic dynamics of this model is a family of FMS over the alphabet $\dot{A} = \{-1, 0, 1\}$ with interaction set

$$\begin{aligned} \Gamma_\nu &= \{s \in \dot{A}^{\mathbb{Z}}; |f(s)| \leq \nu\}, \nu \in \dot{B}^{\mathbb{N}_0}, \quad \dot{B} = \{-2, -1, 0, 1, 2\} \\ f: \dot{A}^{\mathbb{Z}} &\rightarrow \dot{B}^{\mathbb{N}_0}, s \mapsto s_0(s_1 + s_{-1}) \dots (s_i + s_{-i}) \dots \end{aligned} \quad (18)$$

and lexical ordering on $\dot{B}^{\mathbb{N}_0}$.

3 Formal Languages

This section introduced the category of formal languages and some subcategories. It is based on [11, 3].

3.1 Basic definitions

We start from a finite set \dot{A} of abstract symbols, which will be called *alphabet* in the following. Define the free semi-group of words or (finite) strings $\dot{A}^* = \bigcup_{n=0}^{\infty} \dot{A}^n$, which is called the *total language* over \dot{A} . The semi-group operation is the concatenation. Its unit element is the empty string $\$$. Powers correspond to

repetitions of symbols, the length $\text{lg}(s)$ of a string $s \in \dot{A}^*$ yields a formal logarithm. The total language minus the empty string is written as \dot{A}^+ . Any subset L of the total language \dot{A}^* is called a *language*. A word $x \in L$ is a subword or *segment* of a word $y \in L$, written $x < y$ if there are $a, b \in \dot{A}^*$ s.t. $y = axb$. A *homomorphism* h between total languages is understood with respect to the semi-group structures. If, moreover, $h^{-1}(\{\$\}) = \{\$\}$, then h is called *\\$-free*. When we speak of an *inverse homomorphism* h^{-1} we mean the set valued function of preimages of singletons under h .

3.2 Grammars Generating Formal Languages

In this section we shall discuss how to generate a formal language by a grammar. This concept is based on the notion of a rewriting system.

Rewriting systems A *rewriting system* $RW = (\dot{A}, P)$ is given by an alphabet \dot{A} and a finite set $P \subset \dot{A}^* \times \dot{A}^*$. The pairs $(S, T) \in P$ are referred to as *rewriting rules* or *productions* and written as $S \rightarrow T$.

Assume $S \rightarrow T, T \rightarrow U \in P$. We are allowed to apply these productions also to subwords within a word and using the notation " \Rightarrow_{RW} " or in short " \Rightarrow " to generate a new word directly, i.e. in 1 step: $R_1SR_2 \Rightarrow R_1TR_2$ or in finitely many² steps: $R_1SR_2 \Rightarrow R_1TR_2 \Rightarrow R_1UR_2$ which we write as $R_1SR_2 \Rightarrow R_1UR_2$.

Definition 3.1 A *generative grammar* $G = (\dot{A}_T, \dot{A}_V, X_0, P)$ is now a rewriting system (\dot{A}, P) where the alphabet $\dot{A} = \dot{A}_T \dot{\cup} \dot{A}_V$ is partitioned into *terminals* and *variables* or *nonterminals*. The nonterminal alphabet \dot{A}_V contains a distinguished letter, namely the *initial letter* X_0 . In any production $S \rightarrow T \in P$ the word S being "processed" must contain at least one variable, i.e. $S \notin \dot{A}_T^*$.

When we use G to generate its associated language L_G we start with the initial letter X_0 , and do not stop in the generating process before we have produced a terminal word in \dot{A}_T^* :

$$L_G = \{S \in \dot{A}_T^* : X_0 \overset{\cdot}{\Rightarrow} S\} \quad (19)$$

Chomsky hierarchy Grammars are classified by imposing restrictions on the form of productions. The most common classification is the *Chomsky hierarchy*:

Definition 3.2 A grammar $G = (\dot{A}_T, \dot{A}_V, X_0, P)$, $\dot{A} = \dot{A}_V \dot{\cup} \dot{A}_N$, is of the **type i** if the following restrictions (i) are satisfied for the rewriting rules $(X, Y \in \dot{A}_V)$:

- (0) no restrictions;
- (1) the rewriting rules "depend on the context", i.e. they have all the following form: $QXR \rightarrow QAR$, where $Q, R, A \in \dot{A}^*$ and A is not the empty string $\$\$ with the only possible exception: if $X_0 \rightarrow \$\$ appears as a production, then X_0 must not occur on the right side of any production.

² The relation $\overset{\cdot}{\Rightarrow}$ is the reflexive transitive closure of the binary relation \Rightarrow i.e. $\overset{\cdot}{\Rightarrow} := \bigcup_{i=0}^{\infty} (\Rightarrow)^i$ where $(\Rightarrow)^0 := \{(a, a) : a \in \dot{A}\}$.

- (2) All rewriting rules have the form $X \rightarrow A$, where $A \in \hat{A}^*$;
 (3) the rewriting rules concatenate only on one side or replace by a terminal $S \in \hat{A}_T$, i.e. their form is: $X \rightarrow SY$ or $X \rightarrow S$.

Type 1 grammars restrict type 0 grammars by requiring that their productions (with the possible exception of $X_0 \rightarrow \$$) are not length decreasing³ (the length of a word is the number of its primitive (i.e. either terminal or nonterminable) symbols).

A language is of type i , $i \in \{0, 1, 2, 3\}$, if it can be generated by a grammar of type i . The common names for type i languages are *recursively enumerable*, *context sensitive*, *context free* and *regular*, respectively. We denote the set of type i languages by L_i . This classification is properly nested, i.e. $L_{i+1} \subset L_i$. A class properly between type 0 and type 1 languages (type 1/2) are the *recursive* languages. A language L is recursive if both L and its complement $\hat{A}^* \setminus L$ are recursively enumerable. The *Chomsky complexity* $\chi(L)$ of a language L is defined as the type of its simplest generating grammar.

Certain subfamilies of regular languages are of interest, too, such as *finite complement languages*, where a finite set of words $F \subset \hat{A}^*$ are forbidden segments. They correspond to finite subshifts.

In fact, besides these language types many other types have been studied. Many of them have common properties. We shall define just one underlying structure, called trio structures. For this we have to introduce families of languages.

Definition 3.3 Let I be an infinite alphabet and L a set of languages over I , i.e. $L \in 2^I$. The pair (I, L) is called a *family of languages* if

- (i) $\forall L \in L \exists \hat{A} \subset I$ s.t. \hat{A} is finite and $L \subset \hat{A}^*$.
- (ii) $L \neq \emptyset$ for some $L \in L$.

A trio is now a family of languages with certain closure properties:

Definition 3.4 A (*full*) *trio* is a family of languages closed under $\$$ -free (arbitrary) homomorphisms, inverse homomorphisms, and intersections with regular languages.

Example 3.5 The families of regular, context free, and recursive enumerable languages are full trios. The families of context sensitive and recursive languages are trios that are not full.

3.3 Acceptors and Morphisms of Languages

Acceptors Automata are mathematical models of devices that process information by giving responses to inputs. Formal language theory views them as scanning devices or *acceptors* able to recognize, whether a word belongs to a given formal language. A hierarchy of automata corresponds to the Chomsky hierarchy of languages, in the sense that any class i language is recognized by a class i automa-

³ This is even an equivalent characterization of type 1 grammars.

ton and conversely any class i automaton produces a class i language as output. The corresponding automata are called Turing machines (0), linear bounded automata (1), pushdown automata (2) and finite deterministic automaton (3). We refer for details to the literature (e.g. [11, 3, 7]).

Morphisms An intrinsic justification of a classification scheme are closure properties of its classes under naturally associated morphisms. We choose the maps induced by $\$$ -limited generalized sequential machines (GSM). They are generalizations of homomorphisms. Their special importance for our discussion lies in the fact that they correspond to the morphisms of subshifts, i.e. their semi-conjugacies (cf. section 4.3).

Definition 3.6 A *generalized sequential machine (GSM)* is given by the sixtuple $M = (Z, \dot{A}, \dot{B}, \omega, \tau, z_0)$, where Z (state set), \dot{A} (input alphabet), and \dot{B} (output alphabet) are finite sets, $\tau: Z \times \dot{A} \rightarrow Z$ (next state map) and $\omega: Z \times \dot{A} \rightarrow \dot{B}^*$ (output map) are maps. M is called $\$$ -free if the empty string $\$$ is not an assumed value of ω . We extend the output map ω and the next state map τ to $Z \times \dot{A}^*$ by $\tau(z, \$) := z$, $\omega(z, \$) := \$$, and for $s \in \dot{A}^*$, $a \in \dot{A}$ recursively $\tau(z, sa) := \tau(\tau(z, s), a)$, $\omega(z, sa) := \omega(z, s)\omega(\tau(z, s), a)$ by concatenation. The map

$$M: \dot{A}^* \rightarrow \dot{B}^*, \quad s \mapsto \omega(z_0, s) \quad (20)$$

is called a *GSM map*. A GSM M is called $\$$ -limited on a given language $L \subset \dot{A}^*$ if there is a $k > 0$ s.t. $\forall s \in L$: if $s = xyz$ and $\omega(\tau(z_0, x), y) = \$$ for some $x, y, z \in \dot{A}^*$ then $\text{lg}(y) \leq k$.

An important property of $\$$ -limited GSM maps is that they keep trios and hence the Chomsky classes invariant if one neglects the empty string (cf. [3]).

Lemma 3.7 For each trio \mathbf{L} , each $L \in \mathbf{L}$, and each GSM map M $\$$ -limited on L we have:

- (i) $M(L) \setminus \{\$\} \in \mathbf{L}$
- (ii) If \mathbf{L} is a full trio or if \mathbf{L} is closed under union with $\$$ or if $a \in L$, $M(a) = \$$ implies $a = \$$, then even $M(L) \in \mathbf{L}$.
- (iii) $M^{-1}(L) \in \mathbf{L}$.

Corollary 3.8 Let \mathfrak{L} be the language category with languages as objects and $\$$ -limited GSM maps as morphism. Any full trio forms a subcategory.

4 Subshifts and Languages

how they fit together

In this section we are going to study how topological dynamics can be linked with formal language theory. There are two approaches. In the first, which we will not treat further, one extends formal language theory to encompass bi-infinite words. In the second, one establishes a bijective correspondence between subshifts

and so-called central languages. We are going to follow two tracks. The first, introduced in [10], constructs a bijection between the set of all subshifts and the set of all central languages, which will be defined presently. In the second, we restrict our attention to chaotic symbolic dynamics, which we required in particular to be cyclic (cf. section 2.2), and identify those languages which correspond naturally to a cyclic subshift (cf. [14]).

4.1 Dynamically generated languages

First we define the languages we associate to general and cyclic subshifts, respectively. These are the associated *central* language which consists of all segments of permitted sequences and the associated *cyclic* language, which consists of all full periods of a cycle, or more formally:

Define the *periodization map*

$$\zeta_A: A^* \rightarrow \text{Per}(\sigma, A^Z), \quad s \rightarrow \bar{s} \quad (21)$$

where $\bar{s}_i := s_{i \bmod \lg(s)}$.

Definition 4.1 Let $\Sigma, \Sigma' \subset A^Z$ be an arbitrary and a cyclic subshift (i.e. $\Pi(\sigma, \Sigma) = \Sigma$), respectively. Then

(i) $\Omega(\Sigma) = \{a \in A^*; \exists s \in \Sigma: a < s\}$ is called the *associated central language* and

(ii) $\Omega_c(\Sigma') = \zeta_A^{-1}(\text{Per}(\sigma, \Sigma')) \cup \{\$\}$ is called the *associated cyclic language*.

Observe the $\Omega_c(\Sigma')$ contains together with any word w all its repetitions w^i , $i > 0$ and all their cyclic permutations. An easy exercise shows the following

Lemma 4.2 *The maps Ω and Ω_c are injective.*

Therefore different subshifts can still be distinguished on the level of their associated languages.

4.2 Adherences and centers

A map in the opposite direction, i.e. from the set of all languages to the set of all subshifts, is the *adherence*. It maps a language onto the subshift of all sequences whose segments are also segments of words of the language:

Definition 4.3 Let $L \subset A^*$.

$$\text{Adh}(L) = \{s \in A^Z; \forall a < s \exists w \in L \text{ s.t. } a < w\} \quad (22)$$

is called the *adherence* of the language L .

Obviously, $\text{Adh}(L)$ is closed and shift invariant and hence a subshift. The language operators Ω and Ω_c are in fact inverses of the property restricted adherence operator Adh . The right restrictions can be characterized as invariants of the center operators \mathbb{C}, \mathbb{C}_c .

Definition 4.4 We define the following two operations on the set of languages over \dot{A} :

(i) The *center* $\mathfrak{C}(L)$ of a language L is the language

$$\mathfrak{C}(L) = \{a \in \dot{A}^*; \forall N > 0 \exists x, y \in \dot{A}^*, \lg(x), \lg(y) \geq N \text{ s.t. } xay \in L\} \quad (23)$$

A string in $\mathfrak{C}(L)$ is sometimes called *bi-extensible* in L .

(ii) The *cyclic center* $\mathfrak{C}_c(L)$ is

$$\mathfrak{C}_c(L) = \{a \in \dot{A}^*; \forall i > 0 \forall \text{ permutations } \pi(a^i) \exists x, y \in \dot{A}^* \text{ s.t. } x\pi(a^i)y \in L\} \quad (24)$$

We call a language L *central* if $\mathfrak{C}(L) = L$ and *cyclic* if $\mathfrak{C}_c(L) = L$. One sees from this definition immediately that the associated central language of a subshift is central and that the associated cyclic language of a cyclic subshift is cyclic. Conversely, by the following theorem linking subshifts and languages, one sees that any central language is the associated central language of some subshift, and likewise any cyclic language is the associated cyclic language of some cyclic subshift.

We now state the announced theorem, whose first part (i) is due to [10].

Theorem 4.5 *Let L be a language over \dot{A} .*

(i) *If Σ is any subshift over \dot{A} , then*

$$\text{Adh}(\mathfrak{Q}(\Sigma)) = \Sigma \quad (25)$$

$$\mathfrak{Q}(\text{Adh}(L)) = \mathfrak{C}(L) \quad (26)$$

(ii) *If Σ' is a cyclic subshift over \dot{A} , then*

$$\text{Adh}(\mathfrak{Q}_c(\Sigma')) = \Sigma' \quad (27)$$

$$\mathfrak{Q}_c(\text{Adh}(L)) = \mathfrak{C}_c(L) \quad (28)$$

Proof

ad (i) eq. 25: By applying the definitions one sees that $\mathfrak{Q}(\text{Adh}(\mathfrak{Q}(\Sigma))) = \mathfrak{Q}(\Sigma)$. The injectivity of \mathfrak{Q} implies the first assertion.

For eq. 26 let first $a \in \mathfrak{Q}(\text{Adh}(L))$, i.e. $\exists s \in \text{Adh}(L)$ s.t. $a = (a_1 \dots a_n) = (s_1 \dots s_n) < s$. Let $N > 0$ and $b := s_{1-N} \dots s_{n+N}$. We have $b \in \mathfrak{Q}(\text{Adh}(L))$ and $a < b < s$. By the definition of the adherence $\exists c \in L$ s.t. $a < b < c$. Hence $a \in C(L)$. Conversely, let $a \in C(L)$. Then a must have an infinite number of bi-extensions in L . This implies the existence of a nested sequence of words $w^i \in C(L)$, $i \geq 0$, ($w(0) = a$) with strictly increasing length $\lg(w^{(i)}) \rightarrow \infty$. By compactness of \dot{A}^2 the set $W = \bigcap_{i=0}^{\infty} \text{Cyl}(w^{(i)})$ is nonempty, hence a singleton $\{w^{(\infty)}\}$ which by construction is contained in $\text{Adh}(L)$. As $a < w^{(\infty)}$, we get $a \in \mathfrak{Q}(\text{Adh}(L))$.

ad (ii) eq. 27 “ \subset ”: Let $s \in \text{Adh}(\mathcal{Q}_c(\Sigma'))$. Then $\forall a < s \exists w \in \mathcal{Q}_c(\Sigma')$ s.t. $a < w$. But $w \in \mathcal{Q}_c(\Sigma')$ means that $\bar{w} \in \Sigma'$. As Σ' is closed, we conclude $s \in \Sigma'$. “ \supset ”: As $\overline{\text{Per}(\sigma, \Sigma')} = \Sigma'$, it is enough to show $\text{Per}(\sigma, \Sigma') \subset \text{Adh}(\mathcal{Q}_c(\Sigma'))$. So let $s \in \text{Per}(\sigma, \Sigma')$. Then $\exists a \in A^*$ s.t. $s = \bar{a}$ and $\forall i \geq 1: a^i$ and all its permutations are in $\mathcal{Q}_c(\Sigma')$. Therefore $s = \bar{a} \in \text{Adh}(\mathcal{Q}_c(\Sigma'))$.

ad (ii) eq. 28 “ \subset ”: Let $a \in \mathcal{Q}_c(\text{Adh}(L))$. Then $\bar{a} \in \text{Adh}(L)$, i.e. $\forall c < \bar{a} \exists x, y \in A^*$ s.t. $xcy \in L$. In particular $\forall i \geq 1 \forall$ permutation $\pi(a^i) \exists x, y \in A^*$ s.t. $x\pi(a^i)y \in L$. Hence $a \in \mathcal{C}_c(L)$. “ \supset ”: Let $a \in \mathcal{C}_c(L)$. Then $\forall i \geq 1 \exists x, y \in A^*$ s.t. $xa^iy \in L$. Hence $\bar{a} \in \text{Adh}(L)$, so that $a \in \mathcal{Q}_c(\text{Adh}(L))$.

Corollary 4.6 (i) *The pair of maps $(\mathcal{Q}, \text{Adh})$ give a bijective correspondence between central languages and subshifts over A .*

(ii) *The pair $(\mathcal{Q}_c, \text{Adh})$ gives a bijective correspondence between cyclic languages and cyclic subshifts over A .*

4.3 Functors from the category of languages to the category of subshifts

Let \mathfrak{SD} be the category of symbolic dynamics, i.e. its object are the subshifts and its morphism the semi-conjugacies of shift maps. The subcategory of cyclic subshifts will be denoted by \mathfrak{SD}_c . Let \mathcal{L} be the category of languages with languages as objects and \mathcal{S} -limited GSM maps as morphism. We shall now extend the language operators \mathcal{Q} and \mathcal{Q}_c to yield functors.

The key observation is that analogously to CA, (cf. [5, 6]) there is a description of semi-conjugacies by local functions.

Let $\Phi: \sum_A \rightarrow \sum_B$ be a semi-conjugacy, first between arbitrary subshifts. Since \sum_A is compact Φ is even uniformly continuous. Therefore, for each $l \in \mathbb{N}_0$ there exist a minimal odd number $N(l) \geq 0$, s.t. for the discrepancy i_δ of two sequences s, t (i.e. half the length of the longest common middle segment), we find

$$i_\delta(s, t) \geq N(l) \Rightarrow i_\delta(\Phi(s), \Phi(t)) \geq l \quad (29)$$

Since Φ commutes with the shift map, it bears a close resemblance with a cellular automaton (CA) but for the possibility to change the sequence space. Like a CA, Φ is determined by a *local function* albeit defined only on a subset $D \subset A^{2R+1}$, $2R+1 := N(1)$:

$$\begin{aligned} \varphi: D &\rightarrow B, & \varphi(s_{i-R} \dots s_i \dots s_{i+R}) &= (\varphi(s))_i \\ D &= \{a \in A^{2R+1}; \exists s \in \sum_A \text{ s.t. } a < s\} \end{aligned} \quad (30)$$

We call R the *radius of uniform continuity* of Φ .

Conversely, let $r \in \mathbb{N}$, $D \subset A^{2r+1}$, $\varphi: D \rightarrow B$. Denoting by $\sum(F) \subset A^Z$ the finite subshift generated by the forbidden set $F = A^{2r+1} \setminus D$ we get a semiconjugacy $\Phi: \sum(F) \rightarrow \sum_B$, $(\Phi(s))_i = \varphi(s_{i-r} \dots s_i \dots s_{i+r})$ onto some finite subshift $\sum_B \subset B^Z$. Thus, we have shown

Lemma 4.7 *There is a bijective correspondence between semi-conjugacies of finite subshifts and local functions.*

This local representation of Φ can be used to prove the following

Lemma 4.8 *Each semi-conjugacy between subshifts can be simulated by a $\$$ -limited GSM mapping both on the associated language and on the associated cycle language.*

Proof

Let $\Phi: \Sigma_A \rightarrow \Sigma_B$ be a semi-conjugacy between two subshifts $\Sigma_A \subset A^{\mathbb{Z}}$ and $\Sigma_B \subset B^{\mathbb{Z}}$. Let $\varphi: A^{2R+1} \supset D \rightarrow B$, $R \geq 0$, be the associated local function of eq. 30. For convenience, extend φ in some arbitrary way to the whole A^{2R+1} .

(i) First we examine the central language. This part is an adaptation of a proof given by Culik II et al. [10] for cellular automata. Define a map

$$\mathfrak{L}(\Phi): \mathfrak{L}(\Sigma_A) \rightarrow \mathfrak{L}(\Sigma_B), (s_i)_{i=-1}^n \mapsto \begin{cases} \varphi(s_1 \dots s_{1+2R}) \dots \varphi(s_{n-2R} \dots s_n) & \text{if } n > 2R \\ \$ & \text{otherwise} \end{cases} \quad (31)$$

Observe that a string of length $n \geq 2R$ is mapped by $\mathfrak{L}(\Phi)$ to a string of length $n - 2R$. Obviously, $\mathfrak{L}(\Phi(\Sigma_A)) = \mathfrak{L}(\Phi)(\mathfrak{L}(\Sigma_A))$.

Next one defines a GSM, whose associated GSM map will be just $\mathfrak{L}(\Phi)$:

Let $M = (Z, A, B, \omega, \tau, z_0, Z_f)$ with state set $Z = \bigcup_{i=0}^{2R} A^i$, initial state $z_0 = \$$, next state function τ and output function ω defined for $z \in Z$, $a \in A$, as follows:

$$\tau((z_i)_{i=-1}^r, a) = \begin{cases} (z_i)_{i=-1}^r a & \text{if } r < 2R \\ (z_i)_{i=-2}^r a & \text{if } r = 2R \end{cases}; \omega((z_i)_{i=-1}^r, a) = \begin{cases} \$ & \text{if } r < 2R \\ \varphi((z_i)_{i=-1}^r a) & \text{if } r = 2R \end{cases} \quad (32)$$

M is a $\$$ -limited GSM on $\mathfrak{L}(\Sigma_A)$ because only strings shorter than $2R$ are mapped to the empty string $\$$. Obviously, the GSM map $M|_{\mathfrak{L}(\Sigma_A)} = \mathfrak{L}(\Phi)$.

(ii) For the cycle language we proceed similarly. Observe, that we identified repetitions of cycles s^i , $i > 0$, $s \in \mathfrak{L}_c(\Sigma)$. We define a length conserving $\mathfrak{L}_c(\Phi)$ and periodize $s = (s_i)_{i=-1}^n \neq \$$ by $s_{n+i} := (\zeta(s))_{n+1} = s_i$:

$$\mathfrak{L}_c(\Phi): \mathfrak{L}_c(\Sigma_A) \rightarrow \mathfrak{L}_c(\Sigma_B), (s_i)_{i=-1}^n \mapsto \begin{cases} \varphi(s_1 \dots s_{1+2R}) \dots \varphi(s_n \dots s_{n+2R}) & \text{if } (s_i)_{i=-1}^n \neq \$ \\ \$ & \text{otherwise} \end{cases} \quad (33)$$

Again we get $\mathfrak{L}_c(\Phi(\Sigma_A)) = \mathfrak{L}_c(\Phi)(\mathfrak{L}_c(\Sigma_A))$. Next we concatenate to each $s \in A^*$ an end marker \bullet , i.e. we identify s with $s\bullet$. The $\$$ -limited GSM simulating the semi-conjugacy Φ is defined by $Z = \bigcup_{i=0}^{8R^2} A^i$, $z_0 = \$$, $(z_0, a) := [h; t]$,

$h, t \in \dot{A}^*$. The string h is going to store the leading ($2R$ or up to $4R^2$) symbols of an input word s^\bullet (see cases 1, 2 in the eq. below):

$$\begin{aligned} \tau([(h_i)_{i-1}^r; (t_i)_{i-1}^r], a) &= \begin{cases} [(h_i)_{i-1}^r a; (t_i)_{i-1}^r a] & \text{if } r < 2R, a \neq \bullet \\ [((h_i)_{i-1}^r)^{2R}; ((t_i)_{i-1}^r)^{2R}] & \text{if } r < 2R, a = \bullet \\ [(h_i)_{i-1}^{2R}; (t_i)_{i-1}^{2R} a] & \text{if } r = 2R, a \neq \bullet \\ [(h_i)_{i-1}^r; (t_i)_{i-1}^r] & \text{if } r \geq 2R, a = \bullet \end{cases} \\ \omega([(h_i)_{i-1}^r; (t_i)_{i-1}^r], a) &= \begin{cases} \$ & \text{if } r < 2R \\ \varphi((t_i)_{i-1}^{2R} a) & \text{if } r = 2R, a \neq \bullet \\ \varphi(t_1 \dots t_r, h_1) \dots \varphi(t_r, h_1 \dots h_{2R}) & \text{if } a = \bullet \end{cases} \end{aligned} \quad (34)$$

where $\varphi(t) := \varphi(t_1 \dots t_{2R+1})$ if $\lg(t) > 2R$. Again, we get $M|\mathcal{Q}_c(\sum_A) = \mathcal{Q}_c(\varphi)$.

Corollary 4.9 *Definition 4.1 together with equations 31 and 33 defines two functors $\mathcal{Q}: \mathfrak{SD} \rightarrow \mathcal{Q}$ and $\mathcal{Q}_c: \mathfrak{SD}_c \rightarrow \mathcal{Q}$.*

The lemmas 3.7 and 4.8 yield the announced closure properties:

Theorem 4.10 *Any semi-conjugacy Φ between two subshifts preserves any full trio the associated (central or cyclic) language belongs to; in particular, if \sum is an arbitrary subshift, \sum' a cyclic subshift and if Φ is even a conjugacy then:*

$$\begin{aligned} \chi(\dot{\mathcal{Q}}_c(\sum_A)) &= \chi(\dot{\mathcal{Q}}_c(\Phi(\sum_A))) \\ \chi(\dot{\mathcal{Q}}(\sum_A)) &= \chi(\dot{\mathcal{Q}}(\Phi(\sum_A))) \end{aligned} \quad (35)$$

where $\dot{L} := L \setminus \{\$\}$ for any language L .

5 Complexity of dynam. generated languages

In this section we first return to general CA.

5.1 Sofic Systems

The simplest way to generate a language dynamically is by applying some CA a finite number of times to a full shift. As it happens these so-called sofic systems yield just the regular languages:

Definition 5.1 *Let τ be a cellular automaton (CA) over the shift $A^{\mathbb{Z}}$ and \sum_M a subshift of finite type. The subshift $\tau(\sum_M)$ is called a sofic system.*

One has (cf. [8]) the following:

Proposition 5.2 *A subshift \sum is sofic iff $\mathcal{Q}(\sum)$ is regular.*

5.2 Some results for CA and their “standard” invariant sets

We quote here some results from [9]. First one observes the following constraint for limit languages:

Theorem 5.3 *For every cellular automaton $\tau: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ the complement of the forward limit language $A^* \setminus \mathcal{L}(\Lambda^+(\tau))$ is recursively enumerable.*

Apart from this constraint any language is realisable by the limit set of a CA in the following sense:

Theorem 5.4 *If $L \subset A^*$ has a recursively enumerable complement, then there exists a CA $\tau: B^{\mathbb{Z}} \rightarrow B^{\mathbb{Z}}$, a regular language $R \subset B^*$, and a homomorphism $h: B^* \rightarrow A^*$ s.t.:*

$$h(\mathcal{L}(\Lambda^+(\tau)) \cap R) = L \quad (36)$$

In particular examples exist, for which the the central language of the forward limitset is not recursively enumerable.

For the periodic set one has the following result:

Theorem 5.5 (i) *If τ is any CA, then $\mathcal{L}(\Pi(\tau))$ is recursively enumerable.*

(ii) *If $L \subset A^*$ is recursively enumerable, then there exist a CA $\tau: B^{\mathbb{Z}} \rightarrow B^{\mathbb{Z}}$, a regular language $R \subset B^*$, and a homomorphism $h: B^* \rightarrow A^*$ s.t.:*

$$h(\mathcal{L}(\Pi(\tau)) \cap R) = L \quad (37)$$

5.3 Results for some simple truncated families of FMS

For this subsection cf. [13, 14]. Here we treat the examples of subsections 2.3.2 and 2.3.3. In these models one is interested in complexity transitions arising by variation of the family parameter and accompanying transitions to chaos of these models.

Truncated horseshoe The first observation one makes is, that the parameterization map

$$\begin{aligned} \{0, 1\}^{\mathbb{N}_0} &\cong [0, 2] \rightarrow 2^{A^{\mathbb{Z}}} \\ \nu &\mapsto \sum_{\nu} := \{s \in \{0, 1\}^{\mathbb{Z}}; \forall l \in \mathbb{Z}: (s_i)_{i \geq l} \leq \nu\} \end{aligned} \quad (38)$$

is locally constant everywhere but on the set

$$\mathfrak{B} = \{a \in \{0, 1\}^{\mathbb{N}}; \forall i \in \mathbb{N}: \sigma^i(a) \leq a\} \quad (39)$$

which is called the *bifurcation set* of the family of FMS. One observes the following complexity transition:

Theorem 5.6 *Suppose $\nu \in \mathfrak{B} \cap \{1, 2\}$. Then both central language $\mathcal{L}(\sum_{\nu})$ and cyclic language $\mathcal{L}_c(\sum_{\nu})$ are regular iff $\nu \in \mathbb{Q}$.*

Truncated sawtooth This system becomes chaotic at the critical value $\nu_{\text{crit}} = 1 - 10$. For $\nu < \nu_{\text{crit}}$ the trapped set is finite, hence the associated languages regular. At the critical value one finds:

Theorem 5.7 *The associated languages $\Omega(\Sigma_{\nu_{\text{crit}}})$ and $\Omega_c(\Sigma_{\nu_{\text{crit}}})$ are context sensitive but not context free.*

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