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A unified construction of the Alexander- and the Jones-invariant*

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Abstract

The Alexander- and the Jones - invariant both are constructed from the same braid module, extending an observation of V. F. R. Jones. The topological origin of the braid module is explained in terms of holonomy of flat connections and group homology.

1 Introduction

The present article is based on the author's talk given at the 16^{th} winter school at Srni. It contains a short overview on joint work with F. Constantinescu [5] and on the author's work [9]. The missing proofs and further pointers to the literature can be found in these publications.

The topological nature of the Alexander invariant of a tame knot $k : S^1 \to \mathbf{R}^3$ is well understood. One constructs a presentation matrix for the relative homology $H_1(C, C_0; \mathbf{Z})$ of the infinite cyclic covering (and its zero skeleton) (C, C_0) of the knot's complement $\mathbf{R}^3 \backslash k(S^1)$. The relative homology can then be reduced to $H_1(C; \mathbf{Z})$ and the Alexander invariant can be obtained from the presentation matrix, cf. [3]. Thus, by construction, the result is a topological invariant.

A similarly simple and, at the same time, topological construction has not yet been achieved for the Jones invariant. So the point of view taken in this article is to trace back both invariants to a braid representation, the topology of which is easy to understand. This braid representation has been called the 'braid valued' Burau module in [4, 9].

As will be made explicit later on, the 'classical' Burau module enters knot theory at least in two different ways. On the one hand, it can be used to explicitly write down a presentation matrix for $H_1(C, C_0; \mathbf{Z})$. So it produces the Alexander invariant. On the other hand, as was observed by V. F. R. Jones, it can be used to construct a Yang - Baxter matrix. This matrix in turn yields the Alexander invariant, by proceeding along the lines explained e.g. in Ch. Kassel's lectures in this volume.

We will generalise the second approach, by replacing the classical Burau representation by its braid valued preimage.

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2 The topology of the braid valued Burau module

This section defines the braid valued Burau module and shortly explains its topological meaning.

General references on the braid group are [2, 3]. References on the braid valued Burau module are [4, 8, 9], where also the proofs can be found.

The braid group B_n on n strings is the group with generators $\{b_1, \dots, b_{n-1}\}$ obeying the relations $b_i b_j = b_j b_i$ for $\mathrm{abs}(i-j) \geq 2$ and $b_k b_{1+k} b_k = b_{1+k} b_k b_{1+k}$. It is isomorphic to the fundamental group $\pi_1(X_n/S_n)$, where $X_n := \{(x_1, \dots, x_n) \in \mathbf{C}^n \mid x_i \neq x_j, \text{ if } i \neq j\}$ and S_n is the group of n - permutations. There is an imbedding $B_n \hookrightarrow \mathrm{Aut}(F_n)$, where $F_n :=< f_1, \dots, f_n >$ is the free group. The braid generators act according to $b_i(f_j) = \begin{cases} f_i, & j = 1+i \\ f_i f_{1+i} f_i^{-1}, & j = i \\ f_j, & j \notin \{i, 1+i\} \end{cases}$. So it is possible to define the semidirect product group $B_n F_n$ with multiplication $bfb'f' := bb'b'(f)f'$, $b, b' \in B_n$, $f, f' \in F_n$.

The relative augmentation ideal $I := I_{F_n, B_n F_n} \leq \mathbf{Z}[B_n F_n]$ of F_n in the integral group ring of $B_n F_n$ regarded as a left $B_n F_n$ module is free over the set $\{f_1 - 1, \dots, f_n - 1\}$. We consider I as a left $B_n F_n$ right B_n bimodule. This immediately implies the following:

1 (Braid valued Burau module) *The right action of B_n onto the augmentation ideal I is equivalent to the faithful matrix representation $B_n \hookrightarrow GL_n(\mathbf{Z}[B_n F_n])$ which sends a generator b_i to the $n \times n$ block matrix*

$$\begin{pmatrix} b_i \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & b_i(1 - f_i f_{1+i} f_i^{-1}) & b_i f_i & 0 \\ 0 & b_i & 0 & 0 \\ 0 & 0 & 0 & b_i \mathbf{1}_{n-i-1} \end{pmatrix}.$$

We will call the left $B_n F_n$ right B_n bimodule I the 'braid valued Burau module'.

Example 1 (Classical Burau module) *The homomorphism $B_n F_n \to T :=< t >$, $b_i \mapsto 1$, $f_j \mapsto t$, applied to the above matrix yields*

$$\begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}.$$

We will call the free $\mathbf{Z}T$ module $(\mathbf{Z}T)^n$ together with the B_n representation (or equivalently the tensor product $\mathbf{Z}T \otimes_{B_n F_n} I$) the 'classical Burau module'.

Let us explain the topological meaning of the braid valued Burau module. It is closely related to the construction of Hecke algebras given in [1, 7]. The map $Y := X_{n+m}/S_n \to X := X_n/S_n$, $[x_1, \dots, x_{m+n}] \mapsto [x_1, \dots, x_n]$ is a fiber bundle with fiber $Y_x := \{(y_1, \dots, y_m) \in X_m \mid y_i \neq x_j\}$. Given a locally constant sheaf (a local system) L of modules on Y, from the fiber bundle we obtain the associated relative cohomology bundle $p' : H_p^*(Y; L) \to X$ with fibers the vertical cohomology groups $(p')^{-1}(x) =$

$H^*(Y_x; L \mid Y_x)$. This fiber bundle is naturally equipped with a flat connection and the sheaf of its flat local sections is the derived direct image $R^*p_*(L)$ of L. By flatness of the connection, the fiber $H^*(Y_x; L \mid Y_x)$ is a module over $\pi_1(X, x) \simeq B_n$.

The bundle space, the base space and the fibers Y_x are Eilenberg - MacLane complexes of type $(\pi_1, 1)$. We have $\pi_1(Y) \simeq B_{n,m}$, $\pi_1(X) \simeq B_n$, $\pi_1(Y_x) \simeq F_{n,m}$. Here the iterated semidirect products $B_{n,m} := B_n F_n F_{1+n} \cdots F_{m+n-1}$ and $F_{n,m} := F_n \cdots F_{m+n-1}$ are defined via imbeddings $F_n \hookrightarrow B_n F_n \hookrightarrow B_{1+n} \hookrightarrow \mathrm{Aut}(F_{1+n})$. Hence, the fiber of the cohomology bundle is isomorphic to the group cohomology $H^*(F_{n,m}; L \mid Y_x)$ with coefficients in the $\pi_1(Y) \simeq B_{n,m}$ - module $L \mid Y_x$. The holonomy of the flat connection of the cohomology bundle therefore can be described as the natural action of the quotient group $B_{n,m}/F_{n,m} \simeq B_n$ onto $H^*(F_{n,m}; L \mid Y_x)$.

The upshot is that a free resolution $0 \to P_m \to \cdots \to P_0 = \mathbf{Z}[F_{n,m}] \to \mathbf{Z} \to 0$ of $F_{n,m}$ - modules can be constructed using augmentation ideals similar to $I_{F_n, B_n F_n}$. The induced representation of B_n on tensor products $(L \mid Y_x) \otimes_{F_{n,m}} P$ is then given by suitably generalised braid valued Burau matrices. For details see [9].

3 Construction of the invariants

In this section from the braid valued Burau module Yang - Baxter matrices will be obtained. These can be used to compute the Alexander- and the Jones - invariant.

The following example shows the more conventional way by which the Burau module enters into the computation of the Alexander invariant

Example 2 (Homology of knot complement's covering) *The relative homology of the infinite cyclic covering $C \to \mathbf{R}^3 \backslash k(S^1)$ of the knot complement can be described as $H_1(C, C_0) \simeq (\mathbf{Z}T \otimes_{B_n F_n} I_{F_n, B_n F_n})/(\mathbf{Z}T \otimes_{B_n F_n} I_{F_n, B_n F_n}(b-1))$. Here $T := < t >$ is the cyclic covering group and the isotopy class of the knot k is described by a braid representative $b \in B_n$. The ring $\mathbf{Z}T$ via $b_i \to 1$, $f_i \to t$ is a right $B_n F_n$ module such that the tensor product $\mathbf{Z}T \otimes_{B_n F_n} I_{F_n, B_n F_n}$ is defined and yields a left T right B_n module. So the equation presents $H_1(C, C_0)$ as a quotient of left $\mathbf{Z}T$ modules. This e.g. follows from [3, prop. 9.2, p. 123].*

Notice that the relative augmentation ideal $I_{F_n, B_n F_n}$ is the braid valued Burau module defined earlier. Moreover, the tensor product $\mathbf{Z}T \otimes_{B_n F_n} I_{F_n, B_n F_n}$ as a left $\mathbf{Z}T$ right $\mathbf{Z}B_n$ bimodule can be identified with the classical Burau module.

This example is presented in order that the reader might compare it with the following approach to the Alexander invariant, which uses the classical Burau representation in a different way.

Lemma 1 (Alexander invariant from classical Burau matrix) *Let $R := \mathbf{Z}T$ be the integral group ring of the cyclic group $T := < t >$. Let a Burau representation $\rho \in Hom(B_3, GL_3(R))$ of the braid group B_3 on three strings be given by the matrices $b_1 \mapsto \rho_1 := \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ and $b_2 \mapsto \rho_2 := \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$ where $B := \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$. Then the natural extension of ρ to the exterior algebra $\Lambda(R^3)$ (where the ρ_i act as algebra homomorphisms) is equivalent to a Yang - Baxter representation $b_i \mapsto \Upsilon_i^A$, $i \in \{1, 2\}$,* *where* $\Upsilon_1^A := \Upsilon^A \otimes Id_V$, $\Upsilon_2^A := Id_V \otimes \Upsilon^A$, $V := R^2$ *is the 2 dimensional free R module and* $\Upsilon^A \in Aut(V \otimes V)$ *is given by*

$$\Upsilon^A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}$$

Thus, the Yang - Baxter equation $\Upsilon_i^A \Upsilon_{1+i}^A \Upsilon_i^A = \Upsilon_{1+i}^A \Upsilon_i^A \Upsilon_{1+i}^A$ *holds in* $Aut(V \otimes V \otimes V)$.

This lemma is due to Jones. A proof can be found in [6, sect. I.13, pp. 208].

In [6, sect. I.12, pp. 174] it is explained, how the Alexander invariant can be produced from Υ^A, after the matrix has been brought to the form (by rescaling and a change of basis) $\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t-t^- & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t^- \end{pmatrix}$.

The main point to notice about the preceding lemma is that $\dim_R(\Lambda(R^3)) = 2^3 = \dim_R(R^2 \otimes_R R^2 \otimes_R R^2)$. So in order to obtain different Yang - Baxter matrices one might try to replace the exterior algebra $\Lambda(R^3)$ by appropriate tensor products of the braid valued Burau module (which is a module over a noncommutative ring) and then take quotients of the desired dimension. This is indeed possible, as the following proposition shows.

2 (Jones- and Alexander - invariant) *Let* $F^{(j)}$ *be a free group of rank* $3+j-1$, *presented as* $< f_1^{(j)}, \ldots, f_{3+j-1}^{(j)} >$ *for* $j \in \{1,2,3\}$. *Let* $B_{3,j} := B_3 F^{(1)} \cdots F^{(j)}$ *be the iterated semidirect product of the braid group on three strings with the free groups. Let* $I^{(j)} := Lin_{B_{3,j}}(\{f_i^{(j)} - 1 \mid i \in \{1, \ldots, 3+j-1\}\})$ *be the relative augmentation ideal of* $F^{(j)}$ *in the ring* $ZB_{3,j}$. *Let a left* $B_{3,3}$ *right* B_3 *bimodule be defined as the sum of tensor products*

$$M := ZB_{3,3} \oplus I^{(3)} \oplus I^{(3)} \otimes_{B_{3,2}} I^{(2)} \oplus I^{(3)} \otimes_{B_{3,2}} I^{(2)} \otimes_{B_{3,1}} I^{(1)}.$$

Then the right B_3 *module* $ZT \otimes_{B_{3,3}} M$ *as a quotient projects to the representation of* B_3 *defined by the Yang - Baxter matrix*

$$\Upsilon^J := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As a different quotient one can obtain the previously defined Yang - Baxter representation Υ^A *associated to the Alexander invariant.*

The proof proceeds by guessing the appropriate relations and then computing the representation on the quotient. It can be found in [5].

In [6, sect. I.11, pp. 161] it is explained how the Jones invariant can be produced from the matrix Υ^J (again after appropriate rescaling and base change) using a state model on knot diagrams.

Finally we draw the reader's attention to the question, pointed out to us by Ch. Kassel, whether the HOMFLY polynomial can be obtained along similar lines. We strongly believe that this is possible.

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