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THE SOLUTION OF PARABOLIC MODELS BY FINITE ELEMENT SPACE AND A-STABLE TIME DISCRETIZATION

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In papers on solution of parabolic differential equations by the finite element method error bounds are given either in the case that the union of finite elements (straight or curved) matches exactly the given domain (e.g. in Zlámal's papers) or in the case of curved elements which do not cover,in general,the given domain (e.g. in Raviart's papers).In the former case the error bounds are given for fully (i.e. both in space and time) discretized approximate solutions.In the latter case the numerical integration is taken into account,however the error bounds are given only for semidiscrete(not discretized in time) approximate solutions.Error bounds introduced in this lecture are given for fully discretized approximate solutions and for arbitrary curved domains.Discretization in time is carried out by A-stable linear multistep methods.Isoparametric simplicial curved elements in n-dimensional space are applied.Degrees of accuracy of quadrature formulas are determined such that numerical integration does not worsen the optimal order of convergence in L_2 norm of the method.

1.The finite element space discretization of the problem

Let us first introduce the parabolic problem in the variational form. Let $x=(x_1,\dots x_n)\in R^n$. Let Ω be a bounded domain in R^n. Let the functions $g(x)$, $g_{ij}(x)$, $i,j=1,\dots n$ defined on Ω and the function $f(x,t)$ defined on $\Omega\times(0,T]$ be smooth enough. Let

(1) $g_{ij}(x) = g_{ji}(x),\ g(x)\geq g_0(=\text{const})>0,\ \forall x\in\overline{\Omega}$

and let the differential operator

(2) $$L=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(g_{ij}(x)\frac{\partial}{\partial x_i}\right)$$

be uniformly elliptic in Ω.Let $a(u,v)$ be the bilinear form corresponding to operator L,i.e.

(3) $$a(u,v)=\int_{\Omega}\sum_{i,j=1}^{n}g_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx.$$

We study the following problem:

Find a function u(x,t) such that

$u \in L^{\infty}(H_0^1(\Omega)),\ \frac{\partial u}{\partial t} \in L^{\infty}(H^{-1}(\Omega))$,

(4) $(g\frac{\partial u}{\partial t}, v)_{0,\Omega} + a(u,v) = (f,v)_{0,\Omega},\ \forall v \in H_0^1(\Omega)$ and $t \in (0,T]$

$u(x,0) = u_0(x) \in L^2(\Omega)$.

Here $H_0^1(\Omega)$ is the closure of the set $C_0^{\infty}(\Omega)$(i.e. of the set of infinitely differentiable functions with compact support in Ω) in the Sobolev norm $\|\cdot\|_{1,\Omega}$. $H^{-1}(\Omega)$ is the space dual to $H_0^1(\Omega)$ (with dual norm). $L^{\infty}(H^m(\Omega))$ is the space of all functions $v(x,t), x=(x_1,\dots x_n)\in\Omega$, $t \in (0,T]$ such that $v(x,t) \in H^m(\Omega)$, $\forall t \in (0,T]$ and the function $\|v(x,t)\|_{m,\Omega}$ is bounded for almost all $t \in (0,T]$.

First we discretize the problem (4) by the finite element method with respect to x. For this we use a k-regular family of isoparametric simplicial curved elements in n-dimensional space which are constructed in Raviart's paper[1]. Let $\mathcal{C}_h$ be a k-regular triangulation of the set Ω and let V_h be the corresponding finite element space. The union of the elements e from $\mathcal{C}_h$ forms some set Ω_h which, in general, differs from Ω. We extend the functions $g(x), g_{ij}(x), u_0(x)$ to a greater set $\tilde{\Omega} \supset \bar{\Omega}$ such that the conditions (1)and (2) are satisfied. In such a way we obtain the functions $\tilde{g}(x), \tilde{g}_{ij}(x)$ and $\tilde{u}_0(x)$. Obviously, for sufficiently small h, it is true

(5) $\Omega_h \subset \tilde{\Omega}$.

About the solution u of the problem (4) we suppose

(6) $u, \frac{\partial u}{\partial t} \in L^{\infty}(H^{k+3}(\Omega))$.

By the Calderon extension theorem, for every $t \in (0,T]$ there exist extensions $\tilde{u}(x,t)$, $\widetilde{\frac{\partial u}{\partial t}}$. Let us denote

(7) $\tilde{f}(x,t) = \tilde{g}(x)\widetilde{\frac{\partial u}{\partial t}} - \tilde{L}\tilde{u}$,

where

(8) $\tilde{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(\tilde{g}_{ij}(x)\frac{\partial}{\partial x_i})$.

According to (4) we define now the following semidiscrete problem:

Find a function $u_s(x,t)$ such that

$$u_s, \frac{\partial u_s}{\partial t} \in L^\infty(V_h(\Omega_h)),$$

$$(9)\quad (\tilde{g}(x)\frac{\partial u_s}{\partial t}, v)_{0,\Omega_h} + \tilde{a}(u_s, v) = (\tilde{f}, v)_{0,\Omega_h}, \ \forall v \in V_h, \ t \in (0,T],$$

$$u_s(x,0) = u_0 \in V_h,$$

where u_0 is an approximate of $u_0(x)$ and $\tilde{a}(u,v)$ is the bilinear form

$$(10)\quad \tilde{a}(u,v) = \int_{\Omega_h} \sum_{i,j=1}^{n} \tilde{g}_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx.$$

2.The A-stable time discretization of the problem

We called the problem (9) semidiscrete because it is discretized with respect to x only.It is obvious that (9) is a system of ordinary differential equations with an unknown vector function of parameter t .From here we find the way how to discretize the problem with respect to t.We solve the system by ν-step A-stable linear method (for $\nu=1,2$) of order q.We divide the time interval $[0,T]$ into a finite number of intervals of the same length Δt.We introduce the following notation

$$(11)\quad \phi^m = \phi^m(x) = \phi(x, m\Delta t), \ m = 0,1,\ldots$$

for any function $\phi(x,t)$.
If we apply to (9) a ν-step ($\nu = 1,2$) A-stable linear method we get the following discrete problem:

Find a function $u_d(x,t)$ such that

$u_d \in V_h$ for any $t = 0, \Delta t, 2\Delta t, \ldots T$

$$(12)\quad (\tilde{g}(x)\sum_{j=0}^{\nu}\alpha_j u_d^{m+j}, v)_{0,\Omega_h} + \Delta t\tilde{a}(\sum_{j=0}^{\nu}\beta_j u_d^{m+j}, v)$$

$$= \Delta t(\sum_{j=0}^{\nu}\beta_j \tilde{f}^{m+j}, v)_{0,\Omega_h}, \ \forall v \in V_h, \ m = 0,1,\ldots$$

$$u_d^0 = u_0 \in V_h;$$

here (see [11] and [12])
a) for one-step A-stable methods

$\gamma = 1$, $\alpha_1 = 1$, $\alpha_0 = -1$, $\beta_1 = 1-\theta$, $\beta_0 = \theta$, $\theta \leq \frac{1}{2}$ is any real number. If $\theta = 1/2$ then the method is of order $q = 2$, in all the other cases the method is of order $q = 1$.

b) for two-steps A-stable methods
$\gamma = 2$, $\alpha_2 = \theta$, $\alpha_1 = 1-2\theta$, $\alpha_0 = -1+\theta$, $\beta_2 = (1/2)\theta + \delta$, $\beta_1 = (1/2)-2\delta$, $\beta_0 = (1/2) - (1/2)\theta + \delta$, $\theta \geq (1/2)$, $\delta > 0$.

3. The numerical integration

Since it is either too costly or simply impossible to evaluate exactly the integrals $(\cdot,\cdot)_{0,\Omega_h}$, $\tilde{a}(\cdot,\cdot)$, we must now take into account the fact that approximate integration is used for their computation. For this purpose we use the isoparametric numerical integration (see [1]). We remember:

Every finite element $e \in \mathcal{C}_h$ is the image (i.e. $e = F_e(\hat{T})$) of the unit n-simplex $\hat{T}$ through the unique mapping $F_e: \hat{T} \rightarrow R^n$. Let us suppose that we have at our disposal a quadrature formula of degree d over the reference set $\hat{T}$. In other words,

$$(13) \quad \int_{\hat{T}} \hat{\phi}(\hat{x})d\hat{x} \approx \sum_r \hat{\omega}_r \hat{\phi}(\hat{b}_r)$$

for any function $\hat{\phi}(\hat{x})$ which is defined on $\hat{T}$ and for some specified points $\hat{b}_r \in \hat{T}$ and weights $\hat{\omega}_r$. Let $\phi(x)$ be any function defined on e. Then using the standard formula for the change of variables in multiple integrals, we find that

$$(14) \quad \int_e \phi(x)dx \approx \sum_r \hat{\omega}_r |J_e(\hat{b}_r)| \phi(F_e(\hat{b}_r))$$

where $J_e(\hat{x})$ is Jacobian of the transformation $e = F_e(\hat{T})$.
We see that the quadrature scheme (13) over the reference set $\hat{T}$ induces the quadrature scheme (14) over the element e, a circumstance which we call "isoparametric numerical integration".

In agreement with (14) we replace in (12)

$$(15) \quad (\cdot,\cdot)_{0,\Omega_h} \approx (\cdot,\cdot)_h, \quad \tilde{a}(\cdot,\cdot) \approx a_h(\cdot,\cdot)$$

According to (12) and (15) we define the following full discretized problem:

Find a function $u_h(x,t)$ such that

$u_h \in V_h$ for $t = 0, \Delta t, \ldots T$

$$(16)\quad (g(x)\sum_{j=0}^{\nu}\alpha_j u_h^{m+j}, v)_h + \Delta t a_h(\sum_{j=0}^{\nu}\beta_j u_h^{m+j}, v)$$

$$= \Delta t(\sum_{j=0}^{\nu}\beta_j f^{m+j}, v)_h, \quad \forall\, v \in V_h,\ m = 0,1,\ldots$$

$u_h^0 = u_0 \in V_h$.

4. Error bounds

Theorem.

Let $u(x,t)$ be the solution of the problem (5) such that $u, \frac{\partial^r u}{\partial t^r} \in L^\infty(H^{k+3}(\Omega)), r = 1,\ldots q$. Let $\mathcal{C}_h$ be a k-regular triangulation of the set Ω_h where k is a positive integer such that $k > n/2 - 1$. Let the quadrature formulas on the reference set $\hat{T}$ for calculation of the forms $(.,.)_{0,\Omega_h}$ and $\tilde{a}(.,.)$ be of degree $d \geq 2k$ and $d \geq 2k - 1$, respectively. Let a given ν-step time discretization method be A-stable and of order q. Let $\nu = 1$ or 2.

Then the full discrete problem (16) has one and only one solution $u_h(x,t)$ and there exists a constant c independent of t and h such that

$$(17)\quad \|u^s - u_h^s\|_{0,\Omega\cap\Omega_h} \leq c(\Delta t^q + h^{k+1} + |\varepsilon^0|_h + |\varepsilon^{\nu-1}|_h).$$

[Here $\varepsilon^0, \varepsilon^{\nu-1}$ are the errors on the first ν steps, $|\varepsilon|_h = \sqrt{(g\varepsilon, \varepsilon)_h}$.]

Outline of the proof.

Let us denote

$$(18)\quad \tilde{u}^j = \eta^j + \xi^j,$$

where $\eta^j = \eta(x, j\Delta t)$ is the Ritz approximation of the function $\tilde{u}^j = \tilde{u}(x, j\Delta t)$. We recall that $\tilde{u}$ and $\widetilde{\frac{\partial u}{\partial t}}$ are extensions of u and $\frac{\partial u}{\partial t}$ satisfying the inequalities

$$(19)\quad \|\tilde{u}\|_{k+3,\tilde{\Omega}} \leq c\|u\|_{k+3,\Omega}, \quad \|\widetilde{\frac{\partial u}{\partial t}}\|_{k+3,\tilde{\Omega}} \leq c\|\frac{\partial u}{\partial t}\|_{k+3,\Omega}.$$

Next we recall that by the Ritz approximation of the function $\tilde{u}(x,t)$

we mean the function $\eta(x,t) \in V_h$ $(V_h \subset H_0^1(\Omega_h))$, $\forall\ t \in (0,T]$ such that

(20) $(\tilde{g}(x)\ \widetilde{\frac{\partial u}{\partial t}}\ ,v)_{0,\Omega_h} + \tilde{a}(\eta(x,t),v) = (\tilde{f}(x,t),v)_{0,\Omega_h},\ \forall\ v \in V_h$.

It is easy to prove that $\eta(x,t)$ is an orthogonal projection onto V_h of the function $\tilde{u}(x,t)$ in the energy norm given by the bilinear form $\tilde{a}(.,.)$, i.e. that it satisfies

(21) $\tilde{a}(\tilde{u} - \eta, v) = 0,\ \forall\ v \in V_h$.

For the Ritz approximation the following estimate can be derived

(22) $\|\tilde{u} - \eta\|_{i,\Omega_h} \leq ch^{k+1-i}\|u\|_{k+3,\Omega},\ i = 0,1,$

where c is a constant independent of h and t.
From (22) and (18) we get

(23) $\|\xi^j\|_{0,\Omega_h} = \|\tilde{u}^j - \eta^j\|_{0,\Omega_h} \leq ch^k\|u\|_{k+3,\Omega}$.

From (18) it follows

(24) $\|u^j - u_h^j\|_{0,\Omega\cap\Omega_h} \leq \|\tilde{u}^j - u_h^j\|_{0,\Omega_h} \leq \|\xi^j\|_{0,\Omega_h} + \|\eta^j - u_h^j\|_{0,\Omega_h}$.

Hence, it is sufficient to give an estimation of error bounds for

(25) $\varepsilon^j = \eta^j - u_h^j$.

By simple calculation we get from (14) and (16)

$$
\begin{aligned}
&(g\sum_{j=0}^{\nu}\alpha_j\varepsilon^{m+j}\ ,v)_h + \Delta t a_h(\sum_{j=0}^{\nu}\beta_j\varepsilon^{m+j}\ ,v) \\
(26)\qquad &= (\Pi_\nu^m - \omega_\nu^m\ ,v)_{0,\Omega_h} + \Delta t E(v\sum_{j=0}^{\nu}\beta_j\tilde{f}^{m+j}) - E(\tilde{g}v\sum_{j=0}^{\nu}\alpha_j\eta^{m+j}) \\
&\quad - \Delta t E(\sum_{i,j=1}^{n}\tilde{g}_{ij}\frac{\partial v}{\partial x_j}\sum_{r=0}^{\nu}\beta_l\frac{\partial\eta^{m+r}}{\partial x_i}),\ \forall\ v \in V_h\ ,
\end{aligned}
$$

where

(27) $\Pi_\nu^m = \tilde{g}\sum_{j=0}^{\nu}(\alpha_j\tilde{u}^{m+j} - \Delta t\beta_j\widetilde{\frac{\partial u}{\partial t}}^{m+j}),\ \omega_\nu^m = \tilde{g}\sum_{j=0}^{\nu}\alpha_j\xi^{m+j},\ E(\phi) = \sum_{e\in\mathcal{E}_h}E_e(\phi)$

and where $E_e(\phi)$ is according to (14) the error given by the isoparametric integration, i.e.

(28) $E_e(\phi) = \int_e \phi(x)dx - \sum_r \hat{\omega}_r |J_e(\hat{b}_r)|\,\phi(F_e(\hat{b}_r))$.

We denote the expressions in identity (26) by $A_\nu^m(v)$, $B_\nu^m(v)$, $D_\nu^m(v)$, $F_\nu^m(v)$, $G_\nu^m(v)$, $H_\nu^m(v)$ respectively.Next we denote $Q_\nu^m = \Delta t F_\nu^m - G_\nu^m - \Delta t H_\nu^m$. The identity (26) is true for all $v \in V_h$, hence it is also true for

(29) $v = \gamma = \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}$.

From here we get the following basic identity

(30) $$\sum_{m=0}^{s-\nu} A_\nu^m(\gamma) + \Delta t \sum_{m=0}^{s-\nu} B_\nu^m(\gamma) = \sum_{m=0}^{s-\nu} D_\nu^m(\gamma) + \sum_{m=0}^{s-\nu} Q_\nu^m(\gamma)$$

valid for any s such that $s\Delta t \leqslant T$, $s \geqslant \nu$.
Using the similar technique as in [11] we prove

(31) $$\sum_{m=0}^{s-\nu} A_\nu^m(\gamma) \geqslant c_2 \|\varepsilon^s\|^2_{0,\Omega_h} - c_1\left[|\varepsilon^0|_h^2 + |\varepsilon^{\nu-1}|_h^2 \right], \quad \nu = 1,2.$$

To this end we use the inequality

(32) $c_3 \|v\|_{0,\Omega_h} \leqslant |v|_h$, $\forall\, v \in V_h$

valid under the assumption that the quadrature formula on the reference set $\hat{T}$ is of degree $d \geqslant 2k$.In the inequality the notation $|v|_h^2 = (g(x)v, v)_h$ is used.
It is easy to derive the following inequality

(33) $$\sum_{m=0}^{s-\nu} B_\nu^m(\gamma) \geqslant c_4 \sum_{m=0}^{s-\nu} \Big|\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}\Big|^2_{1,\Omega_h}, \quad \nu = 1,2.$$

For this purpose we use the inequality

(34) $c_5 |v|_{1,\Omega_h} \leqslant \|v\|_h$, $\forall\, v \in V_h$

valid under the assumption that the quadrature formula on the reference set $\hat{T}$ is of degree $d \geqslant 2k-2$.In this inequality the notation $\|v\|_h^2 = a_h(v,v)$ is used.
Next, we prove the inequality

(35) $$\sum_{m=0}^{s-\nu} |D_\nu^m(\gamma)| \leqslant c_6 \Delta t(\Delta t^q + h^{k+1}) \sum_{m=0}^{s-\nu} \Big\|\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}\Big\|_{0,\Omega_h}, \quad \nu = j,2.$$

For $Q_\nu^m(\gamma)$ the following estimate can be derived

$$(36)\quad \sum_{m=0}^{s-\nu} |Q_\nu^m(\gamma)| \le c_7 \Delta t h^{k+1} \sum_{m=0}^{s-\nu} \| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \|_{1,\Omega_h}, \quad \nu = 1,2.$$

To this end we use the inequalities

$$|E(wv)| \le c_8 h^{k+1} \|w\|_{k+1,\Omega_h} \|v\|_{1,\Omega_h},$$

$$(37)\quad |E(b\eta v)| \le c_9 h^{k+1} \|u\|_{k+3,\Omega} \|v\|_{1,\Omega_h},$$

$$\left|E\left(b \frac{\partial \eta}{\partial x_i} \frac{\partial v}{\partial x_j}\right)\right| \le c_{10} h^{k+1} \|u\|_{k+3,\Omega} \|v\|_{1,\Omega_h}$$

valid for all $w \in H^{k+1}(\Omega_h)$, $v \in V_h(\Omega_h)$, $u \in H^{k+3}(\Omega)$, $t \in (0,T]$ and $b(x) \in C^{k+1}(\Omega_h)$ under the assumption that the quadrature formula on the reference set $\hat{T}$ is of degree $d \ge 2k-1$.

From (31), (33), (35) and (36),using several times the inequality

$$(38)\quad |ab| \le \frac{1}{2}\tau a^2 + \frac{1}{2\tau} b^2$$

we get

$$(39)\quad \|\varepsilon^s\|^2_{0,\Omega_h} \le c_{11}(\Delta t^{2q} + h^{2(k+1)} + |\varepsilon^0|_h^2 + |\varepsilon^{\nu-1}|_h^2) + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|^2_{0,\Omega_h}.$$

From (39) and from [9] (see Lemma 2.1,p.396) we get the estimate (17).

In the end let us add the following remarks:

Remark 1.

From (17) we see that the L_2-norm of the error is of a magnitude of the order Δt^q $(q = 1,2)$ with respect to Δt and of the order h^{k+1} with respect to h.

Remark 2.

According to our result, for 1-regular triangulation (i.e.for linear isoparametric elements) the quadrature formula on the reference set $\hat{T}$ for calculation of the forms $(.,.)_{0,\Omega}$ and $a(.,.)$ must be,in general,of degree 2 and 1, respectively.It can be proved that using the quadrature formula

$$(40)\quad \int_{\hat{T}} \varphi(\hat{x})d\hat{x} \approx \frac{\text{mes}\,\hat{T}}{n} \left[\varphi(0,\ldots 0) + \varphi(0,1,\ldots 0) + \ldots + \varphi(0,0,\ldots 1)\right]$$

(which is of degree 1) for calculation of the form $(.,.)_{0,\Omega}$ we obtain the same estimate as in (17).In this case the mass matrix is diagonal. In the engineering literature this effect is called the mass lumping.

Remark 3.

For the three-dimensional space the simplicial curved elements have no practical use. For such case the theory using quadrilateral elements must be developed. We are working on this problem now.

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