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In: Jaroslav Kurzweil (ed.): Equadiff 7, Proceedings of the 7th Czechoslovak Conference on Differential Equations and Their Applications held in Prague, 1989. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1990. Teubner-Texte zur Mathematik, Bd. 118. pp. 71--73.

Persistent URL: http://dml.cz/dmlcz/702373

OBLIQUE BOUNDARY VALUE PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS

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Nonlinear elliptic and parabolic equations of the second order represent an important field of general theory of partial differential equations, which has a number of applications.

The investigations in such equations were started in the beginning of the century by S.N.Bernstein. In particular, he made clear the most important role of a priori estimates of solutions for the proofs of existence theorems. Bernstein and, later on, L.Nirenberg obtained also the basic results for nonlinear elliptic equations with two independent variables. The 60-s were marked by a breakthrough in the theory of quasilinear elliptic and parabolic equations with many independent variables (O.Ladyzhenskaya, N.Uraltseva, A.Ivanov, J.Serrin e.a.). General theory of the second order fully nonlinear equations has been developed during last decade and is associated with the names of Krylov, Safonov, Evans, Ivochkina, Lions, Nirenberg, Kaffarelli, Spruck and others. I would like to note that its development was at large extent promoted by the results by Krylov and Evans on the Bellman equations

$$\inf_k(L^k u - f^k) = 0,$$

which appears in theory of controlled stochastic processes. Here L^k are linear elliptic or parabolic operators of the second order.

Originally the Bellman equations were studied exclusively by methods of stochastic theory. Later on, since 1982 methods of the partial differential equations theory have been applied. In this way the results by Krylov and Safonov were essentially used, who obtained estimates of the Hölder norms of the solutions of linear equations with bounded measurable coefficients in terms of the L_q norms of free terms of the equations.

The status of the theory by 1985 is described in the known book by Krylov [1]. As for the boundary value problems, for fully nonlinear equations the solvability of only Dirichlet problem had been investigated by that time.

My talk concerns just the nonlinear problems where the boundary condition is imposed by the first order differential operator. To be more precise, I will discuss oblique derivative problem

$$u_t - F[u] = 0 \quad \text{in } Q, \tag{1}$$

$$b[u] = 0 \quad \text{on } \partial_s Q, \quad u = \phi \quad \text{on } \partial_t Q. \tag{2}$$

Here $Q = \Omega \times (0,T)$, $\partial_s Q = \partial\Omega \times (0,T)$, $\partial_t Q = \Omega \times \{0\}$, Ω is a bounded domain in $\mathbf{R}^n$. We use the standard notations $F[u] = F(u_{xx}, u_x, u, z)$, $b[u] = b(u_x, u, z)$, $z = (x,t)$, $u_x = (u_{x_i})$, $u_{xx} = (u_{x_i x_j})$. We will assume that for any $r \in \mathbf{S}^n$, $p \in \mathbf{R}^n$, $v \in \mathbf{R}$, $z \in Q$ the following conditions hold:

1) F is uniformly elliptic operator,

$$\nu|\xi|^2 \le F_{r_{ij}}(r,p,v,z)\xi^i\xi^j \le \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbf{R}^n.$$

2) $F(r,p,v,z)$ is convex with respect to r, that is $F_{rr} \le 0$.

3) b is oblique operator: $b_p(p,v,z)\cdot\vec{\gamma} > \nu_0$. Here ν, ν_0 are positive constants, $\vec{\gamma}$ is the unit external normal to $\partial\Omega$.

The simplest problem of such form is the Neumann problem, for which

$$F = \frac{da_i(u_x,u,z)}{dx_i} + a(u_x,u,z), \quad b = a_i(u_x,u,z)\gamma_i + \phi(u,z).$$

The existence of the classical solution of this problem was studied by the author in the 60-s. Recently Safonov [5] investigated the $C^{2+\alpha}(\overline{\Omega})$ solvability of oblique boundary value problem for a special class of nonlinear elliptic and parabolic equations including the Bellman equation. Unfortunately, the Safonov's class does not include quasilinear equations. Besides he considered only the case of linear boundary operators b.

More complete results were obtained for elliptic case. By 1984 G.Lieberman [3] investigated the existence of the solution $u \in C^{2+\alpha}(\overline{\Omega})$ for quasilinear elliptic equations of nondivergent form, more precisely for $F = a_{ij}(u_x, u, x)\, u_{x_i x_j} + a(u_x, u, x)$, $b = b(u_x, u, x)$. G.Lieberman and N.Trudinger [4] generalized these results for fully nonlinear elliptic equations under the natural structure conditions on F and b.

Last years new methods were developed by O.Ladyzhenskaya and the author for quasilinear equations [2]. These methods enable us to study the Dirichlet problem for nondivergent quasilinear equations in Sobolev spaces $W^2_q(\Omega)$, $q > n$, under weaker assumptions about smoothness of the data and for some cases to reduce these assumptions just to the optimal ones. The author and K.Tulenbaev [5] have now obtained the same results for elliptic oblique boundary value problems. The corresponding statement more strictly is formulated in Theorem 1.

Theorem 1. Let us suppose that

1°. For all the solutions $u(\cdot, \tau) \in C^{2+\alpha}(\overline{\Omega})$ of the problem

$$\tau[a_{ij}(u_x, u, x)\, u_{x_i x_j} + a(u_x, u, x)] + (1 - \tau)(\Delta u - u) = 0, \quad x \in \Omega,$$

$$\tau b(u_x, u, x) + (1 - \tau)(\frac{\partial u}{\partial \gamma} + u) = \tau \psi(x), \quad x \in \partial\Omega.$$

the uniform estimate $\max_\Omega |u(\cdot, \tau)| \leq M$ holds.

2°. For $x \in \overline{\Omega}$, $|v| < M, p \in \mathbf{R}^n$ the inequalities are satisfied:

$$\nu|\xi|^2 \leq a_{ij}(p, v, x)\xi^i\xi^j \leq \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbf{R}^n,$$

$$|a_{ij,p}|\,(1 + |p|), \quad |a_{ij,v}| \leq \mu,$$

$$|a(p, v, x)| \leq \mu|p|^2 + \Phi(x), \quad |a_{ij,x}| \leq \mu|p| + \Phi(x).$$

3°. For $x \in \partial\Omega$, $|v| \leq M$, $p \in \mathbf{R}^n$ the following inequalities are valid:

$$b_p(p, v, x) \cdot \vec{\gamma} \geq \nu_0; \quad |b_v|, \ |b_{px}| \leq \mu(1 + |p|),$$

$$|b_{vp}| \leq \mu, \quad |b_x| \leq \mu(1 + |p|^2), \quad |b_{pp}| \leq \mu(1 + |p'|)^{-1},$$

where $p' = p - (p \cdot \gamma)\gamma$. Here ν, ν_0, μ are positive constants, $\Phi \in L_q(\Omega), q > n$.

4°. $\partial\Omega \in C^2$, $\psi \in W^{1-1/q}_q(\partial\Omega)$.

5°. There exists a positive function ε, $\varepsilon(R, \rho) \to 0$ as $\rho \to 0$, such that

$$||a(p, v, \cdot) - a(\tilde{p}, \tilde{v}, \cdot)||_{L_q(\Omega)} \leq \varepsilon(R, |p - \tilde{p}| + |v - \tilde{v}|)$$

for any $p, \tilde{p}, v, \tilde{v}$ if $|p| + |v| \leq R$ and $|\tilde{p}| + |\tilde{v}| \leq R$.

Then for any $\tau \in [0, 1]$ there exists a solution $u \in W^2_q(\Omega)$.

The results analogous to Theorem 1 were recently proved by the author for parabolic equations.

Now we formulate the results about the a priori bounds for the solutions of fully nonlinear parabolic equations. Let us consider the solution $u \in C^{4,2}(\overline{Q})$ of (1), (2). Introduce the following notations: $K_M = \{(r, p, v, x, t) : r \in \mathbf{S}^n; p \in \mathbf{R}^n; v \in \mathbf{R}; |v| \leq M; (x, t) \in \overline{Q}\}$, $Y = (p, v, x)$, $z = (x, t)$.

Theorem 2. Let $\partial\Omega \in C^4$, $u \in C^{4,2}(\overline{Q})$ be a solution of (1), (2), $F, b \in C^2(K_M)$, $M \geq \max_Q |u|$. Suppose also that F, b satisfy on K_M the conditions 1)–3) and the following structure conditions:

$$|F|, \ (1 + |p|)|F_p|, \ |F_v|, \ |F_z| \leq \mu(1 + |p|^2 + |r|)$$

$$(1+|r|)|F_{rY}|,\ (1+|r|)|F_{rt}|,\ |F_{YY}|,\ |F_{Yt}| \le \mu_1(|p|)(1+|r|);$$
$$|b(p,v,z)|,\ |b_v|,\ |b_z|,\ |b_{vv}|,\ |b_{vz}|,\ |b_{xx}| \le \mu(1+|p|);$$
$$|b_p|,\ |b_{vp}|,\ |b_{px}|,\ |b_{pt}| \le \mu,$$
$$|b_{pp}| \le \mu(1+|p'|)^{-1},$$

where μ_1 is a continuous function. Then we have an estimate

$$|u|_{2+\alpha,Q} \le C$$

with $\alpha \in (0,1)$ and C depending only on M_0, ν, ν_0, μ, μ_1, $\partial\Omega$, and on the $C^{2+\alpha}(\overline{\Omega})$ norm of ϕ. Here $|\cdot|_{2+\alpha,Q}$ is the norm in the space $C^{2+\alpha}(\overline{Q})$, which is defined by equality

$$|u|_{2+\alpha,Q} = \sup_Q |u| + |u_{xx}|_{\alpha,Q} + |u_t|_{\alpha,Q},$$

where

$$|v|_{\alpha,Q} = \sup_Q |v| + \sup_Q \frac{|v(z)-v(z')|}{|z-z'|^\alpha}.$$

Theorem 2 leads us as usual to the existence theorem for (1), (2).

Theorem 3. Let us suppose that $\phi \in C^{3+\alpha}(\overline{Q})$,

$$b[\phi]=0, \quad \vec{\gamma}\cdot\frac{d}{dx}F[\phi]+F[\phi]=0,$$

$$b_p(\phi_x,\phi,z)\cdot\frac{d}{dx}F[\phi]+b_v(\phi_x,\phi,z)F[\phi]=b_t(\phi_x,\phi,z)=0 \quad \text{on } \partial\Omega\times\{0\}$$

and for any solution $u(\cdot,\tau)$ of the problem

$$u_t - \tau F[u] - (1-\tau)\Delta(u-\phi) = 0 \quad \text{in } Q,$$

$$\tau b[u] + (1-\tau)\left(\frac{\partial u}{\partial n} + u - \frac{\partial\phi}{\partial n} - \phi\right) = 0 \quad \text{on } \partial_x Q,$$
$$u=\phi \quad \text{on } \partial_t Q$$

the estimate $\max|u(\cdot,\tau)| \le M$ holds for any $\tau\in[0,1]$. If Ω, F and b satisfy the conditions of Theorem 2 then for any $\tau\in[0,1]$ there exists a solution $u(\cdot,\tau)\in C^{2+\alpha}(\overline{Q})$ of the above problem.

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