The Multifractal Spectrum of Discrete Measures

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In recent years, multifractals and their $f(\alpha)$-spectrum have become so popular in numerical and experimental studies of strange attractors, diffusion-limited aggregation, turbulence and random resistor networs [1, 2, 5, 7, 8], that it seems necessary to develop solid foundations for these concepts. There are only two types of measures for which the $f(\alpha)$-spectrum was determined rigorously: these are Gibbs states on zero-dimensional hyperbolic attractors in $\mathbb{R}$ ("cookie-cutters") [2, 7] and self-similar measures with respect to two similarity mappings, when the open set condition is fulfilled [5, 8]. In both cases, the thermodynamic formalism was used and the function $f(\alpha)$ has a parabolic shape.

The purpose of this note is to treat analytically some other examples for which the $f(\alpha)$-spectrum is linear. Our methods are quite elementary and all details are proved. We shall restrict ourselves to finite measures μ on $[0, 1]$ which assume positive values on all intervals $[a, b] \subset [0, 1]$. Let us start with some definitions. The local dimension of μ at a point x is defined as

$$(1) \qquad d_\mu(x) = \lim_{\varepsilon \to 0} \frac{\log \mu(U_\varepsilon(x))}{\log \varepsilon}$$

where $U_\varepsilon(x) =]x - \varepsilon. x + \varepsilon[$. $d_\mu(x)$ quantifies "the degree to which x belongs to μ when x is determined more and more accurately". The physicists' "working definition" of the $f(\alpha)$-spectrum is

$$(2) \qquad f(\alpha) = \dim \{x \mid d_\mu(x) = \alpha\}, \quad 0 \leqq \alpha \leqq \infty$$

where dim means Hausdorff dimension (cf. [4] for definitions). Intuitively, μ classifies the parts of $[0, 1]$ where μ is strongly concentrated (small α) or sparsely distributed (large α).

Kahane and Katznelson [6] gave an example of a measure supported by a Cantor

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set in $[0, 1]$ such that for every α, there are at most two x with $d_\mu(x) = \alpha$. Moreover, our examples show that the limit (1) need not exist for many x (the set of all these x has dimension 1). We think that this is also possible in experimental studies. For these reasons we suggest to replace (2) by

$$(3) \qquad f(\alpha) = \lim_{\varepsilon \to 0} \dim \{x \mid \alpha - \varepsilon \leqq \mathbf{d}_\mu(x) \leqq \alpha + \varepsilon\}\,, \quad 0 \leqq \alpha \leqq \infty$$

where $\mathbf{d}_\mu(x)$ denotes the lower local dimension, that is, the liminf in (1). For all measures treated in the literature, it is easy to see that definitions (1) and (3) agree. For $f(\alpha)$-functions, the differences between $d_\mu(x)$, $\mathbf{d}_\mu(x)$ and the upper local dimension $\bar{d}_\mu(x)$ (i.e., the limsup in (1)) have apparently not been studied so far. However, for the dimension distribution of μ which was recently introduced by Cutler and Kahane, it turned out that $\mathbf{d}_\mu(x)$ is the appropriate function [3]. This justifies our definition (3). Without going into details, we note that the dimension distribution of μ classifies $\mathbf{d}_\mu(x)$ by means of the measure μ and the $f(\alpha)$-spectrum of μ classifies in terms of the Hausdorff dimension. The latter is more subtle and difficult.

Our measure μ will be discrete. Their dimension distribution will be trivial since they will be concentrated on the countable set of rational numbers of the form $p/2^n$, p an odd integer. Let $0 < r < 1/2$. Let $\mu\{1/2\} = r$, $\mu\{1/4\} = \mu\{3/4\} = r^2, \ldots$ $\ldots, \mu\{p/2^n\} = r^n$ for $p = 1, 3, 5, \ldots, 2^n - 1$. Then $\mu[0, 1] = r/(1 - 2r)$. Note that for any μ and x, $\mu(\{x\}) > 0$ implies $d_\mu(x) = 0$.

Theorem. For the measure μ defined above, $\bar{d}_\mu(x) = (-\log r)/(\log 2) =: \alpha^*$ whenever $\mu(\{x\}) = 0$. For any α between 0 and α^*, $f(\alpha) = \alpha/\alpha^*$.

Proof. (i) We easily see that $\mu(]p/2^k, (p + 1)/2^k[) = r^{k+1}(1 - 2r)$ for $p = 0, 1, \ldots, 2^k - 1$ and hence $\mu(]y, y + 2^{-k}[) \geqq r^{k+1}(1 - 2r)$ for each y in $[0, 1 - 2^{-k}]$. For $\varepsilon \in [2^{-n}, 2^{-(n-1)}[$ this implies $\log \mu(U_\varepsilon(x))/\log \varepsilon \leqq ((n + 1) \log r + \log (1 - 2r))/(-(n - 1) \log 2)$. Thus $\bar{d}_\mu(x) \leqq \alpha^*$ for arbitrary x.

(ii) Take a point $x \neq p/2^k$ and $\varepsilon > 0$. We determine $\varepsilon' < \varepsilon$ with $|\alpha^* - \log \mu(U_{\varepsilon'}(x))/\log \varepsilon'| \to 0$ for $\varepsilon \to 0$. Let y be the unique number $p/2^k$ in $U_\varepsilon(x)$ with odd p and smallest possible k. Let $\varepsilon' = |x - y|$, $2^{-(n+1)} < \varepsilon' < 2^{-n}$ and $I =]y, y + 2/2^n[$ for $y \leqq x$, $I =]y - 2/2^n, y[$ otherwise. Then $U_{\varepsilon'}(x) \subseteqq I$ implies $\log \mu(U_{\varepsilon'}(x))/\log \varepsilon' \geqq \log \mu(I)/\log 2^{-(n+1)} = (n \log r + \log (1 - 2r))/-(n + 1) \log 2$ which tends to α^* for $n \to \infty$. This proves the first part of the theorem.

(iii) Let $\mathbf{d}_\mu(x) < \alpha < \alpha^*$. We show that x is contained in infinitely many of the sets

$$W_k(\alpha) = \bigcup\{[p/2^k - \delta,\ p/2^k + \delta] \mid p = 1, 3, 5, \ldots, 2^k - 1\}$$

where $\delta = \delta_k(\alpha) = 2^{-k\alpha^*/\alpha}(1 - 2r)^{1/\alpha}$. Note that $\mathbf{d}_\mu(x) < \alpha$ means $\mu(U_\varepsilon(x)) > \varepsilon^\alpha$ for arbitrary small ε. Take an ε for which this inequality holds, and define $y = p/2^k$ as above. Then $\varepsilon^\alpha < \mu(U_\varepsilon(x)) \leqq \mu(](p - 1)/2^k,\ (p + 1)/2^k[) = r^k(1 - 2r)$. Now $r = 2^{-\alpha^*}$ implies $\varepsilon < 2^{-k\alpha^*/\alpha}(1 - 2r)^{1/\alpha}$ and $x \in W_k(\alpha)$.

(iv) Conversely, if x is contained in infinitely many $W_k(\alpha)$ then $\mathbf{d}_\mu(x) \leqq \alpha$. We can assume $\mu(\{x\}) = 0$, so that $x \in W_k(\alpha)$ implies $p/2^k \in U_\delta(x)$ and $\mu(U_\delta(x)) \geqq r^k = 2^{-k\alpha^*} = \delta^\alpha/(1 - 2r)$.

(v) Let us show $\dim \{x \mid \boldsymbol{d}_\mu(x) < \alpha\} \leqq \alpha/\alpha^*$. Using (iii), we verify that the β-dimensional Hausdorff measure is finite for $\beta > \alpha/\alpha^*$. Let $\varepsilon > 0$ and choose k_0 so that $\delta_{k_0}(\alpha) < \varepsilon$. We cover $\bigcup\{W_k(\alpha) \mid k \geqq k_0\}$ by intervals of length $2\,\delta_k(\alpha)$, $k \geqq k_0$. If we write $\beta = (1 + \eta)\,\alpha/\alpha^*$ then $2^{k-1}(2\delta_k(\alpha))^\beta = c \,.\, 2^{-\eta k}$, $c = \frac{1}{2}(1 - 2r)^{\beta/\alpha}$, and the sum for $k \geqq k_0$ is $c \,.\, 2^{-\eta k_0}/(1 - 2^{-\eta})$ which tends to zero for $k_0 \to \infty$.

(vi) Now we show $\dim \{x \mid \boldsymbol{d}_\mu(x) \leqq \alpha\} \geqq \alpha/\alpha^*$, verifying that the β-dimensional Hausdorff measure of this set is positive for $\beta < \alpha/\alpha^*$. Let $\beta = (1 - \eta)\,\alpha/\alpha^*$. We shall construct a sequence $k_1 < k_2 < \ldots$ and a Cantor set $D \subseteqq \bigcap\{W_{k_i}(\alpha) \mid i = 1, 2, \ldots\}$ with $\mu^\beta(D) > 0$. For every k we have $2^{k-1}(2\delta_k(\alpha))^\beta = c \,.\, 2^{\eta k}$ with c from above. Choose k_1 with $c \,.\, 2^{\eta k_1} > 1$, and let $V_1 = W_{k_1}(\alpha)$. Now suppose k_n is constructed and V_n is a union of intervals with Lebesgue measure λ_n $(n \geqq 1)$. Then choose $k_{n+1} > k_n$ such that more than $\frac{3}{4} \,.\, \lambda_n \,.\, 2^{k_{n+1}-1}$ of the intervals of $W_{k_{n+1}}(\alpha)$ are contained in V_n and such that $\frac{3}{4} \,.\, \lambda_n c \,.\, 2^{\eta k_{n+1}} > 1$. Let V_{n+1} be the union of all intervals of $W_{k_{n+1}}(\alpha)$ which are inside V_n. By induction, we built the Cantor set $D = \bigcap\{V_n \mid n = 1, 2, \ldots\}$.

(vii) To estimate $\mu^\beta(D)$, it suffices to consider finite coverings $\mathfrak{C} = \{I_1, \ldots, I_m\}$ by intervals. Assume first that the I_j are intervals from the $W_{k_i}(\alpha)$, $i \leqq n$. Let $\nu(I)$ denote the number of intervals of V_n which are inside I divided by the total number of intervals of V_n. For all I from a fixed $W_{k_i}(\alpha)$, $i \leqq n$, the value $\nu(I)$ and the length $\lambda(I)$ are constant, and since the sum of these $\nu(I)$ is 1 and the sum of the $\lambda(I)^\beta$ is >1 by construction, we have $\lambda(I)^\beta > \nu(I)$. Now taking sums over $j = 1, \ldots, m$ we see that $\sum \lambda(I_j)^\beta > \sum \nu(I_j) \geqq 1$.

(viii) To prove $\mu^\beta(D) > 0$, it remains to check that there are no other "more efficient" coverings of D. Since the intervals of V_n do not always cover the endpoints of I_j, it could be possible to replace the I_j by some smaller I'_j. Nevertheless, the $\frac{3}{4}\lambda_n$ – condition implies $\lambda(I'_j) > \lambda(I_j)/2$, thus $\sum \lambda(I'_j)^\beta > 1/2$.

A more interesting question is whether an interval I from $W_{k_{n-1}}(\alpha)$ of length I can be covered "efficiently" by several intervals J_i, $i = 1, \ldots, t$ smaller than I, but larger than the intervals of V_n. We can assume that the gap length $\varkappa$ between two neighbouring J_i is the same as that between two consecutive intervals of V_n and that the J_i have equal length $l' = (l + \varkappa)/t - \varkappa$. The covering by $J_1, \ldots, J_t$ is "most efficient" if $f(t) = t \,.\, l'^\beta/l^\beta$ is minimal. With $\gamma = \varkappa/l$ we have $f(t) = t \,.\, ((1 + \gamma)/t - \gamma)^\beta$. For $t \in [1, (1 + \gamma)/\gamma]$ there is only one zero of $f'(t)$ which corresponds to a maximum of f. The minimal value of f on $[1, t^*]$, $t^* \leqq (1 + \gamma)/\gamma$ is assumed at one endpoint of the interval. Thus $\sum \lambda(J_i)^\beta$ is minimal if we have either $t = 1$, $J_1 = I$ or the J_i are the intervals of V_n inside I. Consequently, there are no other "more efficient" coverings.

(ix) From (v) and (vi) it follows by standard arguments (involving $\alpha \pm 1/n$) that $\dim \{x \mid \boldsymbol{d}_\mu(x) \leqq \alpha\} = \dim \{x \mid \boldsymbol{d}_\mu(x) < \alpha\} = \alpha/\alpha^*$ for $\alpha < \alpha^*$ and then for $\alpha = \alpha^*$. Hence $f(\alpha) = \alpha/\alpha^*$ by (3).

Remark. The definition of μ can be modified in various ways. Instead of the points $p/2^k$, one can use the endpoints of the construction intervals of a Cantor set, of the points $(p_1/2^k, p_2/2^k)$ in $[0, 1]^2$ (with maximum-metric), or points in a suitable Cantor set in $[0, 1]^n$. The $f(\alpha)$-function is also linear. Is it true that the $f(\alpha)$-function is linear for all discrete measures the weights of which form a geometric series? If the weights go down exponentially, it is clear that $f(\alpha) = 0$ for $\alpha < \infty$. One can also multiply the measure μ with Lebesgue measure on $[0, 1]$ to obtain a non-discrete measure μ' with linear spectrum: $f(1 + \alpha) = 1 + \alpha/\alpha^*$ for $0 \leqq \alpha \leqq \alpha^*$, $f(\gamma) = 0$ for $\gamma < 1$ and $\gamma > 1 + \alpha^*$.

References

[1] Bohr T. and Rand D., The entropy function for characteristic exponents, Physica D 25, 387—398 (1987).

[2] Collet P., Lebowitz J. L. and Porzio A., The dimension spectrum of some dynamical systems, J. Stat. Phys. 47 (1987), 609—644.

[3] Cutler C. D. and Dawson D. A., Estimation of dimension for spatially distributed data and related limit theorems, J. Multivariate Anal. 28 (1989), 115—148.

[4] Falconer K. J.: *The geometry of fractal sets,* Cambridge Univ. Press 1985.

[5] Halsey T. C., Jensen M. H., Kadanoff L. P., Procaccia I. and Shraiman B. I., Fractal measures and their singularities, Phys. Rev. A 33, 1141—1151 (1986).

[6] Kahane J. P. and Katznelson Y., Decomposition des mesures selon la dimension, preprint, Paris 1989.

[7] Rand D. A., The singularity spectrum $f(\alpha)$ for cookie-cutters, Ergod. Th. Dynam. Sys. 9 (1989), 527—541.

[8] Tel T., Fractals, multifractals and thermodynamics, Zschr. f. Naturf. A 43, 1154—1174 (1988).