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A few Remarks on the Set of Finite-to-One Maps of the Cantor set

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1. Introduction

Let $C(2^\infty, I^\infty)$ be the space of continuous mappings from the Cantor set 2^∞ to the Hilbert cube I^∞, equipped with the topology of uniform convergence. A mapping $f: 2^\infty \to X$ is finite-to-one, if all fibers of f are finite.

We shall consider the set

$$\mathscr{C} = \{f \in C(2^\infty, I^\infty): f \textit{ is finite-to-one}\} . \tag{1}$$

One readily checks that the set $\mathscr{C}$ is coanalytic. We shall indicate a natural Lusin-Sierpiński index for $\mathscr{C}$, the transfinite order of a finite-to-one mapping on 2^∞ (sec. 3), and we shall verify that the transfinite order of mappings is related to the transfinite inductive dimension of compacta by a Hurewicz-type formula (sec. 4). Finally, we shall make some observations about Borel-measurable selections of finite-to-one parametrizations on 2^∞ for certain collections of countable-dimensional compacta (sec. 5).

These remarks are related to some open problems about the transfinite inductive dimension, discussed in [Po1] and [Po2; sec. 6].

2. Terminology and some background

Our terminology follows Kuratowski [Ku] and Nagata [Na]. We consider only separable metrizable spaces and by a compactum we mean a compact space. A set $S \subset T$ is *residual* (*non-meager*) in T, if $T \setminus S$ is of first category (S is of second category) in the space T. The spaces of continuous functions are considered with the topology of uniform convergence.

A space is *countable-dimensional* (*strongly countable-dimensional*) if X is a countable union of finite-dimensional sets (compacta)

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The *transfinite inductive dimension ind* is the extension, by transfinite induction, of the classical Menger-Urysohn dimension: $ind\,X = -1$ means that $X = \emptyset$, $ind\,X \leqq \alpha$, if, and only if, each point in X can be separated from a closed set not containing it by a partition L with $ind\,L < \alpha$; we let $ind\,X$ be the smallest ordinal α with $ind\,X \leqq \alpha$, if such an ordinal exists, and we set $ind\,X = \infty$, otherwise. If $ind\,X \neq \infty$, then $ind\,X < \omega_1$.

The following two classical results (proofs can be found in [Na; VI] and [Ku; § 45, II]) provide a link between countable-dimensionality, finite-to-one mappings on the Cantor set, and the transfinite dimension:

2.1. Theorem (Hurewicz). *For a compactum X without isolated points the following conditions are equivalent*:

(i) *X is countable-dimensional,*

(ii) *$ind\,K \neq \infty$,*

(iii) *there is a continuous finite-to-one mapping of 2^∞ onto X,*

(iv) *the set of finite-to-one mappings is dense in the space of continuous mappings of 2^∞ onto X.*

2.2. Theorem (Kuratowski). *Let X be a strongly countable-dimensional compactum without isolated points. Then the set of finite-to-one mappings is residual in the space of continuous mappings of 2^∞ onto X.*

The converse to the Kuratowski's theorem, even with "residual" weakened to "non-meager", also holds true [Po3].

3. The transfinite order of a finite-to-one mapping on 2^∞

Here we shall adopt some general notions from descriptive set theory to the situation we are interested in, cf. Moschovakis [Mo; 2D, 2F], Kuratowski [Ku; § 39].

3.1. The transfinite length of collections of partitions of 2^∞.

Let Ω be the countable collection of all finite partitions of the Cantor set 2^∞ into pairwise disjoint closed-and-open sets. Given $\mathcal{U}, \mathcal{V} \in \Omega$, we write $\mathcal{U} \prec \mathcal{V}$ if $\mathcal{U}$ refines $\mathcal{V}$ and $\mathcal{U} \neq \mathcal{V}$.

Let 2^Ω be the space of all subcollections of Ω with the topology of pointwise convergence (we identify any $\Lambda \subset \Omega$ with its characteristic function). Topologically, 2^Ω is the Cantor set.

Let WF be the set of all collections $\Lambda \subset \Omega$ with the property that there is no infinite descending sequence $\mathcal{U}_1 \succ \mathcal{U}_2 \succ \ldots$ of elements of Λ.

The set $WF \subset 2^\Omega$ is coanalytic. For any $\Lambda \in WF$ the rank function on Λ is defined inductivity as follows, cf. [Mo; pp. 83, 84]: for each $\mathcal{U} \in \Lambda$ we set

$$rank_\Lambda \mathcal{U} = 1 \quad \text{if there is no} \quad \mathcal{V} \in \Lambda \quad \text{with} \quad \mathcal{V} \prec \mathcal{U}\,,$$

and

$$rank_\Lambda \mathscr{U} = sup\,\{rank_\Lambda \mathscr{V} + 1\colon \mathscr{V} \prec \mathscr{U},\ \mathscr{V} \in \Lambda\}\ .$$

The length of $\Lambda \in WF$ is defined by the formula

$$length\ \Lambda = sup\,\{rank_\Lambda \mathscr{U}\colon\ \mathscr{U} \in \Lambda\}\ ,$$

and we set $length\ \Lambda = \infty$ if $\Lambda \notin WF$.

The length is a Lusin-Sierpiński index for the coanalytic set WF.

3.2. The function *ord*. Let $f\colon 2^\infty \to I^\infty$ be a continuous mapping and let

$$\Lambda(f) = \{\mathscr{U} \in \Omega\colon \bigcap \{f(F)\colon F \in \mathscr{U}\} \neq \emptyset\}\ .$$

The mapping $f \to \Lambda(f)$ from the function space $C(2^\infty, I^\infty)$ (see sec. 1) to the Cantor set 2^Ω is Borel-measurable. Since, as one easily checks,

$$\Lambda(f) \in WF \Leftrightarrow f \in \mathscr{C}\ ,$$

where $\mathscr{C}$ is described in (1) sec. 1, the transfinite order defined by the formula

$$ord\,f = length\ \Lambda(f)\ ,\quad \text{for}\quad f \in C(2^\infty, I^\infty)\ ,$$

is a Lusin-Sierpiński index for the coanalytic set $\mathscr{C}$. In particular, the transfinite order is bounded on each analytic set in $\mathscr{C}$, and each set $\mathscr{C}_\xi = \{f \in \mathscr{C}\colon ord\,f \leqq \xi\}$ is Borel, see [Ku; § 39, VIII].

4. A Hurewicz-type formula for the transfinite order

The following fact is a certain substitute for a classical theorem of Hurewicz [Ku; § 45, I, Th. 2].

4.1. Proposition. *Let $f\colon 2^\infty \to X$ be a finite-to-one mapping of the Cantor set onto the compactum X. Then*

$$(*)\qquad ind\,X \leqq ord\,f\ .$$

Proof. The proof is by induction with respect to the transfinite order of the mappings. For the mappings of finite order formula $(*)$ is valid by the classical result. Suppose that $(*)$ holds true for the mappings of order $<\alpha$, α being a countable infinite ordinal, and let $f\colon 2^\infty \to X$ be a continuous surjection with $ord\,f = \alpha$.

Let us split 2^∞ into two nonempty closed-and-open sets K, L, and let

$$Z = f(K) \cap f(L)\ .$$

Since such sets Z separate all pairs of disjoint closed sets in X, it is enough to check that

$$(1)\qquad ind\,Z < ord\,f\ .$$

We can assume that Z has no isolated points, as for the set Z' of points of condensation of Z, either $ind\, Z' = ind\, Z$, or Z is countable. Let S be a minimal compactum such that

$$(2) \qquad S \subset K \quad \text{and} \quad f(S) = Z \,.$$

Since S has no isolated points, there exists a homeomorphism $h\colon 2^\infty \to S$. Let

$$g = f \circ h\colon 2^\infty \to Z \,.$$

We shall check that

$$(3) \qquad ord\, g < ord\, f \,.$$

Let $r\colon K \to S$ be a retraction [Ku; § 26, II, Corollary 2], and let for partition $\mathscr{U} \in \Omega$,

$$\mathscr{U}^* = \{r^{-1}(h(F))\colon F \in \mathscr{U}\} \cup \{L\} \in \Omega \,.$$

The correspondence $\mathscr{U} \to \mathscr{U}^*$ is invertible and preserves the order $\prec$. Moreover, if $\mathscr{U} \in \Lambda(g)$, then $\mathscr{U}^* \in \Lambda(f)$. Therefore, taking into account that $\{2^\infty\}^* = \{K, L\}$, we get (see sec. 3.1): $ord\, f = length\, \Lambda(f) = rank_{\Lambda(f)}\{2^\infty\} > rank_{\Lambda(f)}\{K, L\} \geqq rank_{\Lambda(g)}\{2^\infty\} = length\, \Lambda(g) = ord\, g$, i.e. we obtain (3).

By the inductive assumption, $ind\, g(2^\infty) \leqq ord\, g$, and, since $g(2^\infty) = f(S)$, (1) follows from (2) and (3).

4.2. Remark. *One can define a function $\Psi\colon \omega_1 \to \omega_1$ such that for each compactum X without isolated points, if $ind\, X \leqq \alpha$ then there exists a finite-to-one surjection $f\colon 2^\infty \to X$ with $ord\, f \leqq \Psi(\alpha)$.*

To see this let us fix $\alpha < \omega_1$ and let $u\colon 2^\infty \to K_\alpha$ be a finite-to-one mapping onto a countable-dimensional compactum which contains topologically all compacta with $ind \leqq \alpha$ (see [Po 2 sec. 3]). We let $\Psi(\alpha) = ord\, u$. Now. given a compactum X without isolated points such that $ind\, X \leqq \alpha$ we can assume that $X \subset K_\alpha$ and, for a minimal compactum S in 2^∞ with $u(S) = X$ and for a homeomorphism $h\colon 2^\infty \to S$, we let $f = u \circ h\colon 2^\infty \to X$. Since $ord\, f \leqq ord\, u$, f is the required surjection.

This observation is connected to the assertion of Lemma 2.1 in [Po 1; § 3]; we do not examine, however, the relationship more closely.

4.3. Remark. The remark at the end of sec. 3.2 and Proposition 4.1 yield the following fact:

If $\mathscr{A} \subset \mathscr{C}$ is an analytic set of finite-to-one mappings of 2^∞ in I^∞, then

$$sup\{ind\, f(2^\infty)\colon f \in \mathscr{A}\} < \omega_1 \,.$$

This can be also verified directly. Let $u\colon \omega^\infty \to \mathscr{A}$ be a continuous map of the irrationals ω^∞ onto $\mathscr{A}$ and let $F\colon \omega^\infty \times 2^\infty \to \omega^\infty \times I^\infty$ be defined by the formula $F(t, x) = (t, u(t)(x))$. The map F is perfect and finite-to-one. Therefore, the space $E = F(\omega^\infty \times 2^\infty)$ is completely metrizable and countable-dimensional and, since each $f(2^\infty)$, $f \in \mathscr{A}$, embeds in E, we have $sup\{ind\, f(2^\infty)\colon f \in \mathscr{A}\} \leqq ind\, E < \omega_1$ (cf. [Po 2; sec 6] for similar arguments).

5. Borel-measurable choice of finite-to-one parametrizations

Let $\mathscr{K}(I^\infty)$ be the hyperspace of the Hilbert cube, i.e. the space of compact subsets of I^∞ with the topology induced by the Hausdorff metric.

Let

$\mathbf{C} = \{K \in \mathscr{K}(I^\infty)$: *K is countable-dimensional*$\}$,

$\mathbf{C}^* = \{K \in \mathscr{K}(I^\infty)$: *K is strongly countable-dimensional*$\}$.

5.1. Proposition. *For each analytic set* $\mathbf{A} \subset \mathbf{C}^*$ *there exists a Borel-measurable function* σ *which assigns to each compactum* $K \in \mathbf{A}$ *a finite-to-one continuous mapping* $\sigma(K)\colon 2^\infty \to K$ *onto* K.

Proof. Let $\varphi\colon C(2^\infty, I^\infty) \to \mathscr{K}(I^\infty)$ (see sec. 1) be defined by the formula

$$\varphi(f) = f(2^\infty)$$

By a result of Michael [Mi; Th. 1.1]

(1) the mapping φ is open .

Let us consider the set $\mathscr{C}$ defined in sec 1, (1). By Hurewicz's Theorem 2.1, $\varphi(\mathscr{C}) = \mathbf{C}$ and for each $K \in \mathbf{C}$ the set $\varphi^{-1}(K) \cap \mathscr{C}$ is dense in $\varphi^{-1}(K)$. Therefore, by (1),

(2) $\varphi \mid \mathscr{C}\colon \mathscr{C} \to \mathbf{C}$ is open ,

where $\varphi \mid \mathscr{C}$ is the restriction of φ to $\mathscr{C}$. By Kuratowski's Theorem 2.2, for each $K \in \mathbf{C}^*$ the set $\varphi^{-1}(K) \cap \mathscr{C}$ is residual in $\varphi^{-1}(K)$. Now, the set $\mathscr{C}$ being coanalytic, we can apply, by (2), to the multifunction $F(K) = \varphi^{-1}(K) \cap \mathscr{C}$ defined on $\mathbf{A}$ a selection theorem due to Burgess [Bu; Theorem 3.1] and Cenzer and Mauldin [C-M] which provides a Borel-measurable function $\sigma\colon \mathbf{A} \to \mathscr{C}$ such that $\sigma(K) \in F(K)$, i.e., $\sigma(K)(2^\infty) = K$.

5.2. Remark. By Kuratowski's Theorem 2.2 and the remark following this theorem, $\mathbf{C}^* = \{K \in \mathscr{K}(I^\infty)$: $\varphi^{-1}(K) \cap \mathscr{C}$ *is non-meager in* $\varphi^{-1}(K)\}$. Therefore, the above approach works only for analytic subsets of $\mathbf{C}^*$.

I do not know, if the assertion of Proposition 5.1 is valid for all analytic sets $\mathbf{A} \subset \mathbf{C}$, or even for the analytic sets $\mathbf{C}_\alpha = \{K \in \mathscr{K}(I^\infty)$: $\operatorname{ind} K \leqq \alpha\}$ (cf. the next section).

5.3. Remark. Let $\mathbf{C}_n = \{K \in \mathscr{K}(I^\infty)$: *K is at most n-dimensional*$\}$ and let $\mathscr{C}_n = \{f \in \mathscr{C}$: *the order f is at most n*$\}$. Then $\mathscr{C}_n$ and $\mathbf{C}_n$ are G_δ-sets in $C(2^\infty, I^\infty)$ and $\mathscr{K}(I^\infty)$, respectively [Ku; § 45, IV, Th. 4], and, by a Kuratowski's theorem [Ku; § 45, II, Th. 1], for each $K \in \mathbf{C}_n$, the set $\varphi^{-1}(K) \cap \mathscr{C}_{n+1}$ is dense in $\varphi^{-1}(K)$. It follows that the multifunction $K \to \varphi^{-1}(K) \cap \mathscr{C}_{n+1}$ defined on $\mathbf{C}_n$ is lower-semicontinuous. By a selection theorem due to Kuratowski and Ryll-Nardzewski [K-RN] there exists a first Baire class function $\sigma\colon \mathbf{C}_n \to \mathscr{C}_{n+1}$ such that $\sigma(K)(2^\infty) = K$.

For $n = 0$ such selection σ can be continuous, see Märgerl, Mauldin and Michael [M-M-M; Theorem 5.1(b)].

5.4. Remark. For the analytic set $\boldsymbol{C}_\alpha$, described at the end of sec. 5.2, there is an analytic set $\mathscr{A} \subset \mathscr{C}$ such that $\boldsymbol{C}_\alpha = \{f(2^\infty): f \in \mathscr{A}\}$. Indeed, by Remark 4.2, $\boldsymbol{C}_\alpha \subset \varphi(\mathscr{C}_\xi)$, where $\xi = \Psi(\alpha)$ and $\mathscr{C}_\xi$ is defined at the end of sec. 3.2.

References

[Bu] BURGESS J. P., Careful choices — a last word on Borel selectors, Notre Dame Journal of Formal Logic, 22 (1981), 219—226.

[C-M] CENZER D., MAULDIN R. D., Inductive definability: measure and category, Adv. in Math. 38 (1980), 55—90.

[Ku] KURATOWSKI K., Topology, vol. I and II, Warszawa 1966 and 1968.

[K-RN] KURATOWSKI K., RYLL-NARDZEWSKI C., A general theorem on selectors, Bull. Acad. Pol. Sci. 13 (1965), 397—403.

[M-M-M] MÄGERL G., MAULDIN R. D., MICHAEL E., A parametrization theorem, Topology and Appl. 21 (1985), 87—94.

[Mi] MICHAEL E., On a map from a function space to a hyperspace, Math. Annalen 162 (1965), 87—88.

[Mo] MOSCHOVAKIS Y., Descriptive set theory, Amsterdam 1980.

[Na] NAGATA J., Modern dimension theory, Berlin 1983.

[Po1] POL R., On analytic-measurable collections of countable-dimensional and weakly infinite-dimensional spaces, in: Proc. Fifth Prague Topol. Symp. 1981, Berlin (1982), 548—560.

[Po2] POL R., Selected topics related to countable dimensional metrizable spaces, in: Proc. Sixth Prague Topol. Symp. 1986, Berlin (1988), 421—436.

[Po3] POL R., The converse to a theorem of Kuratowski on parametrizations of compacta on the Cantor set (preprint).