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METRICS CONFORMALLY EQUIVALENT TO BOUNDED GEOMETRY

Jürgen Eichhorn, Jan Fricke, Alexander Lang

1 Introduction

As well known, each open manifold M^n admits a metric g of bounded geometry of infinite order, i.e. $|\nabla^i R^g| \leq C_i$, $i = 0, 1, \ldots$, and $r_{\text{inj}}(M, g) > 0$. This has been proved by Greene in [6]. We used this existence theorem and the assumption of bounded geometry in many papers. For example, we established in [5] the existence of instantons for open 4-manifolds and SU(2)-bundles under certain conditions including bounded geometry of (M^n, g). On the other hand, many manifolds M^n are naturally endowed with a given metric which does not have bounded geometry. Fortunately, instantons depend only on the conformal class of (M^4, g). Hence, for applications in gauge theory and much more general, there arises the question which smooth metricsare conformally equivalent to a metric of bounded geometry. This is a rather delicate question. Already the local question for locally conformal flatness is settled by the Weyl tensor which is a highly nontrivial matter. It is very easy to endow $\mathbf{R}^n$ with a metric g such that g is not conformally equivalent to a metric of bounded geometry. This follows from the fact that the curvature tensor has $\frac{(n-1)n}{2} \cdot \left(\frac{(n-1)n}{2} + 1\right)/2$ components but considering $\tilde{g} = e^u \cdot g$, u and its first and second derivatives produce only $1 + n + \frac{n(n+1)}{2}$ quantities. More carefully spoken, this task leads to the existence problem for the system $R^{\tilde{g}}$ =bounded, $\nabla^{\tilde{g}} R^{\tilde{g}}$ =bounded and so on. Additionally, we have to estimate the injectivity radius. It seems to us that a general attack of this problem in one step is hopeless. Therefore it is adequate to start with simple handable classes of metrics and to enlarge this class step by step. In this paper, we prove the following theorem. Assume (M^n, g) is open with finitely many collared ends ε_i, $i = 1, \ldots, m$ and $g|_{\varepsilon_i} \cong dr^2 + f_i(r)^2 d\sigma^2_{N_i}$, $\int_{a_i}^{\infty} \frac{1}{f_i(r)} dr = \infty$. Then g is conformally equivalent to a metric $\tilde{g} = e^u \cdot g$ satisfying the conditions (B_∞) and (I) below. In a second step we enlarge the class for g rapidly. Finally we derive some applications to gauge theory. As an independent result we proof in this paper that every bounded Riemannian vector bundle $(E,h) \to (M^n,g)$, (M^n,g) with bounded geometry, admits a Riemannian connection ∇^h of bounded geometry.

2 The existence of bounded geometry for bundles

Let (M^n,g) be open. Consider the following conditions (I) and (B_k), $0 \le k \le \infty$,
(I) $\quad r_{\text{inj}}(M) = \inf_{x\in M} r_{\text{inj}}(x) > 0$,
(B_k) $\quad |\nabla^i R| \le C_i$, $0 \le i \le k$,
where $r_{\text{inj}}(x)$ denotes the injectivity radius, R the curvature tensor and $|\cdot|$ the pointwise norm. We say (M^n,g) has bounded geometry of order k if it satisfies (I) and (B_k).
Remark. (I) implies completeness of (M^n,g).□

Theorem 2.1 *Let M^n be open. Then there exists a metric g on M^n satisfying (I) and (B_∞).*

This is the main result of [6].□

Let $(E,h) \to (M^n,g)$ be a Riemannian vector bundle over (M^n,g), ∇ a metric connection associated to h. (E,∇) has bounded geometry of order k if it satisfies the condition
($B_k(E,\nabla)$) $\quad |\nabla^i R^E| \le C_i$, $0 \le i \le k$.
To satisfy the condition ($B_k(E,\nabla)$) is not a property determined by the fibre metric h of E alone. It really depends on the choice of the metric connetion ∇. We present a class of examples. Assume (M^n,g) with (I) and (B_k), $k \ge 0$ and (E,h,∇_1) where ∇_1 is a metric connection satsfying ($B_0(E,\nabla_1)$). If $\eta \in \Omega^1(\mathfrak{g}_E)$ is such that $d^{\nabla_1}\eta = 0$ and η is unbounded then $\nabla_2 = \nabla_1 + \eta$ is metric too but does not satisfy ($B_0(E,\nabla_2)$). This follows immediately from the equation

$$R^{\nabla_2} = R^{\nabla_1} + d^{\nabla_1}\eta + [\eta,\eta]. \tag{1}$$

The existenceof such an η follows under certain conditions. Let $x_1, x_2, \cdots \to \infty$ be a sequence in M with $\text{dist}(x_i,x_j) > r_{\text{inj}}(M)$, $i \ne j$. If there exists a common $0 < \varepsilon < r_{\text{inj}}(M)$ such that $\mathfrak{g}_E|_{U_\varepsilon(x_i)}$ has a parallel section $\xi_i \ne 0$ then we choose $u_i \in C_0^\infty(U_{\varepsilon/2}(x_i))$ such that $\sup\limits_{x\in U_{\varepsilon/2}(x_i)} |du_i \otimes \xi_i| \ge i$ and define

$$\eta(x) = \begin{cases} (du_i \otimes \xi_i)(x), & x \in U_{\varepsilon/2}(x_i) \\ 0, & x \text{ elsewhere.} \end{cases}$$

We obtain $d^{\nabla_1}\eta = 0$ and R^{∇_2} unbounded. This construction goes through, in particular, if ∇_1 is flat. Hence we proved

Proposition 2.2 *If (M^n,g) satisfies (I) and (B_k) and $(E,h,\nabla_1) \to M$ is a flat Riemannian vector bundle then there exists a second metric connection ∇_2 such that (E,∇_2) does not satisfy (B_0).*□

Corollary 2.3 *The setting of (E,h) alone does not determine ($B_k(E,\nabla)$) or not.*□

Assume (M^n, g) with (I) and (B_k). Fix a uniformly locally finite cover $\mathfrak{U} = \{(U_\alpha, u_\alpha^1, \ldots, u_\alpha^n)\}_\alpha$ of M by normal charts of radius $r < r_{\mathrm{inj}}(M)$. Such covers exists according to Calabi, Cheeger. In the sequel we consider only such covers $\mathfrak{U}$. According to [1], [2], there exists an associated decomposition $\{\psi_\alpha\}_\alpha$ of unity satisfying $|\nabla^i \psi_\alpha| \leq C_i$, $0 \leq i \leq k+2$. Let $(E,h) \xrightarrow{\pi} M$ be a Riemannian vector bundle of rank N. $E|_{U_\alpha}$ is trivial, there are N orthonormal sections $e_{1,\alpha}, \ldots, e_{N,\alpha} : U_\alpha \to \pi^{-1}(U_\alpha)$ which define trivializations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times E^N$ and converse. The Φ_α are fibrewise isometric. The Φ_α define transition functions $\Phi_\beta \Phi_\alpha^{-1} : U_\alpha \cap U_\beta \to O(N)$. The Φ_α are far from being uniquely determined. Let $\mathfrak{O}(N)$ be the sheaf defined by $U \to C^\infty(U, O(N))$. Then the bundle (E,h) is determined by an element ξ = class ξ of the cohomology set $H^1(M, \mathfrak{O}(N))$. Here we admit only covers of the kind above and their refinement. We define (E,h) to be by k bounded if there exists a representative cocycle $\{\Phi_\beta \Phi_\alpha^{-1}\}_{\alpha,\beta}$ of ξ such that

$$\sup_{0 \leq i \leq k-1} \sup_{\alpha,\beta} \sup_{x \in U_\alpha \cap U_\beta} |\nabla^i d\Phi_\beta \Phi_\alpha^{-1}|_x < \infty \qquad (k-b).$$

$(k-b)$ makes sense since $U_\alpha \cap U_\beta$, $O(N)$ are Riemannian manifolds.

Theorem 2.4 *Assume (M^n, g) with (I) and (B_k) and $(E,h) \xrightarrow{\pi} M$ a $(k+1)$-bounded Riemannian vector bundle. Then (E,h) has a metric connection ∇ satisfying (B_k).*

Proof: By assumption, there exists bundle atlas $\{(U_\alpha, \Phi_\alpha)\}_\alpha$ such that $\Phi_\beta\Phi_\alpha^{-1}\}_{\alpha,\beta}$ satisfies $(k-b)$. The corresponding orthonormal bases $e_{1,\alpha}, \ldots, e_{N,\alpha}$, $e_{1,\beta}, \ldots, e_{N,\beta}$ are in $U_\alpha \cap U_\beta$ related by the $\Phi_\beta\Phi_\alpha^{-1}$. Define ∇^α by defining the $e_{1,\alpha}, \ldots, e_{N,\alpha}$ to be parallel. ∇^α is a metric connection in $E|_{U_\alpha}$. Set $\nabla = \sum_\alpha \psi_\alpha \nabla^\alpha$. ∇ is metric again. We have to show ∇ satisfies (B_k). Consider first the simplest case $\psi_\alpha + \psi_\alpha = 1$, $R^{\psi_\alpha\nabla^\alpha + \psi_\beta\nabla^\beta} = R^{\nabla^\alpha + \psi_\beta(\nabla^\beta - \nabla^\alpha)}$. Then

$$R^{\nabla^\alpha + \psi_\beta(\nabla^\beta - \nabla^\alpha)} = R^{\nabla^\alpha} + d^{\nabla^\alpha}(\psi_\beta(\nabla^\beta - \nabla^\alpha)) + \psi_\beta^2[\nabla^\beta - \nabla^\alpha, \nabla^\beta - \nabla^\alpha] =$$

$$= d^{\nabla^\alpha}(\psi_\beta(\nabla^\beta - \nabla^\alpha)) + \psi_\beta^2[\nabla^\beta - \nabla^\alpha, \nabla^\beta - \nabla^\alpha] \qquad (2)$$

$[\nabla^\beta - \nabla^\alpha, \nabla^\beta - \nabla^\alpha]$ is bounded if $\nabla^\beta e_{i,\alpha}$ is bounded (or $\nabla^\alpha e_{j,\beta}$). But we can express the $e_{i,\alpha}, \ldots, e_{N,\alpha}$ by $\Phi_\alpha\Phi_\beta^{-1}$ applied to $e_{1,\beta}, \ldots, e_{N,\beta}$. Now $\nabla^\beta e_{i,\beta}$ and $d\Phi_\alpha\Phi_\beta^{-1}$ bounded imply $\nabla^\beta e_{i,\alpha}$ bounded, i.e. $\psi_\beta^2[\nabla^\beta - \nabla^\alpha, \nabla^\beta - \nabla^\alpha]$ bounded. Similarly we conclude for $d^{\nabla^\alpha}(\psi_\beta(\nabla^\beta - \nabla^\alpha))$ and $\psi_{\alpha_1} + \cdots + \psi_{\alpha_r} = 1$. Application of $\nabla, \nabla^2, \ldots, \nabla^k$ to (2) and using $|\nabla^i \psi_\alpha| \leq C_i$, $o \leq i \leq k+2$ and $((k+1)-b)$ establishes the assertion.□

Remark. Not every Riemannian vector bundle (E,h) satisfy $(k-b)$, $k \geq 1$. For example, one only prescribes a cocycle $\{\Phi_\beta\Phi_\alpha^{-1}\}_{\alpha,\beta}$ with $\{d\Phi_\beta\Phi_\alpha^{-1}\}_{\alpha,\beta}$ unbounded. Clearly every cocycle is O-bounded since $O(N)$ is compact.□

3 Warped product metrics

Let (M^n, g) be open with finitely many collared ends ε_i, the collar $[a_i, \infty[\times N_i^{n-1}$ endowed with a warped product metric $g|_{\varepsilon_i} \cong dr^2 + f_i(r)^2 d\sigma_{N_i}^2$, N_i^{n-1} closed, $h_i = d\sigma_{N_i}^2$, $i = 1, \ldots, m$.

Lemma 3.1 *(M^n, g) is conformally equivalent to a metric $\tilde{g} = e^u \cdot g$ satisfying*
$(B_k(\tilde{g}))$ $|(\nabla^{\tilde{g}})^j R^{\tilde{g}}| \leq C_j,\ 0 \leq j \leq k$ *and*
(I) $r_{inj}(\tilde{g}) = \inf\limits_{x \in M} r_{inj}(x, \tilde{g}) > 0$
if and only if this is true for each end.□

Therefore we consider one collar $[a, \infty[\times N^{n-1}$ with warped product metric $ds^2 = dr^2 + f(r)^2 d\sigma^2$.

Example. Let $U_0, U_1, \ldots, U_{n-1}$ be an orthogonal basis in $T_{(r,u)}(]a, \infty[\times N)$ with respect to ds^2, $U_0 = \frac{\partial}{\partial r}$, $U_1, \ldots, U_{n-1}$ orthonormal in $T_u N$ with respect to $h = ds^2$. Then for the curvature tensor and the sectional curvature holds

$$\begin{align} R(U_0, U_i)U_0 &= \frac{f''}{f} U_i \tag{3}\\ R(U_0, U_i)U_j &= -f'' f h_{ij} U_0 \tag{4}\\ R(U_i, U_j)U_0 &= 0 \tag{5}\\ R(U_i, U_j)U_k &= -f'^2 (h_{jk} U_i - h_{ik} U_j) + R_N(U_i, U_j)U_k, \tag{6} \end{align}$$

which implies immediately

$$\begin{align} K(U_0, U_j) &= -\frac{f''}{f}, \tag{7}\\ K(U_i, U_j) &= K_N(U_i, U_j)/f^2 - \frac{f'^2}{f^2}. \tag{8} \end{align}$$

Here $i, j, k = 1, \ldots, n-1$. The easy calculations are performed in [3] and once again below with $u \equiv 1$. It is now easy from (3)–(8) to calculate the general curvature $K(V, W)$.

Example. 1. Take $f(r) = e^{-r}$, N flat, then $K \equiv -1$, ε satisfies (B_0) but $r_{\mathrm{inj}}(\varepsilon) = 0$.
2. Choose $f(r) = e^{-r}$, $K_N \not\equiv 0$, then ε does not satisfy (B_0) and again $r_{\mathrm{inj}}(\varepsilon) = 0$.
3. If $f(r) = e^r$, N flat, then ε satisfies (B_0) and $r_{\mathrm{inj}}(\varepsilon) > 0$.
4. Finally take $f(r) = e^{r^2}$, then ε does not satisfy (B_0) but (I).
Hence all good and bad combinations of properties are possible.□

Since all boundedness properties of curvature and injectivity radius at ∞ of $dr^2 + f(r)^2 d\sigma^2$ are governed by $f(r)$ one should try a conformal transformation $\tilde{g} = e^u \cdot g$ with $u = u(r)$.

Therefore we have to calculate the Christoffel symbols, the curvature tensor and its derivatives for $\tilde{g} = e^u \cdot g$,

$$(\tilde{g}_{kl}) = e^u \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & f^2 \cdot h_{ij} & \\ 0 & & & \end{pmatrix}$$

This are longer calculations and we present the results.

Lemma 3.2 *Let* $(]a,\infty[\times U,(u^0=r,u^1,\dots,u^{n-1}))$ *be a chart of* ε. *Then the Christoffel symbols* $\tilde{\Gamma}^k_{ij}$ *of* $\tilde{g}=e^u\cdot g$ *are given by*

$$\begin{array}{ll}\tilde{\Gamma}^0_{00}=0, & \tilde{\Gamma}^0_{0j}=0,\ \text{for } j>0,\\ \tilde{\Gamma}^k_{00}=0,\ \text{for } k>0, & \tilde{\Gamma}^k_{0j}=\frac{1}{2}(u'+\frac{f'}{f})\delta^k_i,\ \text{for } j,k>0,\\ \tilde{\Gamma}^0_{ij}=-(\frac{1}{2}u'f^2+f'f)h_{ij},\ \text{for } i,j>0, & \tilde{\Gamma}^k_{ij}=\Gamma^{h,k}_{ij},\ \text{for } i,j,k>0.\Box\end{array}$$

Denote $U_0=\frac{\partial}{\partial r}$, $U_i=\frac{\partial}{\partial u^i}$, $i>0$.

Proposition 3.3 *The curvature tensor* $\tilde{R}$ *is given by*

$$\begin{aligned}&\tilde{R}(U_0,U_i)U_0=(\tfrac{1}{2}u''+\tfrac{1}{2}u'\tfrac{f'}{f}+\tfrac{f''}{f})U_i,\\ &\tilde{R}(U_0,U_i)U_j=-(\tfrac{1}{2}u''f^2+\tfrac{1}{2}u'f'f+f''f)h_{ij}U_0,\\ &\tilde{R}(U_i,U_0)U_0=-(\tfrac{1}{2}u''+\tfrac{1}{2}u'\tfrac{f'}{f}+\tfrac{f''}{f})U_i,\\ &\tilde{R}(U_i,U_0)U_j=(\tfrac{1}{2}u''f^2+\tfrac{1}{2}u'f'f+f''f)h_{ij}U_0,\\ &\tilde{R}(U_i,U_j)U_0=0,\\ &\tilde{R}(U_i,U_j)U_k=-(\tfrac{1}{4}u'^2f^2+u'f'f+f'^2)(h_{jk}U_i-h_{ik}U_j)+R_N(U_i,U_j)U_k.\Box\end{aligned}$$

Remark. Setting $u=0$, $e^u=1$ and using $|U_i|^2_{ds^2}=f^2\cdot|U_i|^2_{d\sigma^2}$ yields (3)–(8).□

Proposition 3.4 *For* $u=-2\log f$, *i.e.* $e^u=f^{-2}$, $\tilde{g}=e^u$ *satisfies* (B_∞).

Proof: Inserting into the expressions of proposition 3.3, we obtain $\tilde{R}(U_0,U_i)U_0=\tilde{R}(U_0,U_i)U_j=\tilde{R}(U_i,U_j)U_0=0$, $\tilde{R}(U_i,U_j)U_k=R_N(U_i,U_j)U_k$ and conclude $\tilde{g}$ satisfies (B_0). Next we calculate $\tilde{\nabla}\tilde{R}$. According to lemma 3.2 and $e^u=f^{-2}$, $\tilde{\nabla}_{U_0}U_0=\frac{1}{2}u'\cdot U_0$, $\tilde{\nabla}_{U_0}U_i=0$, $\tilde{\nabla}_{U_i}U_j=\nabla^h_{U_i}U_j$ and we obtain from

$$(\tilde{\nabla}_{U_i}\tilde{R})(U_k,U_\lambda)U_\mu=\tilde{\nabla}_{U_i}(\tilde{R}(U_k,U_\lambda)U_\mu)-\tilde{R}(\tilde{\nabla}_{U_i}U_k,U_\lambda)U_\mu-\tilde{R}(U_k,\tilde{\nabla}_{U_i}U_\lambda)U_\mu-\tilde{R}(U_k,U_\lambda)\tilde{\nabla}_{U_i}U_\mu,$$

$i,k,\lambda,\mu=0,\dots,n-1$, that the only non-zero derivative is given by

$$\tilde{\nabla}_{U_l}\tilde{R}(U_i,U_j)U_k=\nabla^h_{U_l}R_N(U_i,U_j)U_k. \tag{9}$$

The right hand side of (9) is bounded since N is compact. Similarly we conclude for all higher derivatives.□

Concerning the injectivity radius, we have the simple

Proposition 3.5 *For* $u=-2\log f$, $\tilde{g}=e^u(dr^2+f^2d\sigma^2)$ *satisfies the condition (I) at infinity if and only if* $\int_a^\infty\frac{1}{f(r)}dr=\infty$.

Proof: $(]a,\infty[\times N,f^{-2}dr^2+d\sigma^2)$ is the metric product of $(]a,\infty[\times N,f^{-2}dr^2)$ and $(N,d\sigma^2)$ and a curvr is a geodesic if and only if it is the cartesian product of geodesics. $r_{\mathrm{inj}}(N,d\sigma^2)>0$. Hence (I) is satisfied at infinity if and only if it holds for $(]a,\infty[\times N,f^{-2}dr^2)$ at infinity. The geodesics $r(t)$ are of the kind $\dot{r}(t)=\dot{r}(0)/f(r(0))\cdot f(r(t))$ and $r_{\mathrm{inj}}>0$ at infinity if and only if $\int_{r(0)}^\infty\frac{1}{f(r)}dr=\infty$ for all $r(0)$.□

We summarize our hitherto calculations in

Theorem 3.6 *Let* (M^n,g) *be open with finitely many collared ends* ε_i, $g|_{\varepsilon_i}\cong dr^2+f_i(r)^2d\sigma^2_{N_i}$ *a warped product metric,* $i=1,\dots,m$. *Then* g *is conformally equivalent to* *a metric $\tilde{g} = e^u \cdot g$ satisfying (B_∞). If additionally $\int\limits_{a_i}^{\infty} \frac{1}{f_i(r)} dr = \infty$, $i = 1, \ldots, m$, then $\tilde{g}$ satisfies (I).*$\square$

The next step is to enlarge the class of starting metric. For M^n with finitely many collared ends denote by $WP(\infty)$ the class of warped product metrics g at infinity as above. In [4] we introduced for the set $\mathcal{M}$ of all metrics g on M^n a metrizable uniform structure ${}^{b,m}\mathcal{U}(\mathcal{M})$ and calculated the components comp(g) of g in the completion ${}^{b,m}\mathcal{M} = \overline{\mathcal{M}}^{{}^{b,m}\mathcal{U}} \cap C^m\mathcal{M}$ and proved

$$\begin{aligned} \text{comp}(g) = {}^{b,m}\mathcal{U}(g) \equiv \{g' \in {}^{b,m}\mathcal{M} | g \text{ and } g' \text{ are quasiisometric and} \\ {}^{b,m}|g - g'|_g :\doteq \sum_{i=0}^{m} \sup_{x \in M} |(\nabla^g)^i (g - g')|_{g,x} < \infty\}. \end{aligned}$$

Denote

$$\begin{aligned} \mathcal{M}(\mathrm{B}_k) &= \{g \in \mathcal{M} | g \text{ satisfies } (\mathrm{B}_k)\}, \\ \mathcal{M}(\mathrm{I}) &= \{g \in \mathcal{M} | g \text{ satisfies (I)}\} \end{aligned}$$

and

$$\mathcal{M}(\mathrm{I,B}_k) = \mathcal{M}(\mathrm{I}) \cap \mathcal{M}(\mathrm{B}_k).$$

The inclusions $i_1 : \mathcal{M}(\mathrm{B}_k) \hookrightarrow \mathcal{M}$, $i_2 : \mathcal{M}(\mathrm{I}) \hookrightarrow \mathcal{M}$, $i_3 : \mathcal{M}(\mathrm{I,B}_k) \hookrightarrow \mathcal{M}$ induce uniform structures $(i_\lambda \times i_\lambda)^{-1}({}^{b,m}\mathcal{U}(\mathcal{M}))$ on $\mathcal{M}(\mathrm{B}_k)$, $\mathcal{M}(\mathrm{I})$, $\mathcal{M}(\mathrm{I,B}_k)$, $\lambda = 1, 2, 3$, and we obtain spaces ${}^{b,m}\mathcal{M}(\mathrm{B}_k)$, ${}^{b,m}\mathcal{M}(\mathrm{I})$, ${}^{b,m}\mathcal{M}(\mathrm{I,B}_k)$ by the corresponding completion and intersection with $C^m\mathcal{M}$.

Proposition 3.7 *If $g \in \mathcal{M}(B_k)$ or $g \in \mathcal{M}(I,B_k)$, respectively, and comp(g) is the component of g in ${}^{b,k+2}\mathcal{M}$ then comp(g) $\subset {}^{b,k+2}\mathcal{M}(B_k)$ or comp(g) $\subset {}^{b,k+2}\mathcal{M}(I,B_k)$, respectively.*

See [4] for a proof.$\square$

Theorem 3.8 *Assume M^n open with finitely many collared ends ε_i, $i = 1, \ldots, m$. Let comp(g') be a component in ${}^{b,k+2}\mathcal{M}$.*

- ***a.*** *If comp(g') contains a metric $g \in WP(\infty)$ with $\lim\limits_{r\to\infty} f_i(r) = 0$, $\left(\frac{f_i'}{f_i}\right)^{(l)}$ bounded, $0 \le l \le k+1$, $i = 1, \ldots, m$, then all metrics of comp(g') are conformal to metrics satisfying (B_k).*
- ***b.*** *If additionally $\int_{a_i}^{\infty} \frac{1}{f_i(r)} dr = \infty$, $i = 1, \ldots, m$, then all metrics of comp(g') are conformal to metrics satisfying (I) and (B_k).*

Proof: a. According to 3.4, $\tilde{g} = e^u \cdot g$ satisfies (B_∞), $e^u|_{\varepsilon_i} = f_i^{-2}$. We have to show $\tilde{g}$ and $\tilde{g}' = e^u$ ly in the same component of ${}^{b,k+2}\mathcal{M}$. $\tilde{g}$ and $\tilde{g}'$ are quasiisometric since this holds for g and g'. This is equivalent to ${}^b|\tilde{g} - \tilde{g}'|_{\tilde{g}} := \sup\limits_{x \in M} |\tilde{g} - \tilde{g}'|_{\tilde{g},x} < \infty$, ${}^b|\tilde{g} - \tilde{g}'|_{\tilde{g}'} < \infty$. We have to show ${}^b|\tilde{\nabla}^l(\tilde{g} - \tilde{g}')|_{\tilde{g}} < \infty$, $l \le k+2$, where $\tilde{\nabla} = \nabla^{\tilde{g}}$ and start with $l = 1$.

Lemma 3.9 *Let t be a tensor, u times covariant, v times contravariant, $v < u$, $\tilde{g} = e^u \cdot g$, $u = -2\log f$ with $\lim\limits_{r\to\infty} f(r) = 0$. Then $|t|_{\tilde{g}} \le C \cdot |g|$ for the pointwise norms.*

Proof: We have at the end ε $|t|_{\tilde{g},(r,x)} = f^{u-v}(r) \cdot |t|_{g,(r,x)}$.$\square$

We apply 3.9 to the tensors $\tilde{\nabla} - \nabla \equiv \nabla^{\tilde{g}} - \nabla^g$ and $t = \tilde{g} - \tilde{g}'$. $\tilde{\nabla} t$ can be written as

$$\tilde{\nabla} t = (\tilde{\nabla} - \nabla) t + \nabla t.$$

Hence $|\tilde{\nabla}t|_{\tilde{g},x} \leq C \cdot |\tilde{\nabla}t|_{g,x} \leq C \cdot [|(\tilde{\nabla} - \nabla)t|_{g,x} + |\nabla t|_{g,x}]$. By assumption ${}^b|\nabla t|_g = \sup_{x \in M} |\nabla t|_{g,x} < \infty$. Using lemma 3.2, we see that the only non-zero components of $\tilde{\nabla} - \nabla$ are $\frac{1}{2}u' = -\frac{f'}{f}$, $f'f \cdot h_{ij}$, $-\frac{f'}{f}\delta_i^k$. Here we consider one arbitrary chosen end ε. But $f \to 0$, $\frac{f'}{f}$ bounded implies $f' \to 0$, $f \cdot f' \to 0$. Hence all 3 expressions are bounded and we can estimate $|(\tilde{\nabla} - \nabla)t|_{g,x} \leq C_1 |t|_{g,x}$ and obtain

$$ {}^b|\tilde{\nabla}t|_{\tilde{g}} < \infty. $$

Consider now $l = 2$.

$$ \begin{aligned} |\tilde{\nabla}^2 t|_{\tilde{g},x} &\leq |(\tilde{\nabla} - \nabla)(\tilde{\nabla} - \nabla)t| + |\nabla(\tilde{\nabla} - \nabla)t| + |(\tilde{\nabla} - \nabla)\nabla t| + |\nabla^2 t| \leq \\ &\leq C_2'[|t| + |\nabla t| + |\nabla^2 t|] + |\nabla(\tilde{\nabla} - \nabla)t| \end{aligned} \tag{10} $$

The last term of the right hand side of (10) is bounded if $\left(\frac{f'}{f}\right)'$ and $(f \cdot f')'$ are bounded. But this follows from our assumption, $\left(\frac{f'}{f}\right)' = \frac{ff'' - f'^2}{f^2}$ bounded, $f \to 0$, $f' \to 0$ imply $f \cdot f'' \to 0$. The case $\tilde{\nabla}^l t$ leads to the estimation of expressions of the kind

$$ |(\tilde{\nabla} - \nabla)^{i_1} \nabla^{i_2} \cdots (\tilde{\nabla} - \nabla)^{i_{s-1}} \nabla^{i_s} t|, \qquad i_1 + \cdots + i_s = l $$

(cf. [4]). Their boundness follows once again from ${}^b|t|_g < \infty, \ldots, {}^b|\nabla^l t|_g < \infty$, $f \to 0$, $\frac{f'^i}{f}$ bounded, $0 \leq i \leq k+1$. b. follows from a. and proposition 3.5.□

Example.

1. For a warped product metric $ds^2|_\varepsilon = dr^2 + f(r)^2 d\sigma^2$ holds $r_{\mathrm{inj}}(\varepsilon) = 0$ if and only if $\underline{\lim}\, f(r) = 0$.
2. If $f(r) = e^{-r}$ then for $\tilde{g} = f^{-2} dr^2 d\sigma^2$ $r_{\mathrm{inj}}(\tilde{g}) > 0$ and (B_∞) are satisfied.
3. There are warping functions $f(r)$ such that $r_{\mathrm{inj}}(g) = 0$ and $r_{\mathrm{inj}}(f^{-2}dr^2 + d\sigma^2) = 0$. One has only to construct a function $f(r)$ with $\underline{\lim}\, f(r) = 0$ and $\int_a^\infty \frac{1}{f(r)} dr < \infty$, but this is very easy. In this case it is still possible that there exists a conformal factor e^u such that $\tilde{g} = e^u \cdot g$ satisfies $r_{\mathrm{inj}}(\tilde{g}) > 0$ and $(\mathrm{B}_\infty(\tilde{g}))$ but $e^u = f^{-2}$ does not work.□

4 Applications

Let (M^n, g) be open, oriented, $F : M^4 \to S^4$ a fixed grafting map as constructed by Taubes (cf. [5], [7]). Consider the Hopf bundle $P_0 : S^7 \xrightarrow{SU(2)} S^4$, the associated quaternionic line bundle E_0, the t' Hooft connection $\nabla_0^\lambda = d + A^\lambda$, the pull back $P = F^* P_0$, $E = F^* E_0$, $\nabla^\lambda = F^* \nabla_0^\lambda$ and $\mathrm{comp}(\nabla^\lambda) \subset \mathcal{C}_P(\mathrm{B}_k, f)$, where $\mathrm{comp}(\nabla^\lambda)$ is the component of ∇^λ in the completed space $\mathcal{C}_P(\mathrm{B}_k, f)$ of $su(2)$–connections of bounded geometry and finite Yang–Mills action. Denote by $\Delta_{2,-}$ the Laplace operator acting on anti self-dual 2-forms, by σ_e the essential spectrum and by $\bullet_{g,L_2}$ the L_2-intersection pairing. Then we proved in [5]

Theorem 4.1 *Let (M^4, g) be open, oriented with (I) and (B_k), $k > 3$, $\inf \sigma_e(\Delta_{2,-}|_{(\ker \Delta_{2,-})^\perp}) > 0$, $\bullet_{g,L_2}$ positive definite. Then for λ sufficiently small, comp(∇^λ) contains a self-dual connection.*□

Lemma 4.2 *The L_2-intersection pairing $\bullet_{g,L_2}$ is positive definite if and only if this holds for $\bullet_{\tilde{g},L_2}$, $\tilde{g} = e^u \cdot g$.*

Proof: Let $*$ be the Hodge operator acting on 2-forms. Then $*_g = *_{\tilde{g}}$ and $** = 1$. Hence $|\omega|^2_{L_2,g} = \int \omega \wedge *_g\omega = \int \omega \wedge *_{\tilde{g}}\omega = |\omega|^2_{L_2,\tilde{g}}$. Now ω is L_2-harmonic by definition if $|\omega|_{L_2} < \infty$ and $d\omega = 0 = \delta\omega$. But $\delta_g\omega = 0$ if and only if $*_g d *_g \omega = 0$ if and only if $d *_{\tilde{g}} \omega = 0$ if and only if $\delta_{\tilde{g}}\omega = 0$. Finally $\omega \bullet_g \omega := \int \omega \wedge \omega = \int \omega \wedge *_g *_g \omega = \int \omega \wedge *_{\tilde{g}} *_{\tilde{g}} \omega = \omega \bullet_{\tilde{g}} \omega$.□

Theorem 4.3 *Let (M^4, g) be open with finitely many collared ends ε_i, $g|_{\varepsilon_i} \cong dr^2 + f_i(r)^2 d\sigma^2_{N_i}$, $\int\limits_{a_i}^{\infty} \frac{1}{f_i(r)} dr = \infty$, $i = 1, \ldots, m$, and $\bullet_{g,L_2}$ positive definite. Define $comp(\nabla^\lambda)$ as above. Set $\tilde{g} = e^u \cdot g$, $u|_{\varepsilon_i} = -2\log f_i$. If $\inf \sigma_e(\Delta_{2,-}(\tilde{g})|_{(\ker \Delta_{2,-}(\tilde{g}))^\perp}) > 0$. Denote by $comp(\nabla^\lambda, \tilde{g})$ the component of ∇^λ defined by ∇^λ, $\nabla^{\tilde{g}}$, $\tilde{g}$. Then for λ sufficiently small, $comp(\nabla^\lambda, \tilde{g})$ contains a self-dual connection which is also self-dual with respect to g.*

Proof: We perform the conformal transformation 3.6 and obtain a component $\mathrm{comp}(\nabla^\lambda, \tilde{g})$. Then we apply 4.1, 4.2 and use that self-duality is a conformal invariant.□

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