PLANE WAVES, BIREGULAR FUNCTIONS AND HYPERCOMPLEX FOURIER ANALYSIS

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Abstract. In this paper we construct a formula for the biregular extension of an analytic function in $R^m x R^m$. We apply these formulae to the exponetial function $e^{i<\vec{t},\vec{x}>}$, the polynomials $<\vec{x},\vec{t}>^k$ and to plane wave functions $f(<\vec{x},\vec{t}>)$. We show that the biregularity conditions for extensions of plane waves may be expressed by eight equations in five dimensions; the so called biregular plane wave equations.
The complexified biregular exponential function $E(\tau,z)$ is used to define a general hypercomplex Fourier-Borel type transform and we investigate a specialized version of this transform.

Introduction. Let $\Omega\subset R^{m+1} x R^{m+1}$ be open. Then a function $f\in C_1(\Omega;A)$, A being a complex Clifford algebra, is called biregular in Ω if f satisfies $D_x f(x,t)=f(x,t)D_t=0$, where

$D_x=\sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j}$, $D_t=\sum_{j=0}^{m} e_j \frac{\partial}{\partial t_j}$ are generalized Cauchy-Riemann operators

(see [1],[2],[3],[5],[7]).
For this theory of functions, there exists a Cauchy-Kowalewski type theorem, which allows us to construct a formula for the biregular extension of analytic functions in $R^m x R^m$.
First we apply this formula in order to construct the biregular exponential function $E(t,x)$ as the biregular extension of $\exp(i<\vec{t},\vec{x}>)$, $(\vec{t},\vec{x})\in R^m x R^m$.
The explicit calculation of $E(t,x)$ leads to hypercomplex generalizations $L_{k,1}(\vec{t},\vec{x})$ of the classical Laguerre polynomials. Furthermore it turns out that $E(t,x)$ depends only on the five variables $(x_0,t_0,|\vec{x}|^2,|\vec{t}|^2,<\vec{x},\vec{t}>)$.

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Next we define the fundamental biregular polynomials $S_k(t,x)$ as the biregular extension of $<\vec{x},\vec{t}>^k$, $(\vec{x},\vec{t})\in R^m x R^m$. Furthermore we give the expression of $S_k(t,x)$ in terms of the polynomials $(<\vec{x},\vec{t}>-x_0\vec{t})^k$ and the operators $(<\vec{x},\nabla_t>-x_0 D_{0,t})^k$ (see [7],[10]) and the Fueter polynomials (see [1],[5]).
In the third section we establish the equations satisfied by biregular extensions of plane waves $f(<\vec{x},\vec{t}>)$. These equations are expressed in the five variables $(x_0,t_0,|\vec{x}|^2,|\vec{t}|^2,<\vec{x},\vec{t}>)$.
In the fourth section we recall some basic facts about hypercomplex analytic functionals (see [1],[4]) and we define the carrier of a hypercomplex functional.
The final section is devoted to the Fourier-Borel transform of hypercomplex analytic functionals. We study the transform of a functional T :

$$FT(\vec{z})=<T_t, e^{i<\vec{t},\vec{z}>}(\operatorname{ch}t_0[\vec{z}]-\frac{i\vec{z}}{[\vec{z}]}\operatorname{sh}t_0[\vec{z}])>,$$

where $[\vec{z}]=(\sum_{j=1}^{m} z_j^2)^{1/2}$.

Furthermore we give estimates for this transform and we show that, if a holomorphic function f satisfies these estimates; then f is the Fourier-Borel transform of a functional T, for which we can study the carrier in terms of the given estimates of f.

1. A biregular exponential function

Let $\Omega\subseteq R^{m+1}xR^{m+1}$ be open and let $f(x,t)$, $(x,t)\in\Omega$ be a C_1-function in Ω. Then f is called biregular in Ω if

$$D_x f(x,t)=f(x,t)D_t=0,$$

where $D_x=\sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j}$, $D_t=\sum_{j=0}^{m} e_j \frac{\partial}{\partial t_j}$, $e_0=1$.

In the theory of biregular functions, the following Cauchy-Kowalewski type theorem is valid.

Theorem 1 Let f be analytic in an open set $\Omega\subseteq R^m x R^m$. Then there exists a unique biregular extension $\tilde{f}$ of f, defined in a neighbourhood $\tilde{\Omega}$ of Ω in $R^{m+1}xR^{m+1}$.

Put $D_x=\frac{\partial}{\partial x_0}+D_{0,x}$, $D_t=\frac{\partial}{\partial t_0}+D_{0,t}$; then it is easy to see that the biregular extension $\tilde{f}(x,t)$ of $f(\vec{x},\vec{t})$, $x=x_0+\vec{x}$, $t=t_0+\vec{t}$, is given by

$$\tilde{f}(x,t)=\sum_{k,l=0}^{\infty}\frac{x_0^k t_0^l}{k!l!}(-D_{0,x})^k f(\vec{x},\vec{t})(-D_{0,t})^l.$$

Notice that every entire analytic function f in $R^m x R^m$ has an entire biregulier exentension $\tilde{f}$ to $R^{m+1} x R^{m+1}$.
We now introduce the biregular exponential function by

Definition 1. The biregular exponential function $E(t,x)$ is the biregular extension to $R^{m+1} x R^{m+1}$ of the function $f(\vec{x},\vec{t})=\exp(i<\vec{x},\vec{t}>)$.

Notice that

$$E(\vec{t},x)=E(t,x)|_{t_0=0}=e^{i<\vec{t},\vec{x}>}(\mathrm{ch}|\vec{t}|x_0-\frac{i\vec{t}}{|\vec{t}|}\mathrm{sh}|\vec{t}|x_0),$$

(see [7], [10]).
The calculation of $E(t,x)$ may be done in terms of so called generalized Laguerre polynomials.

Definition 2. The generalized Laguerre polynomials $L_{k,1}(\vec{t},\vec{x})$, $(\vec{t},\vec{x})\in R^m x R^m$, are determined by

$$E(t,x)=\sum_{k,1}\frac{x_0^k t_0^1}{k!1!}L_{k,1}(\vec{t},\vec{x})e^{i<\vec{x},\vec{t}>},$$

From the biregularity of $E(t,x)$, $D_x E(t,x)=0$ and $E(t,x)D_t=0$, it follows immediately that

$$L_{k+1,1}(\vec{t},\vec{x})=-(D_{0,x}+i\vec{t})L_{k,1}(\vec{t},\vec{x})$$

(1)

$$L_{k,1+1}(\vec{t},\vec{x})=-L_{k,1}(\vec{t},\vec{x})(D_{0,t}+i\vec{x})$$

As $L_{0,0}=1$, we hence obtain that

$$L_{k,0}(\vec{t},\vec{x})=(-i\vec{t})^k,\ L_{0,1}(\vec{t},\vec{x})=(-i\vec{x})^1,$$

and so,

$$L_{k,1}(\vec{t},\vec{x})=(-1)^{k+1}i^1(D_{0,x}+i\vec{t})^k\vec{x}^1$$

$$=(-1)^{k+1}i^k\vec{t}^k(D_{0,t}+i\vec{x})^1,$$

which is a polynomial of bidegree (k,1) in $(\vec{t},\vec{x})$.
Furthermore, we also have that

$$L_{k,1}(\vec{t},\vec{x})=(-1)^{k+1}i^1 e^{-i<\vec{t},\vec{x}>}D^k_{0,x}(\vec{x}^1 e^{i<\vec{t},\vec{x}>}),$$

a formula which is similar to the definition of the Laguerre polynomials (see [6]) :

$$L_n(x)=\frac{1}{n!}x^{-\alpha}e^x\left(\frac{d}{dx}\right)^n(e^{-x}x^{n+\alpha}).$$

As $(D_{0,x}+it)^2=-\Delta_m+|\vec{t}|^2-2i<\vec{t},\nabla>$, we obtain that

$$L_{2k,2l}(\vec{t},\vec{x})=(-1)^k(\Delta_m+i<\vec{t},\nabla>-|\vec{t}|^2)^k|\vec{x}|^{2l}$$

$$=(-1)^k e^{-i<\vec{t},\vec{x}>}\Delta_m^k(|\vec{x}|^{2l}e^{i<\vec{t},\vec{x}>}),$$

which is a $\mathcal{C}$-valued polynomial, only depending on $|\vec{t}|^2$, $|\vec{x}|^2$ and $<\vec{x},\vec{t}>$.

Hence, in view of the recursion formulae (1) and the definition of $E(t,x)$, it follows that

$$E(t,x) = A + \vec{x}B + \vec{t}C + \vec{x}\wedge\vec{t}D,$$

A,B,C,D being $\mathcal{C}$-valued functions, depending only on five variables, namely $(x_0,t_0,|\vec{x}|^2,|\vec{t}|^2,<\vec{x},\vec{t}>)$.

Hence $E(t,x)$ consists of a scalar part A, a vector part $\vec{x}B+\vec{t}C$ and a bivector part $\vec{x}\wedge\vec{t}D$, $\vec{x}\wedge\vec{t}=\frac{1}{2}(\vec{x}\vec{t}-\vec{t}\vec{x})$.

Functions of the form $A+\vec{x}B+\vec{t}C+\vec{x}\wedge\vec{t}D$, where A,B,C,D depend only on $(x_0,t_0,|\vec{x}|^2,|\vec{t}|^2,<\vec{x},\vec{t}>)=(x_0,t_0,\rho,\tau,\theta)$ are called biregular plane waves.

2. Fundamental biregular polynomials

The fundamental biregular polynomials $S_k(t,x)$ are introduced by

Definition 3. $S_k(t,x)$, $k\in\mathbb{N}$, is the biregular extension of the function $(\vec{x},\vec{t})\to<\vec{x},\vec{t}>^k$, and is called the k th fundamental biregular polynomial.

The polynomials $S_k(t,x)$ occur in the Taylor expansion of biregular plane waves. Let $f(z)=\sum_{k=0}^{\infty} c_k z^k$ be holomorphic. Then the biregular extension of the plane wave $f(<\vec{x},\vec{t}>)$ is given by $\sum_{k=0}^{\infty} c_k S_k(t,x)$.

As an example , we have that

$$(2)\qquad E(t,x)=\sum_{k=0}^{\infty}\frac{i^k}{k!}S_k(t,x),$$

We shall now derive several expressions for the polynomials $S_k(t,x)$. First of all we have

Proposition 1. The polynomials $S_k(t,x)$ are given by

$$S_k(u,x)=\frac{1}{k!}(<\vec{x},\nabla_t>-x_0D_{0,t})^k(<\vec{t},\vec{u}>-u_0\vec{t})^k.$$

Proof. It is clear that the above expression is biregular, since the functions $(<\vec{t},\vec{u}>-u_0\vec{t})^k$ and $(<\vec{x},\nabla_t>-x_0D_{0,t})^k$ are monogenic in u and x. Furthermore the restriction of this expression to $x_0=t_0=0$ equals

$$\frac{1}{k!}<\vec{x},\nabla_t>^k<\vec{t},\vec{u}>^k=<\vec{x},\vec{u}>^k,$$

and so, the conditions of Definition 3 are satisfied. ∎

Next, let $(k_1,\ldots,k_m)\in N^m$ be such that $\sum_{j=1}^{m} k_j=k$. Then we may consider the Fueter polynomials

$$Z_{k_1\ldots k_m}(x)=z_1^{k_1}\odot\ \ldots\ \odot z_m^{k_m}\ ,\ z_j=x_j-e_jx_0,$$

which are the monogenic extensions of $x_1^{k_1}\ldots x_m^{k_m}$ (see [1],[5]). We now give the expression of $S_k(t,x)$ in terms of the Fueter polynomials.

Proposition 2. The fundamental biregular polynomials $S_k(t,x)$ are given by

$$S_k(t,x)=\sum_{(k_1,\ldots,k_m)}\frac{k!}{k_1!\ldots k_m!}\,Z_{k_1\ldots k_m}(x)Z_{k_1\ldots k_m}(t).$$

Proof. The above expression is clearly biregular. Furthermore its restriction to $x_0=t_0=0$ equals

$$\sum_{(k_1\ldots k_m)}\frac{k!}{k_1!\ldots k_m!}(t_1x_1)^k\ \ldots(t_mx_m)^{k_m}=<\vec{x},\vec{t}>^k.$$

Hence, again the conditions of Definition 3 are satisfied. ∎

Next, let us recall that the functions

$$(<\vec{x},\vec{t}>-x_0\vec{t})^k(<\vec{u},\vec{s}>-u_0\vec{s})^k$$

are biregular in $(x,u)\in R^{m+1}xR^{m+1}$, and this for every $(\vec{t},\vec{s})\in S^{m-1}xS^{m-1}$. Hence we wonder if the polynomial $S_k(u,x)$ may be expressed in terms of these polynomials. We indeed have

Proposition 3. There exist real measures $\mu_k(\vec{t},\vec{s})$ on $S^{m-1}xS^{m-1}$ such that

$$S_k(u,x)=\int_{S^{m-1}\times S^{m-1}}(\langle\vec{x},\vec{t}\rangle-x_0\vec{t})^k(\langle\vec{u},\vec{s}\rangle-u_0\vec{s})^k d\mu_k(\vec{t},\vec{s}).$$

Proof. It is easy to see that span $\{\langle\vec{x},\vec{t}\rangle^k \| \vec{t}\in S^{m-1}\}$ contains all homogeneous polynomials of degree k. Hence there exist measures $\mu_{k_1\ldots k_m}(\vec{t})$ on S^{m-1} such that

$$x_1^{k_1}\ldots x_m^{k_m}=\int_{S^{m-1}}\langle\vec{x},\vec{t}\rangle^k d\mu_{k_1\ldots k_m}(\vec{t}).$$

This leads to

$$\langle\vec{x},\vec{t}\rangle^k=\sum_{k_j}\frac{k!}{k_1!\ldots k_m!}(x_1u_1)^{k_1}\ldots(x_m u_m)^{k_m}$$

$$=\int_{S^{m-1}\times S^{m-1}}\langle\vec{x},\vec{t}\rangle^k\langle\vec{u},\vec{s}\rangle^k d\mu_k(\vec{t},\vec{s}),$$

$$d\mu_k(\vec{t},\vec{s})=\sum_{k_j}\frac{k!}{k_1!\ldots k_m!}d\mu_{k_1\ldots k_m}(\vec{t})\otimes d\mu_{k_1\ldots k_m}(\vec{s}).$$

Proposition 3 follows by taking the biregular extension of this formula. ∎

3. The biregular plane wave equations

Let $P(\frac{\partial}{\partial t},D)$ be a differential operator, $D=\nabla_m$, for which a Cauchy-type extension theorem with respect to t is valid. Then we can calculate Cauchy extensions $f(t,\langle\vec{x},\vec{t}\rangle)$ of plane waves $f(\langle\vec{x},\vec{t}\rangle)$, by expressing the system $P(\frac{\partial}{\partial t},D)f=0$ in terms of the variables t and $\langle\vec{x},\vec{t}\rangle$. These equations are called the P-plane wave equations.

Example 1. If $P=\frac{\partial^2}{\partial t^2}-\Delta$, the plane wave equations are simply given by

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)f=0.$$

Example 2. Let $P=\frac{\partial}{\partial x_0}+\sum_{j=1}^{m}e_j\frac{\partial}{\partial x_j}$. Then the plane wave type solutions of $Pf=0$ are of the form

$$g_1(\langle\vec{x},\vec{t}\rangle,x_0|\vec{t}|)-\frac{\vec{t}}{|\vec{t}|}g_2(\langle\vec{x},\vec{t}\rangle,x_0|\vec{t}|),$$

where (g_1,g_2) satisfy the usual Cauchy Riemann equations in the plane (see [10]).

Similar questions may be put for the biregular system. Let $f(\langle\vec{x},\vec{t}\rangle)$

be a plane wave; how to describe the biregular extension $\tilde{f}(x,t)$ of this plane wave and which are the variables needed in order to give such a description?
We shall show that this problem may be solved in five dimensions, namely $(x_0,t_0,|\vec{x}|^2,|\vec{t}|^2,\langle\vec{x},\vec{t}\rangle)=(x_0,t_0,\rho,\tau,\theta)$.
Hence, we generalize the concept of biregular plane wave to

Definition 4. A biregular plane wave is a biregular function of the form $A+\vec{t}B+\vec{x}C+\vec{x}\wedge\vec{t}D$, where A,B,C,D are $\mathcal{C}$-valued functions, depending on the variables $(x_0,t_0,\rho,\tau,\theta)$.

The biregular plane wave equations are the biregularity conditions applied on a biregular plane wave and expressed in terms of the coordinates $(x_0,t_0,\rho,\tau,\theta)$.
We show that this is indeed possible. Let $f=A+\vec{t}B+\vec{x}C+\vec{x}\wedge\vec{t}D$ be a biregular plane wave.
Then we have that

$$\left(\frac{\partial}{\partial x_0}+D_{0,x}\right)f$$

$$=\frac{\partial}{\partial x_0}A+D_{0,x}\rho\cdot\frac{\partial A}{\partial\rho}+D_{0,x}\theta\cdot\frac{\partial A}{\partial\theta}$$

$$+\vec{t}\frac{\partial}{\partial x_0}B+D_{0,x}\rho\cdot\vec{t}\frac{\partial B}{\partial\rho}+D_{0,x}\theta\cdot\vec{t}\frac{\partial B}{\partial\theta}$$

$$+\vec{x}\frac{\partial}{\partial x_0}C+D_{0,x}\rho\cdot\vec{x}\frac{\partial C}{\partial\rho}+D_{0,x}\theta\cdot\vec{x}\frac{\partial C}{\partial\theta}+D_{0x}\vec{x}.C$$

$$+\vec{x}\wedge\vec{t}\frac{\partial D}{\partial x_0}+D_{0,x}\rho\,(\vec{x}\wedge\vec{t})\frac{\partial D}{\partial\rho}+D_{0,x}\theta\,(\vec{x}\wedge\vec{t})\frac{\partial D}{\partial\theta}$$

$$+D_{0,x}(\vec{x}\wedge\vec{t}).D,$$

where

$$D_{0,x}\rho=2\vec{x},\ D_{0,x}\theta=\vec{t},\ D_{0,x}\theta.\vec{t}=-\tau,$$

$$D_{0,x}\rho.\vec{x}=-2\rho,\ D_{0,x}\rho.\vec{t}=2\vec{x}\vec{t}=2(\vec{x}\wedge\vec{t}-\theta),$$

$$D_{0,x}\theta.\vec{x}=\vec{t}.\vec{x}=-(\vec{x}\wedge\vec{t}+\theta),\ D_{0,x}\vec{x}=-m,$$

$$D_{0,x}\rho(\vec{x}\wedge\vec{t})=2\vec{x}(\vec{x}\vec{t}+\theta)=2\ (\theta\vec{x}-\rho\vec{t}),$$

$$D_{0,x}\theta(\vec{x}\wedge\vec{t})=-\vec{t}(\vec{t}\vec{x}+\theta)=(\tau\vec{x}-\theta\vec{t})$$

$$D_{0,x}(\vec{x}\wedge\vec{t})=D_{0,x}(\vec{x}\vec{t}+\langle\vec{x},\vec{t}\rangle)=(1-m)\vec{t}.$$

As a similar expression holds for $f(\frac{\partial}{\partial t_0}+D_{0,t})$, one can easily show that the biregular plane wave equations are given by

$$\frac{\partial A}{\partial x_0}-2\theta\frac{\partial B}{\partial\rho}-\frac{\partial B}{\partial\theta}-2\rho\frac{\partial C}{\partial\rho}-\theta\frac{\partial C}{\partial\theta}-mC=0$$

$$\frac{\partial B}{\partial x_0}+\frac{\partial A}{\partial\theta}-2\rho\frac{\partial D}{\partial\rho}-\theta\frac{\partial D}{\partial\theta}+(1-m)D=0$$

$$\frac{\partial C}{\partial x_0}+2\frac{\partial A}{\partial\rho}+2\theta\frac{\partial D}{\partial\rho}+\tau\frac{\partial D}{\partial\theta}=0$$

$$\frac{\partial D}{\partial x_0}-\frac{\partial C}{\partial\theta}+2\ \frac{\partial B}{\partial\rho}=0$$

$$\frac{\partial A}{\partial t_0}-2\theta\frac{\partial C}{\partial\tau}-\rho\frac{\partial C}{\partial\theta}-2\tau\frac{\partial B}{\partial\tau}-\theta\frac{\partial B}{\partial\theta}-mB=0$$

$$\frac{\partial B}{\partial t_0}+\frac{\partial A}{\partial\theta}-2\tau\frac{\partial D}{\partial\tau}-\theta\frac{\partial D}{\partial\theta}+(1-m)D=0$$

$$\frac{\partial C}{\partial t_0}+2\frac{\partial A}{\partial\tau}+2\theta\frac{\partial D}{\partial\tau}+\rho\frac{\partial D}{\partial\theta}=0$$

$$\frac{\partial D}{\partial t_0}-\frac{\partial B}{\partial\theta}+2\frac{\partial C}{\partial\tau}=0$$

We hence obtain two groups of four equations in five dimensions. The second group follows from the first by replacing x_0 by t_0, ρ by τ, τ by ρ, C by B and B by C.
Next we can wonder whether we can describe biregular plane waves in less than five dimensions. Of course they depend on the variables $(x_0,t_0,\langle\vec{x},\vec{t}\rangle)$. Without the proof we state

<u>Theorem 2</u>. For $m>1$, the biregular plane wave equations can't be formulated in less than five dimensions.

4. Elementary duality theory

Let $K\subseteq R^{m+1}$ be compact and $M_{(1)}(K;A)$ the left A-module of right monogenic functions on K. Then we have the duality theorem (see [4])

<u>Theorem 3</u>. The strong dual $M'_{(1)}(K;A)$ is isomorphic to the space $M_{(r),0}(R^{m+1}\setminus K;A)$ of left monogenic functions in $R^{m+1}\setminus K$, tending to zero at infinity.

The isomorphism is obtained using the Cauchy-Fantappiè indicatrix

$\hat{T}$ of $T\in M'_{(1)}(K;A)$, which is given by (see [4])

$$\hat{T}(x)=\frac{1}{\omega_{m+1}}<T_y,\frac{\overline{x}-\overline{y}}{|x-y|^{m+1}}>.$$

Furthermore for $f\in M_{(1)}(K,A)$ (see [4])

$$<T,f>=\int_{\partial K_\varepsilon} f(x)d\sigma_x\hat{T}(x),$$

K_ε being a suitable ε-neighbourhood of K.
Next, we have that $M_{(1)}(R^{m+1};A)\subseteq M_{(1)}(K;A)$
Hence to every $T\in M'_{(1)}(K,A)$ we can associate $\theta(T)\in M'_{(1)}(R^{m+1};A)$ in a natural way and we have Runge's theorem (see [1])

Theorem 4. θ is injective if and only if K is simply connected in the sense that $R^{m+1}\setminus K$ has only one connected component.

This leads to

Definition 5. Let $T\in M'_{(1)}(R^{m+1};A)$. Then a compact set K is called a carrier of T if
(i) K is simply connected
(ii) T is extendable to $M_{(1)}(K;A)$.

Notice that the indicatrix $\hat{T}$ admits a unique extension to $R_+^{m+1}\setminus K$.
Of course the notion of carrier differs from the notion of support.
The carrier is not unique. Take e.g. $T=\delta_{B_m(0,1)}=e_{(r)}\delta_{S^m\cap R_+^{m+1}}$,

$B_m(0,1)$ the unit ball in R^m, S^m the unit sphere in R^{m+1},

$R_+^{m+1}=\{x\in R^{m+1}\Vert x_0\geq 0\}$ and $e_{(r)}$ the unit normal on S^m. Then T is carried by both $B_m(0,1)$ and $S^m\cap R_+^{m+1}$ but not by $S^{m-1}=B_m(0,1)\cap S^m\cap R_+^{m+1}$, since $\hat{T}$ is not extendable to $R^{m+1}\setminus S^{m-1}$.
Hence, in general , the intersection of two carriers of T is itself not a carrier. There is however a very important exception, which is stated in

Theorem 5. Let $T\in M'_{(1)}(R^{m+1};A)$ be carried by K_1 and K_2 and let $K_1\cup K_2$ be simply connected. Then T is carried by $K_1\cap K_2$.

5. The Fourier-Borel transform

The general hypercomplex Fourier-Borel transform is introduced as follows. Let $E(\tau,z)$ be the complex extension of the biregular exponential function $E(t,x)$ and consider the dual $M'_{(1)}(C^{m+1};A)$ of the space of complex right monogenic functions. Then we introduce

Definition 6. Let $T\in M'_{(1)}(C^{m+1},A)$. Then the general Fourier-Borel transform $FT(z)$ of T is given by $FT(z)=\langle T_\tau,E(\tau,z)\rangle$

Notice that F transforms analytic functionals in complex monogenic sense into left monogenic functions.
For the sake of simplicity, we shall not consider this general transform, but only a specialized version.To that end, notice that the maps

$$\rho:M_{(1)}(C^{m+1};A)\to M_{(1)}(R^{m+1};A)$$

$$\kappa:M_{(1)}(C^{m+1};A)\to O_{(1)}(C^{m};A)$$

induced by the restrictions $f|_{R^{m+1}}$ and $f|_{C^m}$ of a complex monogenic function f are isomorphisms. Hence, the spaces $M'_{(1)}(C^{m+1};A)$, $M'_{(1)}(R^{m+1};A)$ and $O'_{(1)}(C^m;A)$ are in fact the same, but the notion of carrier is of course different (see also [3]). Furthermore, $FT(z)$ is completely determined by $\kappa(FT(z))=FT(\vec{z})$, so that, in principle, it is sufficient to study $FT(\vec{z})$ for $T\in M'_{(1)}(R^{m+1};A)$ or to study $FT(z)$, $T\in O'_{(1)}(C^m;A)$. The last transform has already been studied in [10]. In this paper we study the first specialized Fourier-Borel transform, which is given by

$$FT(\vec{z})=\langle T_t,e^{i\langle\vec{t},\vec{z}\rangle}(\operatorname{ch}t_0[\vec{z}]-\frac{\vec{z}}{[\vec{z}]}\operatorname{sh}t_0[\vec{z}])\rangle,$$

where $[\vec{z}]=(\sum_{j=1}^{n} z_j^2)^{\frac{1}{2}}$, $\operatorname{Re}[z]\geq 0$.

In order to study this transform, we make use of the splitting $E(t,\vec{z})=E_+(t,\vec{z})+E_-(t,\vec{z})$, where $E_\pm(t,\vec{z})=\frac{1}{2}(1\mp\frac{i\vec{z}}{[\vec{z}]})\exp(i\langle\vec{t},\vec{z}\rangle\pm t_0[\vec{z}])$,

and the corresponding transforms

$$F_\pm T(\vec{z})=\langle T_t,E_\pm(t,\vec{z})\rangle.$$

Let K' be a cilindrical domain of the form $K'=K\times[a,b]$, $a<b$, $K\subseteq R^m$ being compact. Then we call $H_K(\vec{y})=\sup_{\vec{t}\in K}(-\langle\vec{t},\vec{y}\rangle)$, the supporting

function of K.
Making use of the fact that $\mathrm{Re}[\vec{z}]<|\vec{x}|$, one can easily obtain the following estimates.

<u>Theorem 6</u>. Let T be represented by a measure in $K\times[a,b]$. Then $F_{\pm}T(\vec{z})$ and $FT(\vec{z})$ satisfy

$$\text{(i)}\quad |\vec{z}F_{+}T(z)|<C|\vec{z}|\exp(H_K(\vec{y})+b|\vec{x}|)$$
$$\text{(ii)}\quad |\vec{z}F_{-}T(\vec{z})|<C|\vec{z}|\exp(H_K(\vec{y})-a|\vec{x}|)$$
$$\text{(iii)}\quad |FT(\vec{z})|<C(1+|\vec{z}|)\exp(H_k(\vec{y})+\max(-a,b)|\vec{x}|).$$

Notice that, if T is carried by K'; then for every ε-neighbourhood K'_ε of K', T is represented by a measure in K'_ε.
We now prove some converse results to Theorem 6. To that end, we shall make use of the classical Fourier-Borel transform, studied by Martineau in [8] and [9]. Let $T\in\mathcal{O}'_{(1)}(\mathcal{C}^m;A)$ be carried by a convex compact set $K\subseteq\mathcal{C}^m$, let $H_K(\vec{z})=\sup_{\vec{\tau}\in K}(-<\vec{t},\vec{y}>-<\vec{s},\vec{x}>)$, $\vec{\tau}=\vec{t}+i\vec{s}$

and consider the classical Fourier-Borel transform

$$\mathcal{FB}(T)=<T_\tau,e^{i<\vec{\tau},\vec{z}>}\quad .$$

Then we shall apply Martineau's theorem to compact sets of the form $K+iB(0,\lambda)$, $K\subseteq\mathcal{R}^m$ being convex compact.
For the general theorem, see [8] and [9].

<u>Theorem 7</u>. Let $f\in\mathcal{O}(\mathcal{C}^m;A)$ be such that $|f(\vec{z})|<C\exp(H_K(\vec{y})+\lambda|\vec{x}|)$. Then $f=\mathcal{FB}(T)$ for some $T\in\mathcal{O}'_{(1)}(K+iB(0,\lambda);A)$.
<u>Proof</u>. It is sufficient to notice that $H_{(K+iB(0,\lambda))}(\vec{z})=H_K(\vec{y})+\lambda|\vec{x}|$ and to apply Martineau's theorem. ∎

Next, consider the isomorphism

$$\kappa\circ\rho^{-1}:M_{(1)}(\mathcal{R}^{m+1};A)\to\mathcal{O}_{(1)}(\mathcal{C}^m;A).$$

Then we shall study the extension of this map to

$$M_{(1)}(K_\lambda;A),K_\lambda=\{x\in\mathcal{R}^{m+1}\,|\,|x_0|^2+d(\vec{x},K)^2<\lambda^2\},$$

which, in view of Runge's theorem, is unique.

<u>Lemma</u>. Let $\lambda>0$ and $K\subseteq\mathcal{R}^m$ be convex compact. Then

$$\kappa\circ\rho^{-1}(M_{(1)}(K_\lambda;A))\subseteq\mathcal{O}_{(1)}(K+iB(0,\lambda);A).$$

Proof. Let $\lambda'>\lambda$ and K_ε be an ε-neighbourhood of K and let $f\in M_{(1)}(K_{\varepsilon,\lambda'};A)$.
Then in a neighbourhood of K in $\mathbb{C}^m$,

$$\kappa\circ\rho^{-1}(f)(\vec{z})=f(\vec{z})=\frac{1}{\omega_{m+1}}\int_{\partial K_{\varepsilon,\lambda'}} f(u)d\sigma_u\,\frac{-\overline{\vec{z}-u}}{[\vec{z}-u]^{m+1}}.$$

As $\mathrm{Re}[\vec{z}-u]^2=u_0^2+|\vec{x}-\vec{u}|^2-|\vec{y}|^2$, a necessary and sufficient condition for $-\frac{\overline{\vec{z}-u}}{[\vec{z}-u]^{m+1}}$ to be holomorphic in $K+iB(0,\lambda)$ is $u_0^2+d(\vec{u},K)^2>\lambda^2$. As this condition is fulfilled on $\partial K_{\varepsilon,\lambda'}$, $f(\vec{z})$ is holomorphic on $K+iB(0,\lambda)$, and this for every $\lambda'>\lambda$ and $\varepsilon>0$. ■

From this, we obtain

Theorem 8. Let $f\in\mathcal{O}(\mathbb{C}^m;A)$ be such that $|f(\vec{z})|\le C\exp(H_K(\vec{y})+\lambda|\vec{x}|)$, $\lambda>0$, $K\subseteq\mathbb{R}^m$ being convex compact. Then f is the Fourier-Borel transform of a functional $T\in M'_{(1)}(K_\lambda;A)$.
Proof. By Theorem 7, $f=\mathcal{F}\mathcal{B}T'$ for some $T'\in\mathcal{O}'_{(1)}(K+iB(0,\lambda);A)$. Let us consider $T=\kappa\circ\rho^{-1}(T')$, where $\langle\kappa\circ\rho^{-1}(T'),f\rangle=\langle T',\kappa\circ\rho^{-1}(f)\rangle$, f being monogenic. Then of course $\mathcal{F}T=\mathcal{F}\mathcal{B}T'$ and by the previous lemma, $T\in M'_{(1)}(K_\lambda;A)$. ■

Next, we shall assume that f is the Fourier-Borel transform of an analytic functional T and we consider the decomposition $f=f_++f_-$, where

$$f_\pm=\frac{1}{2}\Big(1\mp\frac{i\vec{t}}{[\vec{t}]}\Big)f=\mathcal{F}_\pm T.$$

The main result of this section is the following

Theorem 9. Let $f\in\mathcal{O}(\mathbb{C}^m;A)$ be the Fourier-Borel transform of an analytic functional T and assume that

(i) $|f_+(\vec{x})|\le c\exp(b|\vec{x}|)$

(ii) $|f_-(\vec{x})|\le c\exp(-a|\vec{x}|)$.

Then T is carried by a subset of $\mathbb{R}^m\times[a,b]$.
Proof. Let T be carried by $K'\subseteq\mathbb{R}^{m+1}$ and choose $R>0$ and $\alpha<a<b<\beta$ such that K' is in the interior of $B_m(0,R)\times[\alpha,\beta]$. Then $\hat{T}$ is defined on $\Sigma=\partial(B_m(0,R)\times[\alpha,\beta])$ and so

$$f_{\pm}(x)=\frac{1}{2}\int_{\Sigma}(1\mp i\frac{\vec{x}}{|\vec{x}|})e^{i<\vec{t},\vec{x}>\pm t_0|\vec{x}|}\,d\sigma_t\hat{T}(t).$$

First substract from $\hat{T}$ the first term in the Laurent expansion of $\hat{T}$ about the point $\frac{a+b}{2}$ and call this function F. Then we put

$$f'_{\pm}(x)=\frac{1}{2}\int_{\Sigma}(1\mp i\frac{\vec{x}}{|\vec{x}|})e^{i<\vec{t},\vec{x}>\pm t_0|\vec{x}|}d\sigma_t F(t)$$

and as $F(t)=O(|t|^{-m-1})$ if $|t|\to\infty$, by Cauchy's theorem

$$f'_{\pm}(\vec{x})=\frac{1}{2}\int_{t_0=\beta}(1\mp i\frac{\vec{x}}{|\vec{x}|})e^{i<\vec{t},\vec{x}>\pm\beta|\vec{x}|}F(t)d\vec{t}$$

$$-\frac{1}{2}\int_{t_0=\alpha}(1\mp i\frac{\vec{x}}{|\vec{x}|})e^{i<\vec{t},\vec{x}>\pm\alpha|\vec{x}|}F(t)d\vec{t}.$$

But $f_{\pm}(\vec{x})-f'_{\pm}(\vec{x})=FS$, where S is of the form $c\delta_{\frac{a+b}{2}}$, $c\in A$. Hence $f'_{\pm}$ satisfies the same estimates as $f_{\pm}$. Let us investigate f'_{+}. First of all, by Cauchy's theorem,

$$f'_{+}(\vec{x})=\frac{1}{2}\int_{t_0=\beta}(1-\frac{i\vec{x}}{|\vec{x}|})e^{i<\vec{t},\vec{x}>+\beta|\vec{x}|}F(t)d\vec{t}$$

so that

$$e^{-\beta|\vec{x}|}f'_{+}(\vec{x})=\frac{1}{2}\int_{R^m}(1-i\frac{\vec{x}}{|\vec{x}|})e^{i<\vec{t},\vec{x}>}F(\vec{t}+\beta)d\vec{t}.$$

Furthermore, again by Cauchy's theorem,

$$\frac{1}{2}\int_{R^m}(1+i\frac{\vec{x}}{|\vec{x}|})e^{i<\vec{t},\vec{x}>}F(\vec{t}+\beta)d\vec{t}=0,$$

so that

$$e^{-\beta|\vec{x}|}f'_{+}(\vec{x})=\int_{R^m}e^{i<\vec{t},\vec{x}>}F(\vec{t}+\beta)d\vec{t}.$$

Assume that $\beta-b=\varepsilon>0$. Then, as

$$\frac{1}{2}(1+\frac{i\vec{x}}{|\vec{x}|})e^{-\beta|\vec{x}|}f'_{+}(\vec{x})=0,$$

$$F_{+}(t)=\frac{1}{(2\pi)^m}\int_{R^m}e^{-i<\vec{t},\vec{x}>-t_0|\vec{x}|}e^{-\beta|\vec{x}|}f'_{+}(\vec{x})d\vec{x}$$

is left monogenic for $t_0>-\varepsilon$, since $e^{-\beta|\vec{x}|}f'_{+}(\vec{x})$ is of exponential growth $\exp((b-\beta)|\vec{x}|)$. Furthermore,

$$F_+(\vec{t})=\frac{1}{(2\pi)^m}\int_{R^m} e^{-i<\vec{t},\vec{x}>}\int_{R^m} e^{i<\vec{s},\vec{x}>}F(\vec{s}+\beta)d\vec{s}d\vec{x}$$
$$=F(\vec{t}+\beta),$$

which implies that for $t_0>-\varepsilon$, $F_+(t)=F(t+\beta)$ and so F is extendable to $t_0>\beta-\varepsilon=b$.
Similarly, by investigating f'_-, one finds that F is extendable to $t_0<\alpha+\varepsilon=a$.
Furthermore, as $\hat{T}=F+c\delta_{\frac{a+b}{2}}$, $\hat{T}$ is extendable to $R^{m+1}\backslash(B(0,R)x[a,b])$.■

By combining Theorem 8, Theorem 9 and Theorem 5 we obtain

<u>Theorem 10</u>. Let $f\in O(C^m;A)$ be such that

$$\text{(i)}\ |f(\vec{z})|\leq C\exp(H_K(\vec{y})+\lambda|\vec{x}|)$$

$$\text{(ii)}\ |f_+(\vec{x})|\leq C\exp(b|\vec{x}|)$$

$$\text{(iii)}\ |f_-(\vec{x})|\leq C\exp(-a|\vec{x}|).$$

Then f is the Fourier-Borel transform of an analytic functional T carried by $K_\lambda\cap(R^m x[a,b])$.

Notice that if $\lambda=a=b$, T is carried by Kx{a}. This result is very usefull in the theory of boundary values of monogenic functions, where $\lambda=a=b=0$ (see [11],[12]).

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