THE ROLE OF CENTER MANIFOLDS IN ORDINARY DIFFERENTIAL EQUATIONS

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In the last decade center manifold theory turned out to be one of the most useful and widely used concepts of invariant manifold theory. The notion and a first systematic treatment of what nowadays is called center manifold theory appeared 1967 in Kelley's paper [15]. Despite of its modern appearence however some basic aspects of this theory can be traced back to the beginning of qualitative theory of ordinary differential equations. The first treatment was probably given by Bohl [7] in 1904. He essentially constructed the types of invariant manifolds listed in Theorem 1 below. In addition he established one of the basic properties of center manifolds, namely to contain all solutions which are sufficiently small. Lyapunov, on the other hand, proved and used the second basic property (the so-called reduction principle) in his treatment of the critical case of stability with a pair of purely imaginary eigenvalues when he reduced a given system to a two-dimensional one which bears all information concerning stability. The reducing transformation is described by a function representing, in modern language, a center manifold for the original system. Kelley [16] and Pliss [23] developed a corresponding general reduction principle which allows to reduce the dimension of the underlying differential system without losing any information concerning stability. This reducing property is not limited to stability, but also holds for a wider class of local problems in differential equations e.g. in bifurcation theory. Bifurcating objects which are made up of small bounded solutions such as stationary solutions, periodic orbits or tori always lie on center manifolds. Thus, from a bifurcation point of view, the flow of a given system need to be studied on a center manifold only. In most cases treated so far this meant reduction to dimension one or two. Hence the reduction principle is the most useful result of center manifold theory. It is remarkable that it can be extended to certain types of infinite dimensional systems (see Henry [14], Carr [8]). There exists also an analogous notion for mappings (see e.g. Marsden and McCracken [21]) with similar properties and a wide range of applications.

In this report we consider the finite dimensional case only and give first an up to date account of center manifold theory for ordinary differential equations (sections 1 and 2). The main results of this paper are contained in Section 3 and concern an application of center manifold theory to a non-local problem. To be more specific we consider an invariant manifold and rise the question whether the approach to this manifold as a whole implies approach to a particular solution on this manifold. If the manifold is an orbitally asymptotically stable periodic solution the answer is in the affirmative. This is part of the well known Andronov-Witt-Theorem ("Existence of an asymptotic phase"). In Section 4 finally we demonstrate the scope of our method by means of an example which arises in genetic population dynamics.

1. Autonomous systems

In this section we collect the main facts of center manifold theory for first order autonomous differential systems. For general references see Kelley [15], Palmer [22], Carr [8]. Without loss of generality we write the underlying differential system in the form

$$\begin{aligned} \dot{x} &= A^{-}x + p(x,y,z) \\ \dot{y} &= A_{o}y + q(x,y,z) \\ \dot{z} &= A^{+}z + r(x,y,z) \end{aligned} \tag{1}$$

where A^{-}, A_{o}, A^{+} are constant matrices whose eigenvalues have negative, vanishing, positive real parts, respectively, and p,q,r vanish together with their first order partial derivatives at the coordinate origin.

The first theorem is concerned with five types of manifolds which are invariant under the flow of system (1) and which exist under the above hypotheses.

Theorem 1: System (1) admits five types of invariant manifolds whose respective representations near (0,0,0) are

(i) $(y,z) = h^{-}(x)$ (stable manifold H^{-})

(ii) $(x,z) = h_{o}(y)$ (center manifold H_{o})

(iii) $(x,y) = h^{+}(z)$ (unstable manifold H^{+})

(iv) $z = h_{o}^{-}(x,y)$ (center-stable manifold H_{o}^{-})

(v) $x = h_{o}^{+}(y,z)$ (center-unstable manifold H_{o}^{+}).

Each of these functions has the same order of differentiability as the right-hand side of (1) (if it is finite) and vanishes at zero together

with its first order derivative.
Furthermore the following implications hold true:

(I) If a solution $(x(t),y(t),z(t))$ of (1) starts at $t=0$ on H^- (H^+) sufficiently close to the origin then $(y(t),z(t)) = h^-(x(t))$ $((x(t),y(t)) = h^+(z(t)))$ holds for all $t \in [0,\infty)$ $((-\infty,0])$ and the solution tends to $(0,0,0)$ as $t \to \infty$ $(t \to -\infty)$;

(II) If a solution $(x(t),y(t),z(t))$ of (1) stays sufficiently close to the origin for all $t \in [0,\infty)$ $((-\infty,0], \mathbb{R})$ then $z(t) = h_o^-(x(t),y(t))$ $(x(t) = h_o^+(y(t),z(t)),\ (x(t),z(t)) = h_o(y(t)))$ holds for all $t \in [0,\infty)$ $((-\infty,0], \mathbb{R})$.

Remarks: 1. The general definitions of stable, center, unstable, center-stable, center-unstable manifolds, respectively, for system (1) are given in terms of functional representations and approximation properties at the point $(0,0,0)$. In fact, the tangent spaces at $(0,0,0)$ are the corresponding (global) invariant manifolds for the linearized system.
2. It is not true in general that an arbitrary invariant subspace of the linearized system is tangent space at $(0,0,0)$ of an invariant manifold of the nonlinear system (1). As an example consider the linear space given by the equation $y=0$ which is invariant with respect to the linearized equation. The equivalent for the system (1) would be an invariant manifold with equation $y=h(x,z)$ passing through $(0,0,0)$. Such a manifold however does not always exist. Take e.g. x,y,z scalar and consider the equation

$$\dot{x} = -x \quad , \quad \dot{y} = xz \quad , \quad \dot{z} = z \quad .$$

Assume one can find a sufficiently smooth function $h(x,z)$ with $h(0,0)=0$ such that $y=h(x,z)$ defines an invariant manifold. Then h satisfies the partial differential equation

$$xz = -h_x x + h_z z \quad .$$

One arrives then at a contradiction simply from the observation that the Taylor-expansion of the right-hand side does not contain the term xz. We wish to point out however that there may exist other invariant manifolds for (1) than those mentioned in Theorem 1 which are related to linear solution spaces of the linearized equation. One can e.g. consider the space consisting of those solutions of the linear equation which have a decay rate less than or equal to $\exp(-\delta t)$, δ a given positive number. As it has been shown in [18, Kap.V, Satz 9.1] one can indeed construct then an invariant manifold for (1) such that all statements of Theorem, except possibly (II), remain true.

For special cases of system (1) we get this

Corollary: If in system (1) the critical y-component is absent then for a solution $(x(t),y(t),z(t))$ near $(0,0,0)$ the following three statements are equivalent:

(I'a) $(x(0),y(0),z(0))$ belongs to H^- (H^+),

(I'b) $(x(t),y(t),z(t))$ is bounded on $[0,\infty)$ ($(-\infty,0]$),

(I'c) $(x(t),y(t),z(t))$ tends to $(0,0,0)$ as $t\to\infty$ ($t\to-\infty$) .

If in system (1) the unstable z-component (stable x-component) is absent then instead of (II) the following is true:

(II') If a solution $(x(t),y(t),z(t))$ stays sufficiently close to $(0,0,0)$ for all $t\in[0,\infty)$ ($(-\infty,0]$) then $(x(t),z(t)) = h_o(y(t))$ for all $t\in$ $[0,\infty)$ ($(-\infty,0]$).

For system (1) in its general form both the stable and the unstable manifold is unique, in fact, they can be characterized as the set of solutions that approach the origin at an exponential rate as $t\to\infty$, $t\to-\infty$, respectively. The center manifold however and also the center-stable and the center-unstable manifold are not unique in general. In spite of this nonuniqueness each function representing one of the above five types of invariant manifolds has a unique Taylor-expansion which can be determined in a purely algebraic way. This is stated in the next theorem.

Theorem 2: The above five functions satisfy the respective partial differential equation near the corresponding coordinate origin:

(i) $$\begin{pmatrix} A_o & 0 \\ 0 & A^+ \end{pmatrix} h^- + \begin{pmatrix} q(x,h^-) \\ r(x,h^-) \end{pmatrix} = h_x^- \,(\, A^-x + p(x,h^-)\,),$$

(ii) $$\begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix} h_o + \begin{pmatrix} p(h_o^1,y,h_o^2) \\ r(h_o^1,y,h_o^2) \end{pmatrix} = (h_o)_y \,(\, A_o y + q(h_o^1,y,h_o^2)\,)$$

where $h_o = (h_o^1,h_o^2)$,

(iii) $$\begin{pmatrix} A^- & 0 \\ 0 & A_o \end{pmatrix} h^+ + \begin{pmatrix} p(h^+,z) \\ q(h^+,z) \end{pmatrix} = h_z^+ \,(\, A^+z + r(h^+,z)\,),$$

(iv) $$A^+h_o^- + r(x,y,h_o^-) = (h_o^-)_x(A^-x + p(x,y,h_o^-)) + (h_o^-)_y(A_o y + q(x,y,h_o^-)),$$

(v) $$A^-h_o^+ + p(h_o^+,y,z) = (h_o^+)_y(A_o y + q(h_o^+,y,z)) + (h_o^+)_z(A^+z + r(h_o^+,y,z)).$$

Furthermore the Taylor-expansions of h^-,h_o,h^+,h_o^-,h_o^+ are the unique formal power series which solve the respective partial differential equations and start with second order terms.

Remarks: 1. If the right-hand side of system (1) is analytic then both the stable and the unstable manifold is analytic (see the appendix in [1]). The other three types of manifolds however are generally not analytic. Some well known counterexamples can be found in Kelley [15] and Carr [8]. These examples have the property that the coefficient matrix of the linearized equation is singular. We just want to mention another example with nonsingular coefficient matrix which has been used by Lyapunov [19] in a similar context (see also Malkin [20]).

2. In the analytic case a center manifold need not even be of class C^∞ (see van Stien [24] and Carr [8]) although it has, by Theorem 1, derivatives of any order. The explanation is that the domain of definition of the center manifold function may shrink to the origin as the order of differentiability increases.

To conclude the discussion of the autonomous case we describe the reduction principle that generalizes Lyapunov's [19] and Malkin's [20] concept of stability in critical cases. Since stability of the origin for system (1) can occur only without unstable z-component we suppose system (1) to have the form

$$\begin{aligned} \dot{x} &= A^- x + p(x,y) \\ \dot{y} &= A_o y + q(x,y) \ . \end{aligned} \tag{2}$$

Along with this we consider the so-called reduced system

$$\dot{y} = A_o y + q(h_o(y),y) \tag{3}$$

describing the flow of system (2) on a center manifold $x = h_o(y)$. By $(x(t,x_o,y_o),y(t,x_o,y_o))$, $\tilde{y}(t,\tilde{y}_o)$ we denote the respective solution of (2),(3) with initial value (x_o,y_o), $\tilde{y}_o$ for $t=0$.

<u>Theorem 3</u>: The origin of system (2) is stable (asymptotically stable, unstable) if and only if the origin of the reduced system (3) is stable (asymptotically stable, unstable). Furthermore if $\|x_o\| + \|y_o\|$ is sufficiently small then there exists a $\tilde{y}_o = \tilde{y}_o(x_o,y_o)$ such that $\|x(t,x_o,y_o) - h_o(\tilde{y}(t,\tilde{y}_o))\| + \|y(t,x_o,y_o) - \tilde{y}(t,\tilde{y}_o)\|$ decays with exponential rate as $t \to \infty$. The "asymptotic phase" $\tilde{y}_o(x_o,y_o)$ has the same order of differentiability as the right-hand side of (2) if it is finite.

For a precise formulation and the proof of the last statement the reader is referred to [4].

2. Nonautonomous systems

The basic facts about autonomous center manifold theory which were outlined in the previous section are all well documented in the literature by now. In what sense and to what extent they can be generalized to the nonautonomous case is not completely obvious. We give a short description of what has been definitely established in the literature (see [17],[18],[4]). For reasons of simplicity we confine ourselves to the study of the configuration stable/center-unstable manifolds. In terms of differential equations this configuration occurs if the given system can be brought by means of a suitable transformation into the form

$$\begin{aligned} \dot{x} &= A(t)x + p(t,x,y) \\ \dot{y} &= B(t)y + q(t,x,y) \end{aligned} \tag{4}$$

where the matrices $A(t)$,$B(t)$ are continuous and bounded on $\mathbb{R}$ and the nonlinearities p,q are continuous and of class $C^{\nu}, \nu \geq 2$, in x,y. p,q and their first order derivatives are bounded in t and tend to zero uniformly in t as $\|x\| + \|y\| \to 0$. With respect to the linearized equation we require the following kind of exponential dichotomy. The principal fundamental matrices $\Phi_A(t,s)$, $\Phi_B(t,s)$ of $\dot{x} = A(t)x$, $\dot{y} = B(t)y$, respectively, satisfy estimates of the form

$$\|\Phi_A(t,s)\| \leq K\,e^{-\alpha(t-s)} \text{ for } t \geq s, \quad \|\Phi_B(t,s)\| \leq L\,e^{\beta(t-s)} \text{ for } t \leq s \tag{5}$$

with positive constants K,L,α,β such that $\alpha > \beta$.

Theorem 4: Under the above hypotheses system (4) admits two types of invariant manifolds whose respective representations near the t-axis $(x,y) = (0,0)$ are

(i) $\quad y = g^-(t,x) \quad$ (stable manifold)

(ii) $\quad x = g_o^+(t,y) \quad$ (center-unstable manifold).

The functions g^- and g_o^+ are continuous and of class C^{ν} in x,y, respectively, and $g^-, g_x^-, g_o^+, (g_o^+)_y$ tend to zero uniformly in t as $\|x\| + \|y\| \to 0$. Furthermore the following implications hold true:

(I) If, for some $t_o \in \mathbb{R}$, $\|x(t_o)\| + \|y(t_o)\|$ is sufficiently small and $y(t_o) = g^-(t_o, x(t_o))$, then $y(t) = g^-(t,x(t))$ holds identically for $t \geq t_o$ and the solution $(x(t),y(t))$ tends to $(0,0)$ as $t \to \infty$;

(II) If, for some $t_o \in \mathbb{R}$, a solution $(x(t),y(t))$ of (4) stays sufficiently close to $(0,0)$ for all $t \geq t_o$, then $x(t) = g_o^+(t,y(t))$ holds for all $t \geq t_o$.

One can prove a partial analogue to Theorem 2. Since g^-, g_o^+ are smooth functions the invariance property of the corresponding manifolds is equivalent with the fact that these functions satisfy the following partial differential equations, respectively:

$$\begin{aligned} &\text{(i)} \quad B(t)g^- + q(t,x,g^-) = g_t^- + g_x^- \,(A(t)x + p(t,x,y)) \\ &\text{(ii)} \quad A(t)g_o^+ + p(t,g_o^+,y) = (g_o^+)_t + (g_o^+)_y \,(B(t)y + q(t,g_o^+,y)). \end{aligned} \qquad (6)$$

Whether the Taylor-coefficients of g^- and g_o^+, which are now time-dependent functions, can be determined uniquely from (6) is in general not known. If the right-hand side of (4) is periodic in t one can establish uniqueness along the same line of proof as in the autonomous case.

The extension of the reduction principle to the nonautonomous case poses no serious problems provided one strengthens the inequalities (5) to the following type of dichotomy condition

$$\| \Phi_A(t,s) \| \le K\, e^{-\alpha(t-s)} \text{ for } t \ge s, \quad \| \Phi_B(t,s) \| \le L\, e^{\beta|t-s|} \text{ for } t,s \in \mathbb{R}.$$

<u>Theorem 5</u>: Let the strengthened dichotomy condition be satisfied with K, L, α, β positive and β smaller than some number depending only upon ν, α, K, L. Then the following statements hold true:

(i) If a solution $(x(t), y(t))$ of (4) is, for some $t_o \in \mathbb{R}$, sufficiently close to (0,0), then there exists a solution $\tilde{y}(t)$ of (8) below (depending on $x(t_o), y(t_o)$) such that

$$\| x(t) - g_o^+(\tilde{y}(t)) \| + \| y(t) - \tilde{y}(t) \| \le N\, e^{-\alpha' t} \text{ for } t \ge 0 \qquad (7)$$

with positive constants N and α'. In this relation which holds at least as long as $\| y(t) \|$ or $\| \tilde{y}(t) \|$ remain small enough α' can be chosen as close to α as desired. The "asymptotic phase map" $(x(t_o), y(t_o)) \to \tilde{y}(t_o)$ is continuous and $\nu-1$ times continuously differentiable with respect to the state variable (x,y).

(ii) The trivial solution (0,0) of (4) is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable, unstable) if and only if the trivial solution of the reduced system

$$\dot{y} = B(t)y + q(t, g_o^+(t,y)) \qquad (8)$$

is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable, unstable).

<u>Remarks</u>: 1. Note that statement (i) holds without any assumption con-

cerning stability of the solutions on the center manifold. The constants N,α' in (7) do not depend upon the solutions $(x(t),y(t))$ and $\tilde{y}(t)$.
2. Statement (ii) is a straightforward consequence of statement (i).

3. Application: Invariant manifolds with asymptotic phase

We wish to discuss in this section the main result of [5] which has been proved by an elaborate application of the center manifold theory as developed in the previous sections.
Let M be a compact differentiable manifold which is invariant under the flow of an autonomous differential system

$$\dot{x} = f(x) \ . \tag{9}$$

We deal with the following problem. Consider a solution $x(t)$ of (9) which approaches M as $t \to \infty$. Is it true then that $x(t)$ ultimately behaves like a particular solution $\bar{x}(t)$ on the manifold in the sense that $\| x(t) - \bar{x}(t) \|$ decays to zero? If the answer is yes, we say that M is a manifold with asymptotic phase. Examples of asymptotically stable manifolds with asymptotic phase have been known since long (see the references in [5]). The essence of the following theorem however is that one can infer the asymptotic phase property without assuming the asymptotic stability of the manifold. One can even allow the presence of unstable manifolds along trajectories on the manifold.

In the sequel we denote by $x(t,x_o)$ the solution of (9) with initial value x_o for $t=0$. Furthermore we denote by ζ local coordinates on M, i.e. given $\bar{x}_o \in M$ one can find a map $\zeta \to x_o(\zeta)$ which maps a neighborhood N of $\zeta = 0$ onto a neighborhood of $\bar{x}_o$ on the manifold.

<u>Theorem 6</u>: Hypotheses: (i) For every $\bar{x}_o \in M$ the partial derivatives up to second order of $x(t,x_o(\zeta))$ with respect to ζ are bounded for all $t \in \mathbb{R}$ and $\zeta \in N$. (ii) Given a solution $\bar{x}(t)$ on M. Then the linear variational equation along $\bar{x}(t)$ is kinematically equivalent to a decoupled system

$$\dot{x} = A(t)x \ , \ \dot{y} = B(t)y \ , \ \dot{z} = C(t)z \tag{10}$$

and the fundamental matrices of the respective equations satisfy estimates of the form

$$\begin{aligned} &\| \Phi_A(t,s) \| \le K\, e^{-\alpha(t-s)} \text{ for } t \ge s, \ \| \Phi_B(t,s) \| \le L \text{ for } t,s \in \mathbb{R}, \\ &\| \Phi_C(t,s) \| \le M\, e^{\beta(t-s)} \text{ for } t \le s \end{aligned} \tag{11}$$

where K, L, M, α, β may depend upon the solution $\bar{x}(t)$. (iii) The dimension of y in (10) equals the dimension of M.

Conclusion: If $\lim_{t\to\infty} \operatorname{dist}(x(t,x_o),M) = 0$ then there exists a $\varphi(x_o) \in M$ such that $\lim_{t\to\infty} ((x(t,x_o)) - x(t,\varphi(x_o))) = 0$. Furthermore if the z-equation is not present in (10) regardless how $\bar{x}(t)$ is chosen on M, then M is a stable attractor and $x_o \to \varphi(x_o)$ defines a smooth map of the region of attraction onto M.

Remarks: Hypothesis (i) implies that the solutions on the manifold M depend continuously upon initial data, uniformly for $t \in \mathbb{R}$. (ii) is a strengthened form of what usually is called normal hyperbolicity. It allows, by means of a Lyapunov transformation, to describe the flow in the neighborhood of a given solution on M by a system of differential equations of the form

$$\begin{aligned} \dot{x} &= A(t)x + p(t,x,y,z) \\ \dot{y} &= B(t)y + q(t,x,y,z) \\ \dot{z} &= C(t)z + r(t,x,y,z) \end{aligned} \tag{12}$$

where p,q,r are of order $(\|x\|^2 + \|y\|^2 + \|z\|^2)$ uniformly with respect to t. Under the hypotheses concerning the linearized equation (see (11)) one can infer by a repeated application of Theorem 4 the existence of a center manifold for (12) with properties similar to those stated in Section 1. Hypothesis (iii) finally amounts to the condition, that this center manifold can be identified with the intersection of the given manifold M and a sufficiently small neighborhoodof the solution $\bar{x}(t)$ on M. If M is generated by stationary points or periodic orbits one can, of course, express (iii) in terms of eigenvalues or characteristic multipliers of the variational equation. Special cases of the theorem can be found in Malkin [20], Hale and Stokes [12]. These authors deal with stable manifolds only. The hyperbolic case has been treated in [2],[3],[5] and a forthcoming paper by Hale and Massatt [13].

4. An example: Fisher's population model

The Fisher-Wright-Haldane model is the classical selection model in population genetics (for general references see Crow and Kimura [9], Hadeler [11] and Edwards [10], for the particular problem of this section see [6]). The state of the population at each time is described by the vector $p = (p_1,\ldots,p_n)^T$ of the so-called gene frequences. The evolution of the population is governed by the differential equation

$$\dot{p} = PFp - (p^T Fp)p \tag{13}$$

where P is the diagonal matrix $\mathrm{diag}(p_1,\dots,p_n)$ and F, the so-called viability matrix, is constant and symmetric. The appropriate state space of biological interest is the simplex

$$S = \{\, p\in\mathbb{R} : \sum_{i=1}^{n} p_i = 1,\ p_j \geq 0,\ j=1,\dots,n \,\}$$

which is positively invariant for system (13). The positive definite quadratic form $p^T F p$ is a Lyapunov function; its derivative with respect to (13) vanishes precisely on the set of stationary solutions. Thus the ω-limit set Ω of any solution $p(t)$ in S consists of equilibria. A natural question is then whether Ω is a single point, meaning that $p(t)$ always converges to an equilibrium as $t\to\infty$. This question falls into the category of problems we discussed in Section 3, with M being the set of equilibria of (13) in S. Using Theorem 6 one can then establish the following result. If $p(t)$ is a solution of (13) in S with an ω-limit point p_o in the relative interior of S, then $\lim_{t\to\infty} p(t) = p_o$ (see [6], extensions to solutions with ω-limit points on the relative boundary of S can be found there too). The proof requires a careful analysis of the system (13) in order to make sure that all hypotheses of Theorem 6 are fulfilled. One has to establish the following facts about stationary points p_o of (13) in the relative interior of S:

(1) p_o is embedded in a k-dimensional linear manifold M_k of stationary solutions where k equals the defect of the viability matrix F,
(2) F and the Jacobian $J(p_o)$ of the right-hand side of (13) at p_o have the same defect,
(3) $J(p_o)$ is similar to a symmetric matrix and consequently has only real eigenvalues.

It should be noted that the nonzero eigenvalues of $J(p_o)$ depend on p_o and F. In general, nothing can be said about them from the study of the matrix F. Hence the versions of Theorem 6 which have been stated for the case of a stable manifold M can certainly not be applied to the model equation (13), one needs the case of hyperbolic manifolds.

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