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# Minimal Dynamical System for $\omega$-Bounded Groups

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Our aim is to give a direct topological proof of the fact that phase space of a minimal dynamical system with $\omega$-bounded group is coabsolute to a Cantor cube.

A *dynamical system* is a triple $(G, X, \pi)$ where $G$ is a topological group, $X$ is a topological space and $\pi$ is a continuous action on $X$, that is, $\pi : G \times X \to X$ is a continuous map such that:

(1) $\pi(1, x) = x$ for each $x \in X$ (as usual 1 is the neutral element of $G$)

(2) $\pi(gh, x) = \pi(g, \pi(h, x))$ for each $g, h \in G$ and each $x \in X$.

If $(G, X, \pi)$ is a dynamical system then the space $X$ is called a *phase space* of the system $(G, X, \pi)$. We use following notations $\pi^g$ and $\pi_x$ for homeomorphisms $\pi : X \to X$ and continuous maps $\pi_x : G \to X$ defined in the following way $\pi^g(x) := \pi(g, x) =: \pi_x(g)$. Our considerations will be confined only to dynamical systems with compact Hausdorff phase space.

Let $(G, X \pi)$ and $(G, Y, \varrho)$ be dynamical systems and let $\varphi : X \to Y$ be a continuous map. If $\varphi \circ \pi^g = \varrho^g \circ \varphi$ for any $g \in G$ then $\varphi$ is called a *homomorphism* of the system $(G, X, \pi)$ into the system $(G, Y, \varrho)$. If in addition $\varphi$ is a homeomorphism (surjection) of spaces then $\varphi$ is called an *isomorphism* (*epimorphism*) of dynamical systems.

A dynamical system $(G, X, \pi)$ is called *minimal* if there is no proper closed non-empty set $F \subseteq X$ such that $\pi^g(F) \subseteq F$ for each $g \in G$. The system $(G, X, \pi)$ is minimal iff the orbit $\pi_x(G)$ is dense in $X$ for each $x \in X$ and iff for each non-empty open set $U \subseteq X$ there are $g_1, \dots, g_n \in G$ such that $\pi^{g_1}(U) \cup \dots \cup \pi^{g_n}(U) = X$.

A topological group is *$\omega$-bounded* if for any open neighbourhood $V$ of the identity element there is a countable set $F \subseteq G$ such that $G = FV$.

Topological spaces are called *coabsolute* if their Boolean algebras of regular open subsets are isomorphic.

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It was proved by Balcar and Błaszczyk [1] that if $(G, X, \pi)$ is a minimal dynamical system and $G$ is a discrete, countable group then $X$ is coabsolute to a Cantor cube $D^\tau$, where $\omega \leqq \tau \leqq 2^\omega$. Bandlow in [2] has shown the following generalization of the above result:

**Theorem 1 (Bandlow).** *If $(G, X, \pi)$ is a minimal dynamical system and $G$ is $\omega$-bounded then $X$ is coabsolute to a Cantor cube $D^\tau$ for some cardinal $\tau$.*

Bandlow proved this theorem using so-called elementary substructures. We will give a direct topological proof of this fact. Our proof is based on the considerations contained in a paper of Uspenskii [8]. But before we analyze the proof, I would like to add some additional facts crucial to my point.

A map $f : X \to Y$ is *skeletal* if $\operatorname{int} f(U) \neq \emptyset$ for any non-empty open set $U \subseteqq X$.

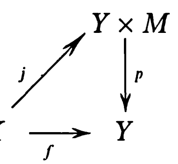
**Lemma.** *If $\varphi$ is an epimorphism of a minimal dynamical system $(G, X, \pi)$ onto a system $(G, Y, \varrho)$ then $\varphi$ is a skeletal map.*

**Proof.** Let $U$ be a non-empty open subset of $X$. Let us choose a non-empty open set $V$ so that $\operatorname{cl} V \subseteqq U$. Since $(G, X, \pi)$ is minimal, then there exist $g_1, \ldots, g_n$ from $G$ such that $\pi^{g_1}(V) \cup \ldots \cup \pi^{g_n}(V) = X$. Thus

$$Y = \varphi(X) = \varphi(\pi^{g_1}(V) \cup \ldots \cup \pi^{g_n}(V)) = \varrho^{g_1}(\varphi(V)) \cup \ldots \cup \varrho^{g_n}(\varphi)V)).$$

Hence $\emptyset \neq \operatorname{int} \operatorname{cl} \varphi(V) \subseteqq \operatorname{int} \varphi(U)$. □

A map $f : X \to Y$ has a *countable weight* if $f$ can be factorized in the following way:




where $j$ is an embedding, $p$ is a projection on the first factor and $M$ is a compact metric space.

A space $X$ is called *Dugundji* if for any closed subset $F$ of the Cantor cube $D^\tau$ and for any continuous map $f : F \to X$ there exists continuous map $\tilde{f} : D^\tau \to X$ such that $\tilde{f} \restriction F = f$. Clearly all Dugundji spaces are dyadic. We shall use the following theorems of Shapiro (see [5] and [6]).

**Theorem 2 (Shapiro).** *A compact space $X$ is coabsulate to a Dugundji space iff $X = \varprojlim \{X_\alpha, p_\alpha^\beta; \alpha < \beta < \kappa\}$ where $\{X_\alpha, p_\alpha^\beta; \alpha < \beta < \varkappa\}$ is spectrum of compact Hausdorff spaces of length $\varkappa$ such that:*

*a)* $|X_0| = 1$

*b)* $X_\lambda = \varprojlim \{X_\alpha, p_\alpha^\beta; \alpha < \beta < \lambda\}$ *for each limit ordinal* $\lambda < \kappa$

*c) all projections* $p_\alpha^\beta$ *are skeletal*

*d)* $p_\alpha^{\alpha+1}$ *has countable weight for each* $\alpha < \kappa$.

A space is *homogenous with respect to weight ($\pi$-weight)* if the weight (resp. $\pi$-weight) of any non-empty open subset equals the weight ($\pi$-weight) of the whole space.

**Theorem 3 (Shapiro).** *A dyadic space homogenous with respect to weight is coabsolute to the Cantor cube $D^{\tau}$ for some cardinal $\tau$.*

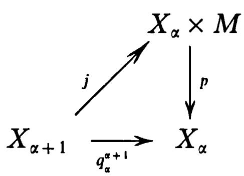
**Proof of the Bandlow's Theorem.** Let $\mathscr{R}$ be a family of all equivalence relations on $X$, closed in $X \times X$ and invariant with respect to family of maps $\{\pi^g \times \pi^g : g \in G\}$. The family $\mathscr{R}$ has the following properties:

(1) $X/R$ is compact Hausorff space for any $R \in \mathscr{R}$
(2) $\mathscr{R}$ is closed under intersection
(3) Group $G$ acts continuously on $X/R$ for each $R \in \mathscr{R}$ in the following way:

$$\tilde{\pi}(g, [x]_R) = [\pi(g, x)]_R$$

Of course, the quotient map $q_R : X \to X/R$ is an epimorphism of the system $(G, X, \pi)$ onto $(G, X/R, \tilde{\pi})$ for each $R \in \mathscr{R}$. Thus by Lemma map $q_R$ is skeletal for each $R \in \mathscr{R}$.

Let $\mathscr{R}_m$ be subfamily of $\mathscr{R}$ consisting of all relations $R$ such that $X/R$ is metrizable. Let $\mathscr{R}_m = \{R_\alpha : \alpha < \kappa\}$ for some cardinal $\kappa$. We define a new sequence of relations $\{Q_\alpha : \alpha < \kappa\}$ where $Q_\alpha = \cap\{R_\beta : \beta < \alpha\}$ and a sequence of compact spaces $\{X_\alpha : \alpha < \kappa\}$ where $X_\alpha = X/Q_\alpha$. Let $q_\alpha : X \to X/Q_\alpha$ be the quotient map. The natural projection $q_\alpha^\beta : X_\beta \to X_\alpha$, $\alpha < \beta$, are skeletal because each $q_\alpha$ is skeletal. Moreover $q_\alpha^{\alpha+1}$ has countable weight. Indeed, if $M = X/R_\alpha$ and $j([x]_{\alpha+1}) := ([x]_\alpha, [x]_{R_\alpha})$ then $j$ is embedding and the following diagram commutes




It is known that if $G$ is $\omega$-bounded group and acts on $X$ then for any continuous function $f : X \to \mathbb{R}$ then set $\{f \circ \pi^g : g \in G\}$ is separable subspace of space of continuous functions $C(X)$ with topology of uniform convergence, see [8]. Thus, if for $x, y \in X$, $x \neq y$ we find continuous function $f \in C(X)$ such that $f(x) \neq f(y)$ and define $R_f = \{(x, y) \in X \times X : f(\pi^g(x)) = f(\pi^g(y))$ for each $g \in G\}$ then $R_f \in \mathscr{R}$ and $X/R_f$ is metrizable. Indeed $X/R_f$ can be embedded in the product $\prod\{h(X) : h \in A\}$, where $A$ is dense, countable subset of $\{f \circ \pi^g : g \in G\}$.

So, we have shown that $X$ is homeomorphic to the $\varprojlim\{X_\alpha, q_\alpha^\beta : \alpha < \beta < \kappa\}$ and by Theorem 2 $X$ is coabsolute to some Dugundji space $Y$. It follows from minimality of our system that space $X$ is homogeneous with respect to $\pi$-weight. One can easily show that spaces have the same $\pi$-weight whenever they are coabsolute. So, $Y$ is also homogeneous with respect to $\pi$-weight. On the other hand, in dyadic spaces weight and $\pi$-weight coincide. Thus space $Y$ is homogeneous with respect to weight. The second theorem of Shapiro implies that $X$ is coabsolute to Cantor cube. The proof is complete. □

A minimal dynamical system $(G, X, \pi)$ is called *universal minimal dynamical system* for a group $G$ if for each minimal dynamical system $(G, Y, \varrho)$ there exists a homomorphism $\varphi : (G, X, \pi) \to (G, Y, \varrho)$. This is known that for every topological group $G$ there is a universal minimal dynamical system which is unique up to isomorphisms, see e.g. [9, IV.3.17, IV.4.43.3]. Let $M(G)$ denotes the phase space of the universal minimal dynamical system for a group $G$.

With aid of Bandlow's Theorem, we can prove the following:

**Corollary.** *The phase space of the universal minimal dynamical system for the group of real numbers with usual topology is coabsolute to the Cantor cube* $D^{2^\omega}$.

**Proof.** Let $b\mathbb{R}$ denotes Bohr compactification of the group of real numbers $\mathbb{R}$. Since any topological group acts on its Bohr compactification in minimal way then there is a homomorphism $\varphi : M(\mathbb{R}) \to b\mathbb{R}$ of the universal minimal dynamical system into the minimal system with the phase space $b\mathbb{R}$. In fact, homomorphism $\varphi$ is epimorphism, then by Lemma $\varphi$ is skeletal. Thus $\pi w(b\mathbb{R}) \leqq \pi w(M(\mathbb{R}))$. It is known that for topological groups $\pi$-weight and weight are equal (c.f. [3, 3.6(ii)]) and weight of Bohr compactification of locally compact Abelian group $G$ equals power of group of characters $\hat{G}$, see [4, Chap. VI] for details, In our case, group of characters of $\mathbb{R}$ is topologically isomorphic to $\mathbb{R}$. Hence

$$\pi w(b\mathbb{R}) = w(b\mathbb{R}) = 2^\omega .$$

On the other hand $\pi w(M(\mathbb{R})) \leqq w(M(P)) \leqq 2^\omega$, because $M(\mathbb{R})$ is separable. So, Theorem 1 implies that $M(\mathbb{R})$ is coabsolute to the Cantor cube $D^{2^\omega}$. □

**Remark.** Let us note that if we consider the group $\mathbb{R}_d$ of real numbers with discrete topology, then $\pi$-weight of $M(\mathbb{R}_d)$ equals $2^{2^\omega}$, see [7]. Thus, in this case $M(\mathbb{R}_d)$ has not to be coabsolute to the cube $D^{2^\omega}$. Therefore the structure of $M(G)$ strongly depends on the topology of the group.

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