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ON SOME QUALITATIVE METHODS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS IN THE LARGE

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1 Introduction

Most of the investigations in the qualitative theory of differential equations are of a local character. The behaviour of integral curves is studied in a sufficiently small neighbourhood of a given solution, e.g. in a neighbourhood of a stationary point or of a periodic solution.

The situation is different if the investigation is made in the whole or in the large. In this case the examined system and a certain domain are given and one has to study all the solutions which are situated in this domain or to find all solutions of a given family which are situated in this domain. Such an approach is due to H. Poincaré [1] who in his second paper included the chapter "Examples of investigation in the whole".*) In this chapter he examined some examples of the behaviour of integral curves of a system of two equations with polynomial right-hand sides.

Among the examples considered by Poincaré there also were those of equations possessing limit cycles both in a finite domain and at infinity. In 1934 these methods were used by A. A. Andronov and A. G. Maĭer to prove the existence of limit cycles for equations of Rayleigh and Van der Pol. To do this they had to use the theory of singular points of higher order. It seems to me that the methods of Poincaré have not yet been fully exploited.

If one analyzes the work concerning either the local theory or even the investigation in the large (Poincaré), one observes that invariants of affine transformations have been studied. However, the classification of singular points given by Poincaré was not affine, but possessed a mixed character. As a matter of fact, numerical values of the eigenvalues of coefficient matrices of linear systems are affine invariants. However, singular points, the corresponding matrix of which possesses real roots with different signs have been called saddle points, and all the saddle points possessing the same number of positive and negative roots of the characteristic equation have been grouped in one class; on the other hand the nodes and foci which from the point of view of homeomorphical mappings belong to a single class in the sense of Poincaré, have been included in different classes. The other founder of modern qualitative theory A. M. Lyapunov, in his classification of stationary solutions in stable and unstable ones, laid the fundamental stress not on geometry, but on time. G. D. Birkhoff classifying va-

*) See [1], Chap. VII: Exemples de discussion complète, pp. 274–283.

rious types of motion also introduced "time" in his definitions. However, subsequent considerations showed that Lyapunov and especially Birkhoff pointed out some topological properties of a family of integral curves [2].

From 1930, under the influence of the excellent book of G. D. Birkhoff "Dynamical systems", papers began to appear where classification was based on topological properties. E.g. at the beginning of the thirties, a problem was formulated concerning the classification of dynamical systems all trajectories of which tend to infinity in both directions. It was shown that their behaviour was closely related to a phenomenon which I called a "saddle point at infinity" or "improper saddle point".

Since in what follows we shall have to work with this concept, its definition will be presented.

Definition. *A family of integral curves will be said to have a saddle point at infinity, if there exist sequences of points* $\{p_n\}$ *and* $\{q_n\}$ *such that*

i) *both* p_n *and* q_n *are situated on a single trajectory*;
ii) $p_n \to p$, $q_n \to q$;
iii) *p and q are situated on different trajectories* ;
iv) *there is a sequence of points* $z_n \in \widehat{p_n q_n}$ *such that* $z_n \to \infty$.

Systems without saddle points at infinity can be mapped to a system of parallel direct lines, i.e. they are parallelizable.

The conception of parallelizable systems has been further developed. Results of mine have been transferred by M. V. Bebutov [4] to the dynamical systems of G. D. Birkhoff and further, by D. Montgomery and L. Zippin [5] to more general systems. Especially, in the latter paper the concept of a dispersive system was introduced which is equivalent to that with a saddle point at infinity. Namely, initially the following definitions were introduced: *A pair of points p and q of a dynamical system is called wandering, if there exist neighbourhoods* U_p *and* U_q *of these points and a T such that for* $|t| > T$ *the map of* U_p *does not intersect* U_q *and the map of* U_q *does not intersect* U_p; *finally, a system is called dispersive, if every pair of points of this system is wandering. Locally compact dispersive systems will be called parallelizable.* E. A. Barbašin [6] presented an analytic criterion of parallelizability which is analogous to the theorems of the "second Lyapunov's method". A dynamical system described by the system of differential equations $dx_i/dt = f_i(x_1, x_2, \ldots, x_n)$ will be parallelizable, if and only if the partial differential equation

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} f_i = 1$$

possesses a uniquely defined solution. Finally, in their recent paper J. Dugundji and H. A. Antosiewicz [7] impressed a geometrical character on this theory. In particular, they proved the following theorem: *The flow of trajectories F will be parallelizable, if and only if F possesses a section for which* t_p *is continuous,* t_p *being the time* *distance of the point p from the section.* Particularly, in the locally compact case in R^n the necessary and sufficient condition for the existence of such a section is that the system is dispersive.

There is an infinite number of topological types of saddle points at infinity. This was particularly studied by Kaplan [8] who enumerated the topological types of systems without singular points in the whole plane.

In 1937 the paper of E. A. Leontovič and A. G. Maĭer [9] appeared in which the problem of topological classification of dynamical systems in the plane was formulated. In this paper a new method of investigation is presented. The problem of topological equivalence of systems is reduced to topological equivalence of a one-dimensional system of orbitally unstable solutions. As a result of the absence of proofs in this paper it has not received the attention it reserved. It is necessary to remark that these proofs have not yet been published. Several results of this paper will be shown below. It is also necessary to mention a little known or studied paper of G. Birkhoff from 1935 [10]. In this paper G. Birkhoff tried to present a topological characterization of transitive systems in three-dimensional space. Such systems include e.g. systems the trajectories of which are everywhere dense in the interior of a torus. More exactly, this class is characterized by the following conditions:

i) These systems are transitive in space R^3, i.e. to any pair of points p_0 and q_0, in an arbitrarily small neighbourhood of these points there exist points p_1 and q_1 lying on the same trajectory.

ii) They admit a regular section, i.e. there exists an analytical manifold S^2 which is cut by all trajectories without contact in one direction; moreover, the trajectories intersect it in every time interval of sufficiently large but fixed length. A finite number of periodic trajectories may be tangent to S^2; in the latter case they may lie on S^2.

iii) There exists at least one periodic solution. The latter is regular, i.e. its neighbourhood does not consist of periodic solutions of the same period only.

G. Birkhoff asserts that for such systems it is possible to find an invariant set E_1 the topological characteristic of which is sufficient for the full topological characterization of the whole system. As to the topological characteristic of E_1, it can be given by means of a *signature* which represents a denumerable set of denumerable sequences of points on the manifold of the section.

2 A topological classification of systems of Poincaré-Lyapunov's type

Systems of the type

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik}x_k + f_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n) \tag{1}$$

where f_i fulfils a Lipschitz condition with a sufficiently small constant, are usually called the systems of Poincaré-Lyapunov. Reading the papers of O. Perron [11] and especially that of I. G. Petrovskiĭ [12] one can have the impression that a complete classification of behaviour in a small neighbourhood of a singular point was given. As usual, their results are formulated as follows: If the roots of the characteristic equation have real parts different from zero, then the behaviour of the integral curves of (1) is the same as that of the reduced system

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} a_{ik} y_k \quad (i = 1, 2, \ldots, n). \tag{2}$$

In this formulation there is the indefinite expression "the same behaviour". First, both I. G. Petrovskiĭ and O. Perron investigated only the behaviour of 0-curves, i.e. of integral curves entering the origin, and the words "the same as" meant that the dimension of the set of 0-curves which possess a certain asymptotical behaviour at the origin is the same for both the systems (1) and (2). However, it is well-known that the dimensions does not represent a full system of invariants and, accordingly, the problem was not solved completely from the point of view of topological classification. At the end of the forties I began to suggest to my pupils that they should pay attention to the investigation in the whole. V. A. Yakubovič [13] proved that there is a topological correspondence between the families of 0-curves of systems (1) and (2); however, it remained unclear whether there exists any one-to-one correspondence between integral curves of other types.

Before giving an answer to this problem let us turn to the linear systems with constant coefficients. È. M. Vaĭsbord [14] showed that if the roots of the characteristic equation have real parts different from zero, then the number of topologically different types in the whole R^n will be finite for a given n and each of these types is defined by the number of positive and negative signs of the real parts of characteristic roots. This simple remark presents the first exact classification for the regular case of systems with constant coefficients.

What about the case where some of the real parts are equal to zero? One is easily convinced that the principle of topological classification can hardly be preserved. As a matter of fact, if one considers e.g. a system with two pairs of imaginary roots, then the set of topological types has the power of a continuum, since every ratio of coefficients in the imaginary case (except when it is rational) defines a topological class. Obviously, here one has to use a rougher classification of solutions than the topological one.

Now let us turn to the systems of Poincaré-Lyapunov's type in the case when imaginary roots are absent. First Vaĭsbord [14] and Minc [15], then Grobman [16], [17] and P. Hartman [18] stated the following final theorem:

Theorem of D. M. Grobman. *If the real parts of all eigenvalues of the matrix A are different from zero and if in a domain G_1 containing the point $x = 0$ the vector*

$f(x), f(0) = 0$, satisfies a Lipschitz condition with a sufficiently small constant, then the trajectories of the systems

$$\frac{dx}{dt} = Ax + f(x)\,, \tag{1}$$

$$\frac{dy}{dt} = Ay \tag{2}$$

are homeomorphic in the domains G_1 and G_2, where G_2 is some domain containing the point 0.

Let us say several words about the methods used in the proofs of this theorem.

First of all system (1) has to be also defined, if necessary, in the whole space; further D. M. Grobman applies his following result [19]:

If the Lipschitz constant of the vector $f(x)$ is sufficiently small, then system (1) will possess only two types of trajectories: the parabolic ones (O^+ – and O^- – curves) and the hyperbolic ones (tending to infinity in both directions). From this it follows that there are no elliptic trajectories. Let the completed system be denoted by

$$\frac{dx}{dt} = Ax + F(x)\,. \tag{1'}$$

The following transformation concerns the systems (1′) and (2) where matrix A is already written in the cannonical form. First, the following relation is proved:

$$y(t) = x(t) - \int_{-\infty}^{t} Y_1(t-\tau)\, F(x(\tau))\, d\tau + \int_{t}^{\infty} Y_2(t-\tau)\, F(x(\tau))\, d\tau$$

i.e. if $x(t)$ is a solution of equation (1), then $y(t)$ defined by the latter equation is a solution of equation (2). In view of this it can be proved that the mapping $y_0 = \Phi(x_0)$ of the space (x) into (y) given by the formula

$$y_0 = x_0 - \int_{-\infty}^{0} Y_1(-\tau)\, F(x_0(\tau))\, d\tau + \int_{0}^{\infty} Y_2(-\tau)\, F(x_0(\tau))\, d\tau$$

transforms the solutions of equation (1′) satisfying the initial condition x_0 into the solutions of equation (2) with the initial condition y_0 and that there exists an inverse mapping. P. Hartman states some time later than D. M. Grobman a less general theorem, as he assumes that the vector $F(x)$ belongs to the class C^2.

In his proof he uses what is undoubtedly a very interesting lemma:

Lemma of P. Hartmann. *Let A be a constant matrix with the eigenvalues $a_1, \ldots, a_N$ satisfying the conditions*

$$0 < |a_i| < 1\,.$$

Let $X(x)$ be a vector function of class C^1 for small $|x|$ and let

$$X = \frac{\partial X}{\partial x_1} = \ldots = \frac{\partial X}{\partial x_n} = 0$$

for $x = 0$; in the neighbourhood of $x = 0$ let X have partial derivatives fulfilling a Lipschitz condition. Then to the mapping

$$T: x' = Ax + X(x)$$

there exists the mapping

$$R: u = x + \varphi(x)$$

which belongs to class C^1 for small x and satisfies the conditions

$$\varphi = \frac{\partial \varphi}{\partial x_1} = \ldots = \frac{\partial \varphi}{\partial x_n} = 0 \quad \text{for} \quad x = 0\,.$$

Moreover, RTR^{-1} can be written as

$$RTR^{-1}: u' = Au$$

i.e.

$$Au = R(Ax + X(x))\, R^{-1}\,.$$

Naturally one can ask whether or not it is possible to state a theorem analogous to that of D. M. Grobman for the whole space. Generally speaking that is not possible without additional assumptions, as e.g. that the saddle point at infinity cannot occur in a bounded domain. In the case of the plane this difficulty can be surmounted. E.g the following theorem of mine [20] is valid: *Let a system of the Poincaré-Lyapunov's type be given, the matrix of the linear part of the system possessing the Jordan form. If the Lipschitz constant of vector $X(x)$ is smaller than $\lambda_{\min}/2\sqrt{2}$, where $\lambda_{\min} = \min(|\lambda_1|^2, |\lambda_2|^2)$, then the topological character of integral curves in the whole plane will be the same as in the case of the linearized system. Generally, if there exists a function $V(x, y)$ such that*

$$\frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q \geqq m\varrho$$

for $x^2 + y^2 \geqq \varrho$, then the system

$$\frac{dx}{dt} = P\,, \quad \frac{dy}{dt} = Q$$

has no saddle point at infinity.

No theorems of this type are as yet available for spaces with more dimensions. Observe that for $n > 2$ there exists another phenomenon which has not been studied at all and which may complicate the situation: the curves tending to infinity may be knotted. As a matter of fact, no examples of systems of differential equations have been constructed up to now where such knots should occur; however, it is apparent that they can be constructed. In the case of linear equations such knots cannot occur. This follows from the mapping of È. M. Vaĭsbord [14], but the theorem of D. M. Grobman does not imply that they do not occur in the case of systems of Poincaré-Lyapunov.

Is it possible to suppose that the problem of classification of solutions for systems of the Poincaré-Lyapunov's type has been completely solved by the theorems of Grobman or Hartman? In a certain sense the answer is in the affirmative. However, there is a circumstance which must be taken into consideration. Namely the continuous mapping of D. M. Grobman must not be considered as a transformation of variables, since it is not differentiable in the general case. For the analytical right-hand sides the following remarkable theorem of C. L. Siegel [21] is known:

With the exception of a zero measure set of eigenvalues every analytical system with eigenvalues different from zero can be analytically transformed to a reduced system.

Let the right-hand sides be differentiable or even analytical. Is it possible to make the mapping of D. M. Grobman differentiable with the determinant different from zero? This problem was solved in the paper of P. Hartman [18] the answer being in the negative: the system

$$x' = \alpha x\,, \quad y' = (\alpha - \gamma)\, y + \varepsilon xz\,, \quad z' = -\,\gamma z$$

was considered and it was proved that there exists no differentiable transformation of variables reducing this system to a linear one of the form

$$x' = \alpha x\,, \quad y' = (\alpha - \gamma)\, y\,, \quad z' = -\,\gamma z\,.$$

From this it follows that from the point of view of the invariants of differentiable transformations a larger number of types will exist which makes the problem more complicated.

However, on the manifolds filled by 0-curves this transformation can be made differentiable; these manifolds being smooth the differential transformation can be realized in the special case when all of the roots of the characteristic equation have negative real parts. This was proved by Lojasiewicz [22] under some additional hypotheses. Since in this case the transformation is differentiable for linear systems, such a transformation can be made especially for the singular node (a bundle of rays). If not all of the roots have real parts of the same sign, the differentiable transformation can be realized if the nonlinear perturbations have a sufficiently high order of smallness. A theorem of this sort was proved by M. Nagumo [23]. What to do in the case of imaginary roots remains an open question.

3 The Case of General Systems in the Plane

Consider some results concerning the general case, i.e. not systems of Poincaré-Lyapunov's type.

For the case $n = 2$ the problem of topological classification has made considerable progress. I have already mentioned the paper of E. A. Leontovič and A. G. Maĭer in 1937. This problem was further studied by E. A. Leontovič, L. Markus and I. Maĭerčik. The basic idea of these papers consists in considering a special class of trajectories the homeomorphism of which defines the homeomorphism of all trajectories. If these singular trajectories form a one-dimensional graph, then evidently the problem becomes easy. This idea was published in 1937 [9].

E. A. Leontovič and A. G. Maĭer [25] took the orbitally unstable trajectories as the mentioned class of singular trajectories. Let L be a trajectory completely contained in the examined domain and let M be any point on it. The point M divides trajectory L in two half-trajectories L_M^+ and L_M^- respectively. Trajectory L is called orbitally stable as $t \to \infty$, if for any point M and every $\varepsilon > 0$ there is a $\delta > 0$ such that all trajectories passing at $t = t_0$ through the δ-neighbourhood of point M do not leave the ε-neighbourhood of the half-trajectory L_M^+ for $t > t_0$. An orbitally stable trajectory for $t \to -\infty$ is similarly defined.

It is easy to prove that orbital stability is a topological invariant. As early as 1937 it was proved that if the sets of orbitally stable trajectories are homeomorphic, then the sets of all trajectories are homeomorphic.

In his remark from 1955 and in his dissertation E. A. Leontovič present the way to describe the family of orbitally stable trajectories for the case that the number of orbitally stable trajectories is finite. This way of description is called *a scheme* and the following fundamental fact is established: If the schemes of two systems are identical, then both systems are homeomorphic. In particular this description shows that for polynomial righthand sides, the degrees of the polynomials not exceeding n, there is only a finite number of topological types, if one considers bounded domaines with a specially selected boundary. (The latter assumption is not essential.) The boundedness of the domain plays a well-known role.

The things are a little more involved in the case of an unbounded domain where the number of orbitally stable trajectories may be unbounded. I. Maĭerčik [27] using one of the results of L. Markus [26] established that the number of topological types on the whole sphere of Poincaré is finite.

Not that I. Maĭerčik using essentially the result of Markus [26], showed that forming the family of singular curves on the whole sphere one has to add even the limits of orbitally unstable trajectories to the orbitally unstable trajectories. Consequently, if the number of orbitally unstable trajectories in the whole plane is finite, then their configuration, i.e. one-dimensional graph consisting of these trajectories, defines the topology in the plane.

In my recent paper [28] I pointed out a class of systems called systems similar to the homogeneous ones for which this graph can be defined very easily and the topology of the family of integral curves is determined by enumerating in a cyclic order both the hyperbolic and elliptic domains.

This class of systems fulfils the three following conditions:

i) There is only one singular point.

ii) There are no points except the singular one which should be either α or ω limiting points for other trajectories.

iii) Neither the system itself nor the system obtained from the given one by means of radius-vectors contains saddle points in ∞.

Also it should be mentioned that Bratkovskiĭ [29] presented a classification of trajectories for systems with polynomial right-hand sides. Particularly, he considered systems with the only singular point in the origin of coordinates of the Poincaré's sphere and showed that in the "general case" it is possible to find such a neighbourhood of the equator that the topological structure of the family of the trajectories in this neighbourhood also defines the topological structure of the whole system of trajectories.

One has to keep in mind that the construction of the graph of orbitally unstable trajectories representing a verification of the conditions for the scheme of E. A. Leontovič cannot always be carried out by means of a finite number of operations. This concerns not only the well-known difficulties connected with determining the number and location of limit cycles, but also some difficulties of a fundamental character. M. I. Voĭlokov established recently that by a finite number of operations, in general, it is only possible to define the topological character of structurally stable system (this concept will be defined below). As to this class of systems A. N. Bellyustina [30] and M. I. Voĭlokov showed that an effective verification of the above mentioned conditions can be made, if one uses the method of qualitative integration with the help of piecewise linear lines of Euler which was presented by myself [31]. However, this does not mean that it is possible in fact to carry out the examination of the topological structure for a wide class of analytically given systems, since to establish that an analytically given system is structurally stable does not represent an effective operation.

Now the results of the theory of structurally stable systems will be briefly examined.

The concept of structurally stable system was introduced by A. A. Andronov and L. S. Pontryagin in 1937 [32]: *The system of two equations* $\mathrm{d}x_i/\mathrm{d}t = P_i(x_1, x_2)$ *is called structurally stable in the domain G, if to any* $\varepsilon > 0$ *there is an* $\eta > 0$ *such that for any* $\bar{P}_i(x_1, x_2)$, $i = 1, 2$ *from the class* C^1 *satisfying the conditions*

$$|\bar{P}_i(x_1, x_2)| < \eta\,, \quad \left|\frac{\partial \bar{P}_i}{\partial x_j}\right| < \eta \quad (i, j = 1, 2)$$

there exists a homeomorphic mapping T of the domain G satisfying the following conditions:

i) *the distance of the corresponding points is smaller than* $\varepsilon > 0$;

ii) *the points of the trajectory of the given system are transformed into the points of the trajectory of the system*

$$\frac{dx_i}{dt} = P_i(x_1, x_2) + \bar{P}_i(x_1, x_2), \quad i = 1, 2.$$

In other words, in these systems "small" perturbations do not change the topological character of the pattern of trajectories in the plane.

L. S. Pontryagin and A. A. Andronov established the following necessary and sufficient conditions for a system to be structurally stable:

A_1. There is a finite number of singular points and they are simple. (The roots of characteristic are simple and they have real parts different from zero.)

A_2. The trajectories contained in G cannot pass from one saddle point to another.

A_3. There is a finite number of periodic solutions and every limit cycle has index

$$h(\gamma) = \int_\gamma \operatorname{div} X \, dt \neq 0.$$

The proofs of the theorems of L. S. Pontryagin and A. A. Andronov remained unpublished for 15 years and as late as in 1952 H. F. De Baggis [33] published a full treatment of the problem omitting at the same time the superfluous assumptions on the analycity of perturbations $\bar{P}_i(x_1, x_2)$. Besides L. S. Pontryagin, A. A. Andronov and H. F. De Baggis assumed that the domain under consideration has the boundary which is an arc without contact, all of the trajectories crossing it inside G. Further the structurally stable systems in the plane were examined by M. M. Peixoto and M. C. Peixoto [34] and [35]. They showed that it is possible to omit this latter restrictive condition and to examine more general boundaries of domains. Besides they showed that for $n = 2$ the set of structurally stable system is everywhere dense in the set of all systems of class C^1. This theorem makes a little less unsymmetrical the assertion of M. I. Voĭlokov that only for structurally stable system is it possible to find the topological structure by means of a finite number of operations. For $n > 2$ the investigation of structurally stable systems is in its initial stage. Definitions and several examples of n-dimensional structurally stable systems were given by M. M. Peixoto [34]. Some further advance in this field was made by L. Markus [36], [37].

Consider the system:

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n)$$

and a space with the norm

$$\|S\| = \max \|f_i(P)\| + \max \|\operatorname{grad} f_i(P)\|.$$

Two systems S and S' are called homeomorphic, if there exists a homeomorphism Φ of manifolds where these systems are defined satisfying the following conditions:

i) Φ transforms the curves of the system S into curves of S' and Φ^{-1} transforms the curves of S' into the curves of S;

ii) every point is displaced by Φ not more than an $\varepsilon > 0$.

Definition. *A system S is structurally stable on* $M^{(n)}$, *if to any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that condition* $\|S - S'\| < \delta$ *implies that* S' *and* S *are homeomorphic.*

L. Markus established several necessary conditions for a system to be structurally stable. E.g. all singular points are to be simple, periodical solutions isolated and, consequently, the periodical solutions may form at most a denumerable set. In paper [37] there is another group of necessary conditions the proofs of which have not yet been published.

A further investigation of structurally stable systems for $n > 2$ is an open problem. It seems to me that it is necessary to relax the definition of structural stability a little in order to obtain sufficient conditions.

4 Systems of Differential Equations with Elementary Structure

(Method of Lyapunov Functions)

What has been said above shows in a sufficiently convincing manner that a qualitative investigation in the whole is impossible in more complicated cases if one insists on topological principles of classification. Consequently, one has to choose from two possibilities: either to roughen the desired classification or to take into account some elementary classes of systems only. In the following I suggest a method called the method of Lyapunov functions, in which both of these possibilities are used.

Now turn to the second method of Lyapunov to examine it from a geometrical point of view. The following classification of trajectories will be presented:

1) Tending to infinity in both directions (hyperbolical trajectories).

2) Tending to infinity in one direction and to a singular point in the other (parabolical trajectories).

3) Tending to a singular point in both directions (elliptic trajectories).

4) Singular points.

5) Complex trajectories.

In the latter class all the remaining trajectories will be included.

Consider the system

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n).$$

N. N. Krasovskiĭ established a sufficient condition in an analytic form for the system not to possess complex trajectories.

Theorem of N. N. Krasovskiĭ [38]. *A necessary and sufficient condition that in a bounded domain containing at most one singular point there are no trajectories (different from the singular point) completely contained in this domain, is that there exists a continuous function* $V(x_1, x_2, \ldots, x_n)$ *whose time derivative with respect to the given system, i.e. the expression*

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_i \frac{\partial V}{\partial x_i} f_i(x_1, x_2, \ldots, x_n)$$

is positive everywhere except at the singular point where it vanishes.

If the conditions of this theorem are fulfilled, only hyperbolic and parabolic trajectories may occur in the system and the classification of such systems is related to the structure of the set of zero points and of the domains where function $V(x_1, x_2, \ldots, x_n)$ is positive or negative respectively.

From all possible situations of zero points of V I selected three of them by means of the following definitions:

1) $V(x, y)$ is called an *elliptic function*, if $V(0, 0) = 0$, $V(x, y) \neq 0$ for $x^2 + y^2 \neq 0$, $V = c$ representing simple closed lines.

2) $V(x, y)$ is called a *hyperbolic function*, if the curve $V(x, y) = 0$ represents a finite number of curves coming out of the origin and dividing the plane into a finite number of unbounded regions. In neighbouring regions $V(x, y)$ has different signs. The remaining level curves $V(x, y) = \text{const.}$ consist of a finite number of branches each of which divides the plane in two regions.

3) $V(x, y)$ is called a *parabolic function*, if it has no zero points at all and the level curves $V(x, y) = \text{const.}$ represent one-to-one images of direct lines dividing the plane into two regions.

Assuming a priori that there exists a Lyapunov function of one of the above mentioned types, it is possible to characterize the situation of integral curves; e.g. in the case of a hyperbolic function the pattern of integral curves is of saddle point type. In any angular region bounded by the branches of $V(x, y) = 0$ there is at least one parabolic domain. If one makes an additional assumption that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} \geqq m_R > 0$$

outside the circle of radius R, then the number of parabolic domains will be exactly equal to the number of branches of the level curve $V(x, y) = 0$ and there will be no saddle points at infinity.

From this theorem it can be seen that even in the plane the Lyapunov function yields rougher results than the topological characteristics, as the case of parabolical sectors will not be discerned from the case when these sectors reduce to curves. For a higher-dimensional space the results obtained will probably be still rougher. Such an elementary approach allows one to solve e.g. the problem of whether or not the pattern of integral curves of a linear system in the whole plane will be preserved under nonlinear perturbations. It is also possible to proceed in a slightly different way.

Only positive definite functions $V(x_1, x_2, \ldots, x_n)$ will be taken into account; however, the derivative of such a function with respect to the system will be allowed to vanish and change the sign on certain manifolds called neutral manifolds. These ideas are explained in the paper of P. N. Papuš [39]. It is assumed that the investigated domain contains only one singular point and can be divided by topological cones H_l, $l = 1, 2, \ldots, m$ with the vertices in the origin in ω_s domains, $s = 1, 2, \ldots, m$. These manifolds represent neutral manifolds on which $\mathrm{d}V/\mathrm{d}t = 0$. To obtain certain results it is assumed that the second derivative of $z = V(x_1, x_2, \ldots, x_n)$ with respect to the given system has the same sign on the whole neutral manifold.

The sign of the second derivative $\mathrm{d}^2V/\mathrm{d}t^2$ allows one to determine whether the system is elliptic, i.e. such that all integral curves are either elliptic or parabolic, or whether it is hyperbolic, i.e. such that the integral curves are either hyperbolic or parabolic. It is interesting to remark, as noted by M. B. Kudaev [40], that the conditions in the theorem of P. N. Papuš also represent necessary conditions for sufficiently smooth systems. In order to describe mixed systems of elliptic-hyperbolic type by means of the Lyapunov functions it is necessary to admit that the second derivative has different signs on different cones.

M. B. Kudaev showed one of the principles for distribution of signs for which such systems can be obtained. As yet such systems have been examined which possessed no bounded trajectories except singular points and elliptic trajectories.

The question arises whether the method of Lyapunov functions cannot be used also in these cases. As yet two approaches appeared in this direction.

In the first one the same Lyapunov functions were used as by P. N. Papuš, but it is necessary to assume that the manifolds on which such a function changes its sign are closed and are the boundaries of domains containing a singular point. This method is widely used by Yoshizawa [41] to establish conditions implying that the solutions are ultimately bounded i.e. that every solution enters earlier or later some bounded domain (Levinson's class D).

Evidently, if it is now guaranteed by some condition that the unique singular point in the origin is of a repelling type, then one obtains conditions for the existence of a recurrent family of motions.

In the case of the plane this idea was realized by B. I. Železnov [42] who established conditions for the existence of limit cycles for the systems of the type

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x) + ay\,,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = bx + f_2(y)\,.$$

Essentially the same method was used by È. M. Vaĭsbord [43] to establish the existence of a periodic solution of nonlinear equations of the third and fourth orders. In this connection one should not forget the more refined methods of section manifolds and point transformations developed by H. Poincaré and G. Birkhoff and employed under difficult and complicated circumstances by A. A. Andronov [44] and V. A. Pliss [45].

Another approach to this problem was suggested by myself [46]. It is the method of a rotating Lyapunov function consisting in considering functions $V(x_1, x_2, \ldots, x_n)$ posessing properties of a tangent, the level manifolds of which represent a family of manifolds with a common axis. As yet this method has not been sufficiently employed. Perhaps a recent paper of A. Halanay [47] could be mentioned in this connection.

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