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SOME GEOMETRIC ASPECTS OF SOBOLEV SPACES

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The extremely rich literature on Sobolev spaces by no means can be considered as covering all the main trends on the theory in recent time. In the current mathematical literature on non-linear p.d.e., especially of variational and differential geometric origin various observations and results appear which reveal some new geometric aspects of the Sobolev theory and which have, so far, not found sufficient description in the newest monographic literature. They are on one hand related with elementary geometric decomposition properties of open subsets of euclidean spaces, à la Whitney type, with some topics of real harmonic analysis and quasiconformal mappings, on the other with geometric topology and homotopy theory. New interesting phenomena have been observed in that context. In the "category" of Sobolev maps, even discontinuous transformations are able to carry some homotopically non trivial information, producing e.g. a well defined action on homotopy and homology groups. What we have in mind is, hopefully, made clearer by the examples described below.

In what follows we shall mainly speak about linear Sobolev spaces $W^{l,p}(\Omega, R^k)$ $(= W^{l,p}(\Omega),\ k=1)$ where l is an integer, $p \geqq 1$, and Ω an open subset of euclidean space R^m, the corresponding local spaces ${}^wW^{l,p}_{loc}$ and the non-linear Sobolev spaces $W^{l,p}(M,N)$ of maps $f : M \to N$, M,N compact Riemannian smooth manifolds, possibly with boundary, $\dim M = m$, $\dim N = n$. For N imbedded in R^k the space

$$W^{l,p}(M,N) = \left\{ f \in W^{l,p}(M,R^k) : f(x) \in N \quad \text{for a.e.} \quad x \in M \right\}$$

is defined as a closed subset of the linear space $W^{l,p}(M,R^k)$.

I. Sobolev imbedding inequalities for domains with irregular boundary

The imbedding $W^{l,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $lp<m$, p^*-the Sobolev conjugate to p, is described by the basic integral inequality

$$(1) \qquad \left(⨍_\Omega |f-P_f|^{p^*}dx\right)^{\frac{1}{p^*}} \leqq C(m,p,\Omega)|\mathrm{diam}\ \Omega|^l \left(⨍_\Omega |\nabla^l f|^p dx\right)^{\frac{1}{p}},$$

$$\left(⨍_\Omega f \equiv \frac{1}{|\Omega|}\int_\Omega\right)$$

where P_f is a polynomial of degree $\leqq (l-1)$, $\nabla^l f$ - the l-th gradient

of f . The constant $C(n,p,\Omega)$, **depends** only on the conformal class **of the** domain Ω . The basic problem studied is to describe possibly broad classes of domains Ω for which (1) holds and to discuss, possibly in some explicit way, the dependence of the constants $C(n,p,\Omega)$ on the geometric properties of the domains. Starting point is the model local inequality for balls (or cubes) $\Omega = B_R$, Q_R - the cube of sidelength R ,

$$(2)\qquad \left(\fint_{Q_R} |f-P_{Q_R}|^{p^*}dx\right)^{\frac{1}{p^*}} \le C(m,p,l)R^l\left(\fint_{Q_R} |\nabla^l f|^p dx\right)^{\frac{1}{p}} , \quad f\in W^{l,p}(Q_R) .$$

An open set $\Omega \subset R^m$ satisfies the chain condition $F(\sigma,B,C)$, $\sigma>1$, $C\ge 1$, $B>1$, if there exist a covering $F=\{Q\}$ of Ω by open cubes Q , with one cube Q_0 fixed, s.t. for every $Q\in F$, there exists a connecting chain $Q_0,Q_1,\dots,Q_N = Q$ of cubes from F satisfying some overlap conditions, see [2] .

The length $N = N(Q)$ of the connecting chain may depend on the particular cube] and can be not bounded for $Q\in F$. Bounded Lipschitz domains and bounded John domains satisfy the chain condition.

Theorem [2] . Let the bounded domain $\Omega\subset R^m$ satisfy the chain condition $F(\sigma,B,C)$. For every $f\in W^{l,p}_{loc}(\Omega)$ the global inequality holds

$$(3)\qquad \left(\int_\Omega |f-P_\Omega f|^{p^*}dx\right)^{\frac{1}{p^*}} \le C(\Omega)\left(\int_\Omega |\nabla^l f|^p dx\right)^{\frac{1}{p}}$$

with the constant $C(\Omega)$ depending on the parameters C,B,σ,p and the dimension m only. Moreover $|P_\Omega(f)(x)| \le \tilde{C}(\Omega)\int_\Omega |f|dx$, $x\in\Omega$ with the constant C not depending on f . In particular this implies that the local condition $f\in W^{l,p}_{loc}(\Omega)$ and the finiteness condition $\int_\Omega |\nabla^l f|^p dx < +\infty$ imply the Sobolev imbedding inequality (1) with the constant $C(n,p,\Omega)$ - depending only on the chain condition parameters. For a fixed open $\omega\subset\Omega$ the polynomial $P_\Omega(f)$ may be chosen such that $|P_\Omega(f)(x)| \le \tilde{C}(\Omega,\omega)\int_\omega |f|dx$. Then in (3) $C = C(\Omega,\omega)$.

The proof of the theorem sketched in [2] uses the local Sobolev imbedding inequality (2) for every cube Q of the covering F , the Markov inequality for the family of polynomials $P_Q(f)$, $Q\in F$ and a version of Hardy-Littlewood maximal function theorems as auxiliary tools. It shows that the local estimates of the form (2) in a domain Ω satisfying the chain condition for all cubes $Q \in F$ may be "integrated" to the global estimate (3).

II. Continuity and approximation

The parameter $\gamma = pl-m$ measures the modulus of continuity of a Sobolev map $f \in W^{1,p}(M,N)$: e.g. for $\gamma > 0$, $l=1$ we have $f \in C^{0,\alpha}(M,N)$ with $\alpha = \frac{\gamma}{p}$. The map f induces then the transformation of the basic algebraic functors of M and N. For $\gamma \leqq 0$ the mapping $f : M \to N$ may admit discontinuities. Nevertheless in some situations the invariants of continuous mappings f can be correctly defined for arbitrary maps in $W^{l,p}(M,N)$, $pl \leqq m$.

The simplest example is the Hopf degree, $\deg f$ of a mapping $f : M \to N$ from the class $W^{1,m}(M,N)$, $(\dim M = m = \dim N)$, M, N oriented with the orientation form ω on N, dx on M. The formula

$$\deg f = \frac{1}{|N|}\int_M J_f dx = \frac{1}{|N|}\int_M f^*\omega \ , \quad (J_f \text{ - the Jacobi determinant})$$

valid for smooth maps, is meaningful for any $f \in W^{1,m}(M,N)$ and defines the integer valued function $\deg : W^{1,m}(M,N) \to Z$. If a sequence $f_l \to f$ in $W^{1,m}(M,N)$ then $\deg f_l \to \deg f$. The degree $\deg f$ cannot be defined for an arbitrary f in $W^{1,p}(M,N)$ if $1 \leqq p < m$. This example illustrates a rather general and delicate fact that for $\gamma \geqq 0$, including the limiting value $\gamma = 0$, the topological behaviour of the Sobolev maps in the class $W^{l,p}(M,N)$ is similar to that of continuous maps. New phenomena will appear only for $\gamma < 0$. This is closely related with the following density result: for $\gamma \geqq 0$ the space $\mathrm{Lip}(M,N)$ of Lipschitzian maps f, or the class of smooth maps $C^\infty(M,N)$ is dense in $W^{l,p}(M,N)$. For $\gamma < 0$ this is no longer the case. Define $H^{l,p}(M,N) = \overline{C^\infty(M,N)}$ in $W^{l,p}(M,N)$. In contrast with the basic fact for the linear Sobolev spaces $W^{l,p}(\Omega)$, in which $\overline{C^\infty(\Omega)}^{W^{l,p}(\Omega)} = W^{l,p}(\Omega)$ it may happen that $H^{l,p}(M,N) \not\subseteq W^{l,p}(M,N)$. The counter example [9], [10] is given by the retraction of the ball B^{n+1} onto the boundary sphere S^n

$$\chi(x) = \frac{x}{|x|}\ , \quad x \in B^{n+1}, \quad (\chi) \in W^{l,p}(B^{n+1}, S^n) \quad \text{for } lp < n+1 .$$

The basic questions here are: Under what conditions on M and N $H^{l,p}(M,N) = W^{l,p}(M,N)$? Describe $H^{l,p}(M,N)$ as a subspace of $W^{l,p}(M,N)$. Partial answers are known.

Theorem [1]. For $1 \leqq p < n$ $W^{1,p}(M,S^n) = H^{1,p}(M,S^n)$. For $n \leqq p < n+1$ every map $f \in W^{1,p}(B^{n+1},S^n)$ can be approximated in the $W^{1,p}$ norm by a sequence f_k, $f_k \to f$, smooth, except at most at a finite number of isolated points.

Theorem [8]. If a map $f \in W^{1,p}(M,N)$, $p > m-1$, may be approximated by smooth maps locally, then $f \in H^{1,p}(M,N)$.

The obstacles to global smooth approximations thus depend on topological actually homotopical properties of the target [1], [3], [13].

III. Homotopy classes and functional properties

The discussion of homotopy properties of the classes $W^{1,p}(M,N)$ as well as the smooth approximation properties is most naturally done by composition with the deformation of the self-mappings of the source manifold M and the target manifold: $f \longrightarrow \varphi \circ f \circ \psi$, $\psi : M \longrightarrow M$ and $\varphi : N \longrightarrow N$.

Theorem [3]. Let M be compact and $p < m$. For any s, $0 < s \leq 1$, any $\varepsilon > 0$ and any **regular**, sufficiently fine triangulation $T = \{\triangle\}$ of M, ($\triangle$-simplices in T) with $\operatorname{diam} \triangle < \eta$, for η small enough, there exists an $f_{s,\varepsilon} \in W^{1,p}(M,M)$ such that a) $\| f_{s,\varepsilon}(x) - x \| < \varepsilon$ for each $x \in M$; b) $\| f_{s,\varepsilon} - I \|_{W^{1,p}(M,M)} \leq Cs$ for some constant $C = C(M,p)$, (I - the identity map); c) $f_{s,\varepsilon}$ is a Sobolev retraction of M onto a tubular neighbourhood $U_{p,s,\varepsilon}$ of the $[p]$ - skeleton $T^{[p]}$ of T. The mapping $f_{1,\varepsilon}$ can be chosen as a Sobolev strong deformation retraction of M onto $T^{[p]}$; d) The singularity set $\sum_f$ of $f_{1,\varepsilon}$ lies on a subpolyhedron of M of dimension $\leq m - [p] - 1$; e) The singularity set $\sum_f$ and the skeleton $T^{[p]}$ are "transverse" in the sense of Borsuk [5], i.e. $T^{[p]}$ is a strong deformation retract of $M \setminus \sum_f$ and $\sum_f$ is a deformation retract of $M \setminus T^{[p]}$.

B. White in his discussion of homotopy classes in Sobolev spaces considers a closure $H_w^{1,p}(M,N)$ of $C^{\infty}(M,N)$ (or $\mathrm{Lip}(M,N)$) in the weak-bounded topology of $W^{1,p}(M,N)$: $f \in H_w^{1,p}(M,N)$ iff there exists a sequence $f_i \in \mathrm{Lip}(M,N)$ such that $\| f - f_i \|_{L^p} \longrightarrow 0$, $\| Df_i \|_{L^p} \leq K(f)$ for some K and all i, $f(x) \in N$ for all $x \in M$.

Theorem [13]. Let d be the greatest integer strictly less then p. Any two sufficiently close in the sense of $H_w^{1,p}(M,N)$ Lipschitz maps have the same d-homotopy type. Each $f \in H_w^{1,p}(M,N)$ has a well-defined d-homotopy type and d-homotopy types are preserved by bounded weak convergence.

We recall that the d-homotopy type of a continuous map from M to N is the homotopy class of its restriction to the d-dimensional skeleton of a triangulation T of M. This implies that maps in $H_w^{1,p}(M,N)$ determine conjugacy class of homomorphisms from $\Pi_d(M)$ to $\Pi_d(N)$. Maps in $W^{1,p}(M,N)$ determine these homomorphisms for $d = [p-1]$.

The composition of Sobolev maps is a delicate matter.

The strongest non trivial result in this direction is

Theorem [7], [10]. Let φ be a map $\varphi : M \to M$, $m = \dim M$. Then the induced map $\varphi^*(f) = f \circ \varphi$, $\varphi^* : W^{1,p}(M,R) \to W^{1,p}(M,R)$ is an isomorphism of Banach spaces iff a) for $p \neq m$, φ - is a homeomorphic quasi-isometry; b) for $p=m$, φ - is quasiconformal [3], [7].

The discussed examples show that the class of Sobolev maps produces a variety of specific geometric effects and phenomena, which arise in several important areas of mathematical research.

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