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BIFURCATION IN MATHEMATICAL MODELS OF SOCIAL AND ECONOMIC INTERACTION

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1. Introduction

In the last few years we have seen a growing use of mathematical tools in the social sciences. But mathematics is not a newcomer to the social sciences. The first mortality tables were published by John Graunt in 1662, and the first calculations of life annuities, based on such tables, were carried out by the astronomer Halley in 1693. Cournot's pioneering work in mathematical economics, which provided foundations for the theory of monopoly and imperfect competition, appeared in 1838.

Mathematical models within which a national economy could be described by a closed self-determining system of equations began to appear in the 1870's. The limited class of models initially proposed (by Leon Walras in Switzerland and Vilfredo Pareto in Italy) was substantially broadened in the late 1920's and early 1930's in a series of papers by John von Neumann and Oscar Morgenstern.

Of course, since the quantitative material, motivational, and planning factors which determine the motion of the real economy are enormously varied, the development of quantitatively reliable models are far from perfect. Nevertheless, carefully constructed linear econometric models generally give good forecasts within a forecasting range of about a year. These models are recognized as being highly approximate in nature and not to be used for dètailed and long-range prediction. Moreover, most activities in socioeconomic systems are nonlinear, i. e. the mathematical description makes nonlinear changes more comprehensible and sometimes suggests an unexpected explanation or relationship of the observed phenomena. For example, we regard an agricultural production as the transformation of given inputs (land, seed, fertilizer, tractor and machinery, labour) into one output (wheat). Sometimes, this process can be represented by an input-output matrix, but only where the relationship is linear. In general, production is represented by a nonlinear operator f mapping a vector of inputs $(x_1,x_2,\dots,x_n)$ into a vector of outputs $(y_1,y_2,\dots,y_m)$.

2. Global models

Since the publication of The Limits to Growth [1], the interest in global models has increased rapidly. As is well-known, the book warns our society against a number of crises that might, conceivably, develop within half a century if current trends, such as economic growth and increasing pollution, are going to continue as in the past.

The term "global model" refers to a mathematical model describing in an abstract way those aspects of human society that are considered to be relevant. The first global model, Forrester's [2] has relatively little real-life value; the world model subsequently developed at the Massachusetts Institute of Technology by Meadows et al. [3] is more detailed and appears to be better founded; Kaya [4] constructed a "distributed" world model in which not only the time but also Gross Product per capita was employed as an independent parameter. More recently, global models constructed by teams of the Fundacion Bariloche (The Latin American model [5]) and Mesarovic-Pestel [6] are characterised by a division into regions, each region being described by a separate model.

3. Extrapolative and normative global models

All global models can be divided into two groups - extrapolative and normative - depending on their purpose. The Latin American model is an example of a normative model. It is described by the following set of finite-difference equations:

$$Y(t_{k+1}) = Y(t_k) + t \cdot F[Y(t_k), \alpha, U(t_k)],$$

$$Y(t_0) = Y_0,$$

where Y is the vector of model's state variables, α is the vector of model parameters, U is the vector of control actions.

Of the well-known global models, those of Forrester, Meadows, and Mesarovic-Pestel are extrapolative models. The first two models can be described by the following set of nonlinear differential equations

$$\dot{Y} = F[Y, \alpha, U(Y, \beta)], \quad (1)$$

$$Y(t_0) = Y_0,$$

where Y is the vector of model's state variables, U is the vector function of control actions, α, β are the vectors of model parameters which can be determined internally.

The study of the system in this case consists of sorting out the important policies and analyzing their impact on system behaviour. By policies we mean the different combinations of hypotheses about the functional relationships, the structure of the control mechanism $U(Y, \beta)$, and the numerical values of the parameters α and β. Such an approach makes it possible in the early stages of analysis to improve one's understanding of the dynamics of the processes as well as to identify feasible development trends for the system, because the dynamics of global models (1) can be extraordinarily complex as numerous authors have displayed in numerical simulations.

The model (1) is intended to answer a broad range of policy questions, such as: What are the causes of the 4-year, 8-year and 50-year business cycles? Can a specific government action (or any government action) significantly affect the business cycle? To what extent do a growing population, increasing capital investment, and government efforts to sustain a rising standard of living create inflationary pressures? What effect will wage and price controls have? Where does economic growth lead? What tax policies will achieve desired actions?

4. Bifurcations in global models

The main purpose of our work was to develop a way of describing easily and clearly primary and secondary branching of solutions of (1). The Hopf Bifurcation Theorem was originally proved for bifurcations of singular points of vector fields [9]. The theorem was generalized by Ruelle and Takens [10] to deal with bifurcations of diffeomorphisms about a fixed point or a periodic orbit. However, the theorem of Hopf is a local one and the results of Hopf are valid only on an extremely small neighborhood of a parameter value where degeneracies occur. For this reason it has become clear that the study of various kinds of degenerate Hopf bifurcations was desirable and we shall study those degeneracies which still allow the Lyapunov-Schmidt method [11] or the alternative method [12] to determine a real function $G(x, \alpha)$ whose nontrivial zeros correspond to periodic solutions to (1). Given the function G, one can use [13] the singularity theory methods developed in [14] and [15] to understand the structure of the periodic solutions as α is varied.

An advantage of the singularity theory approach is [13] that it allows one to give - through the notion of codimension - the beginnings of a hierarchy of those degeneracies which are "likely" to occur, and to study with no extra effort the qualitative effects of arbitrary small perturbations of parameters. For example, they help to identify those parameters which can most effectively be used in tuning the model to track recorded behaviour of the economy and to bring the model's behaviour in line with the expectations of accepted theory and knowledgeable intuition.

5. Conclusions

Finally, we must address the practical relevance of our results:

a) First of all, as we have seen in our simulations, economists should be concerned not only with the more obvious codimension-1 problems, but also with the more hidden codimension-2 problems.

b) A system that possesses these three characteristics - rapid growth, environmental limits, and feedback delays - is inherently unstable.

c) Short-term technologies designed to mask the initial signals of impending limits and to promote further growth will not be effective in the long term.

d) Social value changes that reduce the forces causing growth, institutional innovations that raise the rate of technological or social adaptation, and long-term forecasting methods that shorten feedback delays may be very effective in reducing system instability.

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