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VARIATIONAL AND BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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The systems with after-effect that are described by differential equations with deviating arguments have the following characteristic property: for the estimation of the future in such a system it is necessary to know the past for the time equal to the time-lag. It means that the initial value space for such a system is a functional space S (with a given norm), and a natural analog to the simple variational problem is a problem of minimizing a functional with deviating argument on trajectories connecting two points of the space S. Such variational problems are named by us the infinite defect variational problems. In the same way, the boundary value problems for differential equations with deviting argument when the trajectories connect the points of the space S are called the infinite defect boundary value problems.

Various problems involving infinite defect are studied extensively now. N.N.Krasovskii [1] , § 45 has formulated and solved the problem of the quieting of a system with time-lag. For the quieting of a system without time-lag $x'(t) = Ax(t) + Bu(t)$ it is sufficient to find a control function $u(t)$ such that $x(t_1) = 0$ for a $t_1 > t_0$ and then put $u(t) = 0$ for $t \geq t_1$. In contrast, for the quieting of the system with time-lag

$$x'(t) = Ax(t) + Gx(t-\tau) + Bu(t)$$

a control function $u(t)$ such that $x(t) = 0$ for $t_1 \leq t \leq t_1 + \tau$ is needed. This problem, as is not difficult to see, is an infinite defect problem.

A.W. Krjashimskii and Yu.S. Osipov [2] studied a difference-differential game with a target set in a functional space. H.Banks and G.A.Kent derived Pontrjagin type maximum principle for control of neutral type difference-differential equations with a functional target set. (See[3]). The variational problems for functional with deviating argument were investigated by G.A. Kamenskii [4, 5] . All mentioned papers deal with infinite defect problems.

The boundary value problems for differential equations with deviating argument have been studied by A.Halanay [6], L.J. Grimm and K.Schmitt [7], G.A.Kamenskii and A.D.Myshkis [8] and others. In the last works was described the essential difference between the boundary value problems for equations with the deviations in the highest order derivatives and for equations without such deviations.

In this paper we investigate the variational problems for the functional with deviating argument of the more general type than in [4, 5] . We consider also analogous problems for functionals depending on functions of many arguments and finite difference method for solving the boundary value problem arising in the onedimentional case.

1. Variational problems for functionals with deviating argument

Consider the problem of the extremum of the functional

$$(1)\qquad J(y) = \int_{\alpha}^{\beta} F(x,\, y(\omega_0(x)),\dots,\, y(\omega_m(x)),\, y'(\omega_0(x)),\dots,\, y'(\omega_m(x)))\, dx,$$

where $-\infty < \alpha < \beta < \infty$, $F: [\alpha,\beta] \times (R^n)^{2m+2} \to R$, $m \geq 1$, in the class $\overset{\circ}{H}_p$ of the functions $y: R \to R^n$, $y(x) \equiv 0$ ($x \bar{\in} [a,b]$), $\alpha \leq a < b \leq \beta$ (a nonhomogeneous boundary value problem may be reduced to the homogeneous one by the standard change of variables), $y(x)$ is absolutely continuous, $y' \in L_p$, $1 \leq p \leq \infty$ with the natural norm. It is supposed that $F \in C_1$, $\omega_0(x) \equiv x$, all $\omega_j \in C_1[\alpha,\beta]$, $\omega_j'(x) \neq 0$, $\omega_j([\alpha,\beta]) \supseteq [a,b]$ and for $p < \infty$

$$(2)\qquad |F(x, y_0,\dots,y_m,z_0,\dots,z_m)| + \sum_{j=0}^{m} \| F_{y_j}(\dots) \| \leq K(\|y_0\|,\dots,\|y_m\|)\,(1 + \sum_{j=0}^{m} \|z_j\|^p)$$

$$(3)\qquad \sum_{j=0}^{m} \| F_{z_j}(x, y_0,\dots,y_m,z_0,\dots,z_m) \| \leq L(\|y_0\|,\dots,\|y_m\|)\,(1 + \sum_{j=0}^{m} \|z_j\|^{p-1})$$

$$(\forall\, x \in [\alpha,\beta],\quad y_0,\dots,y_m,\, z_0,\dots,z_m \in R^m)$$

with continuous K, L; F_{y_j} is a n-dimentional vector.

Denote $\gamma_j = \omega_j([\alpha,\beta]) \to [\alpha,\beta]$ the inverse functions to the ω_j. It is easy to prove that the functional $J(y)$ under our assumptions is differentiable and by simple changes of variables and by integrating by parts we can get the first variation of $J(y)$ in the form

$$\delta J = \int_b^a \Big[\Big[\sum_{j=0}^m \gamma_j'(x) \; F_{y_j}(\gamma_j (x), y(\omega_o(\gamma_j(x))), \dots, y(\omega_m(\gamma_j(x))),$$
$$y'(\omega_o(\gamma_j(x))), \dots, y'(\omega_m(\gamma_j(x)))) - \frac{d}{dx} \sum_{j=0}^m \gamma_j'(x) F_{z_j}(\dots) \Big] \delta y(x) \cdot$$
$$\cdot \, dx$$

(the subtrahend here is necessary to understand in the terms of the theory of distributions). By standard methods we get the proof of the following

Theorem 1. If the function y is a stationary point for the functional (1) (in particular, the point of extremum), then $y \in H_p$ satisfies almost everywhere on $[a, b]$ the equation

$$(4) \quad \sum_{j=0}^m \gamma_j'(x) \; F_{y_j}(\gamma_j(x), \; y(\omega_o(\gamma_j(x))), \dots, y(\omega_m(\gamma_j(x))),$$
$$y'(\omega_o(\gamma_j(x))), \dots, y'(\omega_m(\gamma_j(x)))) - \frac{d}{dx} \sum_{j=0}^m \gamma_j'(x) F_{z_j}(\dots) \; = \; 0.$$

It follows that the expression in (4) standing after the sign $\frac{d}{dx}$ has to be absolutely continuous. (Mark that $y'(x)$ in general case does not belong to that class of functions).

Thus the $y(x)$ is the generalized solution of the equation (4) though the equation (4) is satisfied by $y(x)$ almost everywhere. Remind that you have to put in (4) $y(\omega_j(\gamma_\ell(x))) = y'(\omega_j(\gamma_\ell(x))) = 0$ every time when $\omega_j(\gamma_\ell(x)) \overline{\in} [a, b]$; and that $y(a) = y(b) = 0$. Suppose in addition that $F \in C_2$, $p \geq 2$ and for $p < \infty$ the matrices $F_{y_j y_1}$ satisfy (2), the matrices $F_{y_j z_1}$ satisfy (3) and the matices $F_{z_j z_1}$ - the analoguous inequality with the power p-2. Then by usual methods we may get the following representation of the increment of the functional (1)

$$(5) \qquad \Delta J = \delta J + \frac{1}{2} \delta^2 J + o(\| \delta y \|^2_{\overset{\circ}{H}_p}),$$

where

$$\delta^2 J = \int_\alpha^\beta \sum_{j,1=0}^m \Big[(F_{y_j y_1} \delta y(\omega_j(x))) \cdot \delta y(\omega_\ell(x)) +$$
$$(F_{y_j z_1} \delta y(\omega_j(x)) \cdot \delta y'(\omega_\ell(X)) \; + \; (F_{z_j z_1} \delta y'(\omega_j(x)) \cdot \delta y'(\omega_\ell(x)) \Big] dx.$$

Suppose also that mes $\{ x | \omega_j(x) = \omega_l(x) \} = 0$ $(\forall j, l, j \neq l)$. Then we may state the following analog to the necessary condition of Legendre.

Theorem 2. Suppose that the above mentioned conditions are satisfied and the functional (1) attains on y the local minimum in the space H_p. Then for almost all $x \in [a, b]$ the matrix

$$\sum_{j=0}^{m} F_{z_j z_j}(\gamma_j(x), y(\omega_o(\gamma_j(x))), \ldots, y(\omega_m(\gamma_j(x))), y'(\omega_o(\gamma_j(x))), \ldots$$
$$\ldots, y'(\omega_m(\gamma_j(x)))) \gamma'_j(x)$$

is non-negative.

For the proof it is necessary for any $x_0 \in]a, b[$, $x_0 \bar{\in} \bigcup_{j \neq l} \{ x | \omega_j(x) = \omega_l(x) \}$ to put $\delta y = \frac{1}{M} g(M(x - x_0))$, $g \in H_\infty$, $M \longrightarrow \infty$ and to use the arbitrarity of the finite function g.

2. A variational problem for the quadratic functional depending on the functions of many deviating arguments.

Let S and $Q \subset S$ be non-empty open bounded sets in R^n $(n \geq 2)$ and on $\bar{S}$ are given the functions $\omega_k: \bar{S} \to \omega_k(\bar{S}) \subset R^n$ having the inverse functions $\omega^{-1}_k = \gamma_k: \omega_k(\bar{S}) \to \bar{S}$ and $\omega_o(x) \equiv x$, $Q \subset \omega_k(S)$, $\omega_k \in C^2(\bar{S})$, $\gamma_k \in C^2(\omega_k(\bar{S}))$ $(k = 0, \ldots, m, m \geq 1)$.

Consider the problem of the minimum of the functional

$$J(u) = \iint_S \Big[\sum_{i,j=1}^{n} \sum_{k,l=0}^{m} a_{ijkl}(x) u_{x_i}(\omega_k(x)) u_{x_j}(\omega_l(x)) +$$
$$2 \sum_{i=1}^{n} \sum_{k,l=0}^{m} b_{ikl}(x) u_{x_i}(\omega_k(x)) u(\omega_l(x)) + \sum_{k,l=0}^{m} c_{kl}(x) u(\omega_k(x)) \cdot$$
$$\cdot u(\omega_l(x)) + 2 \sum_{i=1}^{n} \sum_{k=0}^{m} d_{ik}(x) u_{x_i}(\omega_k(x)) + 2 \sum_{k=0}^{m} e_k(x) u(\omega_k(x)) \Big] dx$$

in the subspace H of the space $W_2^1(R^n)$ that is the closure of the set H_o of the infinitely differentiable functions that are finite on Q. With other words we may say that u belongs to the space $\overset{o}{W}{}_2^1(Q)$ and $u_{x_i}(\omega_k(X)) = u(\omega_k(x)) = 0$ by $\omega_k(x) \bar{\in} Q$. Here a_{ijkl}, b_{ikl}, $d_{ik} \in C^1(\bar{S})$, $c_{kl} \in C^o(\bar{S})$, $e_k \in L_2(S)$ $(i,j = 1, \ldots, n;$ $k,l = 0, \ldots, m)$. Without loss of generality we shall suppose that $a_{ijkl} = a_{jilk}$, $c_{kl} = c_{lk}$. Let u be the extremal point for $J(u)$. Then for any $v \in H$

$$(6) \qquad \delta J(u, v) = 0.$$

By a change of variables in the integral representation of (6) we obtain

$$(7) \int_Q \Big\{ \sum_l \Big[\sum_{i,j,k} a_{ijkl}(\gamma_l(x)) u^{kl}_{x_i}(x) v_{x_j}(x) + \sum_{i,k} b_{ikl}(\gamma_l(x)) u^{kl}_{x_i}(x) \cdot$$
$$\cdot v(x) + \sum_{i,k} b_{ilk}(\gamma_l(x)) u^{kl}(x) v_{x_i}(x) + \sum_k c_{kl}(\gamma_l(x)) u^{kl}(x)\, v(x) +$$
$$+ \sum_i d_{il}(\gamma_l(x)) v_{x_i}(x) + e_l(\gamma_l(x))\, v(x) \Big] \Big| \gamma_l'(x) \Big| \Big\}\, dx = 0,$$

where $\bullet(\omega_k(\gamma_l(x))) = \bullet^{kl}(x)$, $\gamma_l'(x)$ - Jacobian $\frac{D(\gamma_{l1}, \dots, \gamma_{ln})}{D(x_1, \dots, x_n)}$. If the function $u \in H$ satisfies the equation (7) for any $v \in H$, we shall call u the generalized solution of the differential equation

$$(8) - \sum_{i,j,k,l} \Big[A_{ijkl}(x) u^{kl}_{x_i}(x) \Big]_{x_j} + \sum_{ikl} B_{ikl}(x)\, u^{kl}_{x_i}(x) +$$
$$+ \sum_{k,l} C_{kl}(x)\, u^{kl}(x) = F(x) \quad (x \in Q).$$

Here

$$A_{ijkl}(x) = a_{ijkl}(\gamma_l(x)) \big| \gamma_l'(x) \big|, \quad B_{ikl}(x) = b_{ikl}(\gamma_l(x)) \big| \gamma_l'(x) \big| -$$
$$- \sum_{r,s=1}^{n} b_{ikl}(\gamma_l(x)) \big| \gamma_l'(x) \big| \cdot (\omega_{kr})_{x_s}(\gamma_l(x)) \cdot (\gamma_{ls})_{x_i}(x),$$

$$C_{kl}(x) = - \sum_i \Big[b_{ikl}(\gamma_l(x)) \big| \gamma_l'(x) \big| \Big]_{x_i} + c_{kl}(\gamma_l(x)) \big| \gamma_l'(x) \big|,$$

$$F(x) = \sum_{i,l} \Big[d_{il}(\gamma_l(x)) \big| \gamma_l'(x) \big| \Big]_{x_i} - \sum_l e_l(\gamma_l(x)) \big| \gamma_l'(x) \big|,$$

$$\omega_k = (\omega_{k1}, \dots, \omega_{kn}), \gamma_l = (\gamma_{l1}, \dots, \gamma_{ln}).$$

We proved the following

Theorem 3. If the functional $J(u)$ attains on the function u the extremum in the space H then u is the generalized solution of the equation (8).

It is easy to show on simple examples (not like for the equations without deviations of arguments) that any requirements on the smoothness of the right hand parts cannot guarantee the existence of twice differentiable solutions. Therefore it is necessary to use the above mentioned definition of the solution in all cases.

Consider now the boundary value problem for the equation (8) in the space H. The boundary condition has the form $u|_{\partial Q} = 0$ and $u^{kl}_{x_i}(x) = u^{kl}(x) = 0$ by $\omega_k(\gamma_l(x)) \overline{\in} Q$.

Define bounded operators

$$A: L_2^n(Q) \to L_2^n(Q),\ (Au)_i(x) = -\sum_{j,k,l} A_{jikl}(x)\, u_j^{kl}(x);$$

$$R: H \to L_2(Q),\ (Ru)(X) = \sum_{i,k,l} B_{ikl}(x) u^{kl}_{x_i} + \sum_{k,l} C_{kl}(x) u^{kl}(x)$$

and adjoint operators

$$A^+: L_2^n(Q) \to L_2^n(Q), (A^+u)_i(x) = \sum_{j,k,l} A^{lk}_{ijkl}(x) \left|\omega_l'(\gamma_k(x))\, \gamma_k'(x)\right| u_j^{lk}(x)$$

$$R^+: H \to L_2(Q),\ (R^+u)(x) = -\sum_{i,k,l} \Big[B^{lk}_{ikl}(x) \left|\omega_l'(\gamma_k(x))\, \gamma_k'(x)\right| \cdot u^{lk}(x)\Big]_{x_i} + \sum_{k,l} C_{kl}(x) \left|\omega_l'(\gamma_k(x)) \cdot \gamma_k'(x)\right| u^{lk}(x).$$

Denote by $(.,.)$ the scalar product in $L_2(Q)$ and by $(.,.)_n$ - the scalar product in $L_2^n(Q)$. Suppose that for a $C > 0$

(9) $\quad (Au, u)_n \geq C(u,u)_n \quad (\forall u \in L_2^n(Q)),$

in this case it is natural to name the equation (8) elliptic. By definition the function $u \in H$ is a solution of the stated boundary value problem for the equation (8), if

(10) $\quad (A\nabla u, \nabla v)_n + (Ru, v) = (F, v) \quad (\forall v \in H).$

Consider also in the space H the homogeneous boundary value problem

(11) $\quad (A\nabla u, \nabla v)_n + (Ru, v) = 0 \quad (\forall v \in H)$

and adjoint boundary value problem

(12) $\quad (A^+\nabla u, \nabla v)_n + (R^+u, v) = 0 \quad (\forall v \in H).$

By means of reducing the equations (8) - (12) introduced above to the equations in the Hilbert space H and using the theory of compact operators in Hilbert spaces we obtain the following

Theorem 4. If the boundary value problem (11) has only zero solution, then the problem (10) has one and only one solution u_F for any $F \in L_2(Q)$, and $\| u_F \|_H \leq C_1 \| F \|$.

If the boundary value problem (11) has non-zero solutions, then the problem (10) has solutions if and only if $(F, \tilde{u}) = 0$, for all solutions $\tilde{u}$ of the problem (12). The dimensions of the solutions spaces of (11) and (12) are finite and equal.

In obtaining of the results of this section took part A.L. Skubachevskii.

3. The finite differences method of the numerical solution of the boundary value problem for the linear equtions with many senior members.

We describe here the finite difference method for linear equations with many senior members and with deviations commensurable with the length of the interval on wich we search for the solution. Such equations may be reduced to the equations with integer deviations of the form

$$(13)\quad \sum_{k=-m}^{m} \Big[(a_k(x)y'(x-k))' + b_k(x)y'(x-k) + c_k(x)y(x-k) \Big] = f(x),$$
$$0 \leq x \leq b \quad (b - \text{integer}),$$

and boundary condutions

$$(14)\quad y(x) = 0 \quad \text{for} \quad x \leq 0 \quad \text{and} \quad x \geq b .$$

Suppose that all a_k', b_k, c_k, $f \in C'[0, b]$ - the space of piecewise continuous functions with the possible jumps in the integers, The equation (13) may be written as the operator equation

$$(15)\quad Ly = DQDy + RDy + Sy = f ,$$

where

$$(Qz)(x) = \sum_{k=-m}^{m} a_k(x)z(x-k) ,$$

$$(Rz)(x) = \sum_{k=-m}^{m} b_k(x)z(x-k),$$

$$(Sz)(x) = \sum_{k=-m}^{m} c_k(x)z(x-k)$$

with boundary conditions (14), D is the operator of differentiation, operators Q, R, S act in $L_2[0, b]$. We suppose that the operator Q has alwas the bounded inverse operator Q^{-1}. By M_n we denote the space of functions defined on $T_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, b\right\}$. The operators Δ^+_n, Δ^-_n, $Q_n : M_n \to M_n$ are defined by the formulae:

$$(\Delta^+_n \xi)(s) = \begin{cases} n(\xi(s+h) - \xi(s)) & (s \in T_n \setminus \{b\}),\ h = \frac{1}{n}, \\ n(\xi(b) - \xi(b-h)) & (s=b), \end{cases}$$

$$(\Delta^-_n \xi)(s) = \begin{cases} n(\xi(h) - \xi(0)) & (s = 0), \\ n(\xi(s) - \xi(s-h)) & (s \in T_n \setminus \{0\}), \end{cases}$$

$$(Q_n \xi)(s) = \sum_{k=-m}^{m} a_k(s)\, \xi(s-k) \quad (s \in T_n,\ \xi(s-k) = 0 \text{ for } s-k \bar{\in} T_n).$$

The operators R_n and S_n are definied similarly. Define operator $[\,\cdot\,]_n : C[0, b] \to M_n$ by equality $[y]_n(s) = y(s)$, $s \in T_n$ and define norms in M_n by formulae

$$\|\xi\|^{o}_{n} = \Big(\sum_{\nu=1}^{bn-1} \xi^2(h\nu)\Big)^{\frac{1}{2}}, \quad \|\xi\|_n = \Big(\sum_{\nu=0}^{bn} \xi^2(h\nu)\Big)^{\frac{1}{2}},$$

$$\|\xi\|^{C}_{n} = \max_{s \in T_n} |\xi(s)|.$$

The approximate solution of the boundary value problem (13), (14) is a net function $\xi(s)$ satisfying the equation

(16) $$(L_n \xi)(s) = (\Delta^-_n Q_n \Delta^+_n + R_n \Delta^+_n + S_n)\, \xi(s) = [f]_n(s),$$
$$s \in T_n \setminus \{0, b\}; \quad \xi(0) = \xi(b) = 0.$$

It is easy to prove that if $y \in H_\infty$, $y' \in C'[0, b]$, y satisfies the condition (14), then

$$\lim_{n\to\infty} \| [Ry']_n - R_n \Delta^{\pm}[y]_n \|_n = 0.$$

If we put now $Ry = y$ and insert Qy instead of y, we prove that if on every interval $]0, 1[, \ldots,]b-1, b[$ exist uniformly continuous y' and y'' and $Qy' \in H_\infty$, $(Qy')' \in C'[0, b]$, then

$$\lim_{n\to\infty} \| [(Qy')']_n - \Delta^-_n Q_n \Delta^+_n [y]_n \|^{o}_{n} = 0.$$

Thus we have prooved the following theorem of the approximation of the operator L :

Theorem 5. If $Ly = f$, then

(17) $$\| L_n [y]_n - [f]_n \|^{o}_{n} \longrightarrow 0 \quad \text{for } n \longrightarrow \infty.$$

The following theorem states the stability of the finite difference scheme.

Theorem 6. For the injectivity of the operator L it is necessary and sufficient that there exist $C > 0$ and n_0 such that

$$(18) \quad \| L_n \xi \|^{\circ}_n \geq C \{ \| \Delta^-_n Q_n \Delta^+_n \xi \|_n + \| Q_n \Delta^+_n \xi \|^C_n + \| \Delta^+_n \xi \|_n + \| \xi \|^C_n \} \quad (n \geq n_0,\ \xi(0) = \xi(b) = 0).$$

The necessity is prooved by the assumption of the contrary by using the piecewise linear interpolation of the functions for which the expression in parenthesis in (18) is equal to 1. The sufficiency follows from the theorem 5.

From (18) it follows in particular that (16) has an exactly one solution for each $n \geq n_0$. If we put in (18) $\xi = [y]_n - [\xi]_n$ and apply (17), we prove the theorem of the approximation of the solution :

Theorem 7. If the operator L is injective, y is a solution of $Ly = f$ and ξ_n is a solution of (16), then

$$\| [y]_n - \xi_n \|^{\circ}_n \longrightarrow 0 \quad \text{for} \quad n \longrightarrow \infty.$$

In obtaining the results of this section took part A.G. Kamenskii.

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