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MOLLIFYING OPERATORS WITH VARIABLE STEP AND THEIR APPLICATION TO APPROXIMATION BY INFINITELY DIFFERENTIABLE FUNCTIONS

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1. Mollification with variable step

The mollification of functions plays an important part in various problems of the modern analysis. The operation of mollification makes it possible to construct a sequence of infinitely differentiable functions converging in a prescribed sense to the given function. By a special choice of the mollifying kernel we can achieve that the sequence possesses some further properties. The properties of mollification are presented in detail in the books of S. L. Sobolev [1], [2]. The mollification can also be applied to other purposes: in order to construct partitions of unity (see, e.g., Sec. 1.3 below), to find integral representations of a function in terms of its derivatives or differences (this problem is dealt with in detail in the book by O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ [3], Chap. II), to extend functions to spaces of larger dimension: the parameter of mollification can be regarded as another variable (such a method of extension, involving the mean functions of V. A. Steklov, was presented by A. A. Dezin [4]).

In the present section we describe the method of mollification with a variable step in the form in which it will be applied in Chapters 2, 3 to the proof of theorems on approximation by infinitely differentiable functions.

In 1.1 we establish a general lemma on the partition of unity similar to that of Whitney [5]. In 1.2 we describe the nonlinear mollifiers with a variable step, which was introduced by Deny and Lions [6], while in 1.3 we describe the linear mollifiers with a variable step, studied by the author [7]. Sec. 1.4 deals with another variant of linear mollification with a variable step, the construction of which makes use of the regularized distance. For open sets of a special form, similar mollifiers were introduced and studied in detail by L. D. Kudryavcev [8], [9], Chap. I.

The main object of our study will be the Sobolev spaces of functions defined on an open set $\Omega \subset E_n$, where E_n is the n-dimensional Euclidean space of points $x = (x_1,\dots,x_n)$.

Throughout the text we shall use the following notation: Ω - an open set in E_n , $C^\infty(\Omega)$ - the family of all functions $f(x) = f(x_1,\dots,x_n)$, infinitely differentiable in Ω ; $C_0^\infty(\Omega)$ - the family of all functions infinitely differentiable in Ω , whose support supp f is compact and included in Ω ; $L^{loc}(\Omega)$ - the family of all functions locally integrable in Ω . Given $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_i \geqq 0$ integers, we denote by $D^\alpha f \equiv \frac{\partial^{\alpha_1+\dots+\alpha_n}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}} f$ the generalized derivative, and by $W_p^\ell(\Omega)$ - the Sobolev space of functions f which have generalized derivatives $D^\alpha f$, $|\alpha| \leqq \ell$, and with a finite norm *)

$$||f||_{W_p^\ell(\Omega)} = \sum_{|\alpha|\leqq\ell} ||D^\alpha f||_{L_p(\Omega)} .$$

Other function spaces will be introduced when necessary.

1.1. Partitions of unity

In this section we introduce a general lemma on the partitions of unity that was proved by the author in [10] and whose particular cases will be used in the present paper. A countable partition of unity with the corresponding estimates for the derivatives was for the first time constructed and employed by Whitney [5].

In what follows we shall use the notations ($X \subset E_n$)

$$X_{(\delta)} = \{x \in X : Q(x,\delta) \subset X\} , \quad X^\delta = \bigcup_{x\in X} Q(x,\delta) ,$$

where

$$\delta = (\delta_1,\dots,\delta_n) , \quad \delta_i > 0 ;$$

$$Q(x,\delta) = \{y : |x_i - y_i| < \delta_i , \quad i = 1,\dots,n\} .$$

If $\delta_i = \delta > 0$, $i = 1,\dots,n$, we replace the rectangle $Q(x,\delta)$ by the ball $B(x,\delta) = \{y : |y - x| < \delta\}$.

Let X be a set in E_n and let a finite or countable system of sets X_n satisfy

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*) As usual, we denote

$$||\phi||_{L_p(\Omega)} = \Big(\int_\Omega |\phi(x)|^p dx\Big)^{1/p} , \quad 1 \leqq p < \infty ,$$

and

$$||\phi||_{L_\infty(\Omega)} = \operatorname{vrai\,sup}_{x\in\Omega} |\phi(x)| .$$

(1) $$X \subset \bigcup_m X_m .$$

Further, let $X'_m \subset X_m$ similarly satisfy

(2) $$X \subset \bigcup_m X'_m ,$$

and let $\rho_m = (\rho_{m1},\dots,\rho_{mn})$, $\rho_{mi} > 0$, be such a vector that

(3) $$X''_m = (X'_m)^{\rho_m} \subset X_m .$$

For $x \in \bigcup_m X_m$ let us denote by $\kappa(x)$ the number of sets X_m which contain the point x , and let us call the number $\kappa = \sup_{x \in \bigcup_m X_m} \kappa(x)$ the multiplicity of the covering $\{X_m\}$.

Further, for $x \in \bigcup_m X_m$ denote by $\tilde{\kappa}(x)$ the least of all positive integers with the following property: there is a neighbourhood U_x such that the number of sets X_m that intersect U_x is equal to $\tilde{\kappa}(x)$. The number $\tilde{\kappa} = \sup_{x \in \bigcup_m X_m} \tilde{\kappa}(x)$ will be called the regular multiplicity of the covering $\{X_m\}$. Notice that always $\kappa(x) \leq \tilde{\kappa}(x)$ and $\kappa \leq \tilde{\kappa}$. On the other hand, it is not difficult to give an example of a covering with $\kappa < \infty$ but $\tilde{\kappa} = \infty$. (Let $n = 1$, $X_0 = (-1, \frac{1}{2})$, $X_m = (2^{-m-1}, 2^{-m+1})$, $m = 1,2,\dots$; then $\kappa = 3$, $\tilde{\kappa} = \infty$ since $\tilde{\kappa}(0) = \infty$.)

Let us notice that $\tilde{\kappa}(x) = \kappa(x)$ for $x \in \bigcup_m X_m$ provided

$$r_x = \inf_{\ell:\, x \notin X_\ell} \operatorname{dist}(x, X_\ell) > 0 ,$$

since in this case the ball $B(x, \frac{r_x}{2})$ intersects only those X_m which contain x . In the opposite case we have $\kappa(x) < \tilde{\kappa}(x)$.

Finally, let $\tilde{\rho}_m = (\tilde{\rho}_{m1},\dots,\tilde{\rho}_{mn})$, where

(4) $$\tilde{\rho}_{mi} = \inf_{\ell:\, X_\ell \cap X_m \neq \emptyset} \rho_{\ell i} > 0 , \quad i = 1,\dots,n .$$

<u>*LEMMA 1.1.*</u> *Let* (4) *be fulfilled for all* m *in question,* $\tilde{\kappa}(\{X''_m\}) < \infty$ *and* $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)$, $0 < \varepsilon_i < 1$, $i = 1,\dots,n$. *There are such nonnegative functions* $\psi_m \in C^\infty(E_n)$ *that*

(i) $\sum_m \psi_m(x) = 1 \;,\quad x \in \bigcup_m X'_m \;;\quad \sum_m \psi_m(x) = 0 \;,\quad x \in {}^c\!\left(\bigcup_m X_m\right) \;;$

(ii) $X'_m \subset \operatorname{supp} \psi_m \subset (X'_m)^{\varepsilon\rho_m} \subset X_m \;;$

(iii) $X \subset \bigcup_m \operatorname{supp} \psi_m$, *and the multiplicity of the covering* $\{\operatorname{supp} \psi_m\}$ *does not exceed* $\kappa = \kappa(\{X''_m\})$;

(iv) *for any vector* $\alpha = (\alpha_1,\dots,\alpha_n)$ *with nonnegative integer components,*

$$|D^\alpha \psi_m(x)| \leq c(\alpha,\varepsilon,\kappa)\tilde{\rho}_m^{-\alpha} \;,$$

holds with $c(\alpha,\varepsilon,\kappa)$ *depending only on* α , ε *and* κ ($\tilde{\rho}_m^{-\alpha} = \tilde{\rho}_{m1}^{-\alpha_1}\dots\tilde{\rho}_{mn}^{-\alpha_n}$) .

REMARK 1. If $X = \bigcup_m X_m = \bigcup_m X'_m$ and if there is a positive integer N such that for any m in question the number of sets X_ℓ that intersect X_m (including X_m itself) does not exceed N , then

$$\kappa(\{X''_m\}) \leq \tilde{\kappa}(\{X''_m\}) \leq N \,.$$

REMARK 2. The constant $c(\alpha,\varepsilon,\kappa)$ in (iv) satisfies the inequality

$$c(\alpha,\varepsilon,\kappa) \leq c_1(\alpha,n)\kappa^{|\alpha|}\varepsilon^{-\alpha} \;.$$

1.2. Nonlinear mollifiers with variable step

Let us consider a function $\omega(x)$, $x \in E_n$ (the kernel of the mollifier) that possesses the following properties:

(1) $$\omega(x) \in C^\infty(E_n) \;,$$

(2) $$\omega(x) = 0 \quad \text{for} \quad |x| \geq 1 \;,$$

(3) $$\int_{E_n} \omega(x)dx = 1 \;.$$

Let a locally integrable function f be given on an open set $\Omega \subset E_n$. Consider the mollifier of the function f with a radius of mollification $\delta > 0$

(4) $$(A_\delta f)(x) = \int_{E_n} f(x - \delta z)\omega(z)dz = \frac{1}{\delta^n}\int_{E_n} \omega\left(\frac{x-y}{\delta}\right)f(y)dy =$$

$$= \left(\omega_{(\delta)} * f\right)(x) ,$$

where $\omega_{(\delta)}(x) = \delta^{-n}\omega(\delta^{-1}x)$. The functions $(A_\delta f)(x)$ given by the identity (4) are defined for $x \in \Omega_\delta$. In order to define $(A_\delta f)(x)$ for arbitrary $x \in \Omega$ (and, in general, for arbitrary $x \in E_n$) we put $f(x)$ equal to zero outside Ω , preserving the original notation. Thus we can write

(5) $$(A_\delta f)(x) = \frac{1}{\delta^n}\int_\Omega \omega\left(\frac{x-y}{\delta}\right) f(y)dy .$$

It is well known (see S. L. Sobolev [1], [2]) that the function $(A_\delta f)(x)$, $x \in E_n$, possesses the following properties: if $f \in L^{loc}(\Omega)$, then

(6) $$(A_\delta f) \in C^\infty(\Omega)$$

(and, in general, $(A_\delta f) \in C^\infty(E_n)$) and for almost all $x \in \Omega$,

(7) $$\lim_{\delta\to 0} (A_\delta f)(x) = f(x) ;$$

if $f \in L_p(\Omega)$, then

(8) $$||A_\delta f||_{L_p(\Omega)} \leq c||f||_{L_p(\Omega)} , \quad 1 \leq p \leq \infty$$

(here $c = \int_{E_n} |\omega(x)|dx$, the condition (3) need not hold for the inequality (8)) and

(9) $$\lim_{\delta\to 0} ||A_\delta f - f||_{L_p(\Omega)} = 0 , \quad 1 \leq p < \infty .$$

Generally speaking, the mollifiers $(A_\delta f)(x)$ do not satisfy the identity $\lim_{\delta\to 0} ||A_\delta f - f||_{W_p^\ell(\Omega)} = 0$. Using them we can only prove that for an arbitrary $\varepsilon > 0$,

(10) $$\lim_{\delta\to 0} ||A_\delta f - f||_{W_p^\ell(\Omega_\varepsilon)} = 0 .$$

This situation is caused by the fact that when introducing these mollifiers, the original function $f(x)$ was extended outside Ω by zero. In the case of the spaces $L_p(\Omega)$, the extension by zero preserves the class; however, this is not the case for the spaces $W_p^\ell(\Omega)$. Obviously, it is possible to extend the function from Ω to E_n in such a way as to preserve the class. However, this possibility occurs only provided the boundary $\Gamma(\Omega)$ of the set Ω possesses a certain degree of smoothness. On the other hand, the very possibility of an arbitrarily accurate approximation of a function $f \in W_p^{\ell}(\Omega)$ by infinitely differentiable functions will be seen to be independent of the properties of the boundary $\Gamma(\Omega)$.

In order to obtain an approximation in the case of an arbitrary open set, it is suitable to employ the following mollifiers, which can be naturally called the mollifiers with a variable step (depending on the point $x \in \Omega$). The construction given below is due to Deny and Lions [6] (see also Meyers, Serrin [11]).

Consider a sequence of bounded open sets $V_1, V_2, \dots$, such that

$$V_1 \subset \overline{V}_1 \subset V_2 \subset \overline{V}_2 \subset \dots , \quad \bigcup_{m=1}^{\infty} V_m = \Omega .$$

We construct a partition of unity (see Lemma 1.1 with $X = \Omega$, $X_m' = V_m \setminus V_{m-1}$, $X_m = V_{m+1} \setminus V_{m-1}$, $\rho_m = \operatorname{dist}(X_m', E_n \setminus X_m)$, $\varepsilon = \frac{1}{2}$)

$$\sum_{m=1}^{\infty} \psi_m(x) = 1 , \quad x \in \Omega ,$$

such that $\psi_m \in C_0^{\infty}(E_n)$, $m = 0,1,2,\dots$ and

$$\operatorname{supp} \psi_m \subset (V_m \setminus V_{m-1})^{\frac{\rho_m}{2}} \subset V_{m+1} \setminus V_{m-2} , \quad m = 1,2,\dots$$

($V_0 = V_1 = \emptyset$). Now let $\overline{\delta} = (\delta_1, \delta_2, \dots)$. Put

$$(11) \qquad (B_{\overline{\delta}} f)(x) = \sum_{m=1}^{\infty} \int_{E_n} \psi_m(x - \delta_m z) f(x - \delta_m z) \omega(z) dz =$$

$$= \sum_{m=1}^{\infty} \left(\omega_{(\delta_m)} * \psi_m f\right)(x) = \sum_{m=1}^{\infty} \left(A_{\delta_m} (\psi_m f)\right)(x) ,$$

where $\omega(z)$ is the above mentioned kernel of the mollifier. The numbers $\delta_m > 0$ are chosen so that $\delta_m < \frac{1}{2} \rho_m$; then for $|z| \leq 1$,

$$(12) \qquad \operatorname{supp} \psi_m(x - \delta_m z) \subset V_{m+1} \setminus V_{m-2} .$$

We assume that $\psi_m(y) f(y) = 0$ for $y \notin \operatorname{supp} \psi_m$, even if $f(y)$ is not defined ($y \notin \Omega$).

By virtue of (12) the functions $(B_{\overline{\delta}} f)(x)$ are defined for $x \in \Omega$. Moreover, (11) actually represents a finite sum: if $x \in \Omega$ and the number $s = s(x)$ is chosen so that $x \in V_s$ but $x \notin V_{s-1}$, then

$$(13) \qquad (B_{\bar\delta} f)(x) = \sum_{m=s-1}^{s+1} \int_{E_n} \psi_m(x - \delta_m z) f(x - \delta_m z) \omega(z) dz .$$

Further, there exists a neighbourhood of the point x such that y belonging to this neighbourhood also satisfies the identity (13). Hence $(B_{\bar\delta} f)(x)$, $x \in \Omega$, are infinitely differentiable functions.

By means of the mollifiers (11) it is possible (see Deny and Lions [6] and also Chap. 2 below) to choose numbers $\bar\delta^{(s)} = (\delta_1^{(s)}, \delta_2^{(s)}, \ldots)$, dependent on f, such that the sequence $\phi_s(x) = (B_{\bar\delta^{(s)}} f)(x)$ has the property

$$(14) \qquad ||f - \phi_s||_{W_p^\ell(\Omega)} \longrightarrow 0 , \quad s \longrightarrow \infty .$$

Generally speaking, the functions $\phi_s(x) \to f$ obtained with the help of the mollifiers (11) depend on f nonlinearly (for s fixed, $\delta_i^{(s)}$ are chosen in dependence on f). Therefore, the mollifiers (11) will be called nonlinear mollifiers with a variable step. (Although the operator $B_{\bar\delta} f$ for fixed $\bar\delta = (\delta_1, \delta_2, \ldots)$ is essentially linear.)

When proving the property (14), we substantially use the fact that the functions $f \in W_p^\ell(\Omega)$ $(1 \leq p < \infty)$ exhibit continuity with respect to translation. In Chap. 2 we will consider general function spaces $Z(\Omega)$ and, using the mollifiers $B_{\bar\delta} f$, we will establish some conditions for the infinitely differentiable functions to be dense in such spaces. It will be shown that, under some apriori assumptions on $Z(\Omega)$, the density of the infinitely differentiable functions is equivalent to the continuity with respect to translation for any function $f \in Z(\Omega)$ with compact support $\operatorname{supp} f \subset \Omega$. It will be also proved that the sequence of infinitely differentiable functions ϕ_s that converges to f in the norm $||\cdot||_{Z(\Omega)}$ can be selected so that the functions $\phi_s(x)$ have the same boundary values as $f(x)$.

In a number of cases it is more suitable to use mollifiers which are close to (11), given by

$$(15) \qquad (C_{\bar\delta} f)(x) = \sum_{m=1}^{\infty} \psi_m(x) \int_{E_n} f(x - \delta_m z) \omega(z) dz =$$

$$= \sum_{m=1}^{\infty} \psi_m(x) \left(\omega_{(\delta_m)} * f\right)(x) = \sum_{m=1}^{\infty} \psi_m(x) \left(A_{\delta_m} f\right)(x) .$$

(For instance, for $f(x) \equiv 1$ and arbitrary $\delta_1, \delta_2, \dots$, $(C_{\bar\delta}f)(x) \equiv 1$.)

Starting from the mollifiers $C_{\bar\delta}f$ and choosing both the partition of unity and the δ_m (in the form $\delta\nu_m$) in a special way, we will construct linear mollifiers with a variable step, which will enable us to amend the results of Chap. 2 for the spaces $W_p^\ell(\Omega)$.

1.3. Linear mollifiers with variable step

We divide the open set $\Omega \subset E_n$ into "layers" in the following manner: denote

$$\rho(x) = \mathrm{dist}(x, {}^c\Omega)$$

and for $a > 1$, $m = 0, \pm 1, \pm 2, \dots$ put

(1) $$\Omega_m = \Omega_{(a^{-m-1})} \setminus \Omega_{(a^{-m})} = \{x \in \Omega,\ a^{-m-1} < \rho(x) \le a^{-m}\} .$$

First we construct a partition of unity.

LEMMA 1.2. There exists such a sequence of nonnegative functions $\psi_m \in C^\infty(E_n)$, $m = 0, \pm 1, \pm 2, \dots,$ *that*

(2) (i) $$\sum_{m=-\infty}^{\infty} \psi_m(x) = 1 , \quad x \in \Omega ; \quad \sum_{m=-\infty}^{\infty} \psi_m(x) = 0 , \quad x \in {}^c\Omega ;$$

(3) (ii) $$\Omega_m \subset \mathrm{supp}\, \psi_m \subset \tilde\Omega_m \equiv \Omega_{m-1} \cup \Omega_m \cup \Omega_{m+1} ;$$

(iii) $\Omega = \bigcup_{m=-\infty}^{\infty} \mathrm{supp}\, \psi_m$ *with the multiplicity of the covering* $\{\mathrm{supp}\, \psi_m\}$ *equal to* 2 ;

(iv) *for an arbitrary vector* $\alpha = (\alpha_1, \dots, \alpha_n)$,

(4) $$|D^\alpha \psi_m(x)| \le c(\alpha, a) a^{m|\alpha|} ,$$

where $c(\alpha,a)$ *depends only on* α *and* a .

Lemma 1.2 follows from Lemma 1.1 with $X = \Omega$, $X'_m = \Omega_m$, $X_m = \tilde\Omega_m$, $\rho_m = \mathrm{dist}(\Omega_m, {}^c\tilde\Omega_m) = a^{-m-2}(a - 1)$, $\varepsilon = \frac{1}{4}$. Moreover, $\mathrm{supp}\, \psi_m \subset (\Omega_m)^{\rho_m/4}$, which implies that the multiplicity of the covering $\{\mathrm{supp}\, \psi_m\}$ equals 2 .

REMARK. In most cases it is sufficient to consider $a = 2$; nonetheless, when proving Theorem 3.1 we have to take into account numbers a arbitrarily close to one.

Using the partition of unity from Lemma 1.2 we construct mollifiers of a function $f \in L^{loc}(\Omega)$ in the following way:

(5) $$(E_\delta f)(x) = \sum_{m=-\infty}^{\infty} \psi_m(x) \int_{E_n} f(x - \delta\nu_m z)\omega(z)dz =$$

$$= \sum_{m=-\infty}^{\infty} \psi_m(x) \left(\omega_{(\delta\nu_m)} * f\right)(x) = \sum_{m=-\infty}^{\infty} \psi_m(x) \left(A_{\delta\nu_m} f\right)(x) ,$$

where the kernel $\omega(x)$ of the mollifier satisfies the conditions (1) - (3) from Sec. 1.2 as well as the condition

(6) $$\int_{E_n} \omega(x)x^k dx = 0 , \quad 0 < |k| < \ell ,$$

$\left(x^k = x_1^{k_1} \dots x_n^{k_n}\right)$. Here ν_m, $m = 0, \pm 1, \pm 2, \dots,$ is a given sequence of positive numbers which satisfy the inequality

(7) $$\nu_m \leqq a^{-m} .$$

Concerning δ we will always assume that

(8) $$0 < \delta \leqq \delta_0 , \quad \delta_0 = a^{-3}(a - 1) .$$

For the above introduced ν_m and δ, we have

(9) $$\delta\nu_m \leqq a^{-m-3}(a - 1) = \operatorname{dist}(\Omega_{m+1}, {}^c\tilde{\Omega}_{m+1}) .$$

Let $x \in \Omega$. Then there is $s = s(x)$ such that $x \in \Omega_s$ ($a^{-s-1} < \rho(x) \leqq a^{-s}$); in virtue of the property (3) of the function ψ_m, for this x the only nonvanishing summands in the sum (5) may be those with $m = s-1, s, s+1$, which means that

(10) $$(E_\delta f)(x) = \sum_{m=s-1}^{s+1} \psi_m(x) \int_{E_n} f(x - \delta\nu_m z)\omega(z)dz ,$$

the identity being valid not only for the point x but also for y belonging to a certain neighbourhood of x.

Thus the values of the mollifier $(E_\delta f)(x)$ are determined by the values of the function $f(y)$ for $y = x - \delta\nu_m z$ ($m = s-1, s, s+1$, $|z| \leqq 1$) satisfying for $0 < \delta < \delta_0$ the inequality

(11) $$|x - y| < \operatorname{dist}(\Omega_s, {}^c\tilde{\Omega}_s)$$

(in virtue of (9)) and

$$(12) \qquad |x - y| < \delta a^2 \rho(x) < \rho(x) ,$$

since $|\delta \nu_m z| \leq \delta a^{-s+1} < \delta a^2 \rho(x) < \rho(x)$.

In particular, this implies that for δ introduced above the mollifiers $(E_\delta f)(x)$ are correctly defined for any $x \in \Omega$.

It follows from (10) that the mollifiers $(E_\delta f)(x)$, $x \in \Omega$, are infinitely differentiable, similarly to those defined by the identity (11) in Sec. 1.2.

The mollifiers (5) possess the same properties (6) - (9) from Sec. 1.2 as the usual mollifiers: see Chap. 3 below (the condition (6) is not used).

Making use of the mollifiers $E_\delta f$ we complete the result of Chap. 2 for the spaces $W_p^\ell(\Omega)$, constructing such a sequence $\phi_s \in C^\infty(\Omega)$, $\phi_s \to f$, $s \to \infty$ (in $W_p^\ell(\Omega)$), that ϕ_s depend linearly on f , ϕ_s do not depend of p , the growth of the derivatives $D^\alpha \phi_s$, $|\alpha| > \ell$, is bounded from above when approaching the boundary.

Changing in (5) the variable according to the rule $x - \delta \nu_m z = y$ we obtain

$$(13) \qquad (E_\delta f) = \int_\Omega K(x,y,\delta) f(y) dy ,$$

where

$$(14) \qquad K(x,y,\delta) = \sum_{m=-\infty}^{\infty} \psi_m(x) (\delta \nu_m)^{-n} \omega\left(\frac{x - y}{\delta \nu_m}\right) .$$

Comparing the formulas (13), (14) with the formula (5) from Sec. 1.2 we see that, analogously to the usual mollifiers, the linear mollifiers represent an integral operator, however, with a more complex kernel $K(x,y,\delta)$ instead of $\delta^{-n} \omega\left(\frac{x - y}{\delta}\right)$.

1.4. Regularized distance and mollifiers with a variable step

We turn once more to the formula (5) from Sec. 1.3. Since the values $(E_\delta f)(x)$ are determined by the values $f(y)$ from the ball $|x - y| < \delta a^2 \rho(x)$, which means that for a fixed x the radius of mollification does not exceed $\delta a^2 \rho(x)$, we may attempt to construct mollifiers with a variable step in a simpler way, namely, putting

(1) $$(H_\delta f)(x) = \int_{E_n} f\big(x - \delta\rho(x)z\big)\omega(z)dz = \big(\delta\rho(x)\big)^{-n} \int_\Omega \omega\Big(\frac{x-y}{\delta\rho(x)}\Big) f(y)dy$$

for $\delta < 1$.

In the case $\Omega = E_n \setminus E_{n-1}$ a similar mollifier with a fixed δ and a certain special kernel ω was studied in detail by L. D. Kudryavcev [8], [9], Chap. 1, being applied to the construction of such extensions of functions from E_{n-1} to E_n , which are the best possible from the point of view of the growth of their derivatives when approaching E_{n-1} .

For $\Omega = E_n \setminus E_{n-1}$ the distance $\rho(x) = x_n$ represents an infinitely differentiable function. However, for a general open set this is generally not the case. Accordingly, the function $(H_\delta f)(x)$ need not be infinitely differentiable.

This drawback can be removed by replacing the distance $\rho(x)$ by the so-called regularized distance $\Delta(x)$ which is an infinitely differentiable function.

LEMMA 1.3. *Let* Ω *be an open set in* E_n *and* $\rho(x)$ *the distance of* $x \in \Omega$ *from the boundary* $\Gamma(\Omega)$. *For an arbitrary* $0 < \varepsilon < 1$ *there is such an infinitely differentiable function* $\Delta(x) \equiv \Delta(x,\Omega,\varepsilon)$ *that for* $x \in \Omega$,

(2) $$(1 - \varepsilon)\rho(x) \leqq \Delta(x) \leqq (1 + \varepsilon)\rho(x)$$

and

(3) $$|D^\alpha \Delta(x)| \leqq c_\alpha \varepsilon^{-|\alpha|} \rho(x)^{1-|\alpha|} ,$$

where c_α *depends only on* α .

This lemma, which is of versatile usefulness, was proved by Calderon and Zygmund [12] with the relations

$$c_1\rho(x) \leqq \Delta(x) \leqq c_2\rho(x) , \quad c_1 = \frac{1}{5} , \quad c_2 = \frac{4}{3}\, 12^n$$

instead of (2). (See also the detailed account in the book by Stein [13], Chap. VI.) The above version of Lemma 1.3 appears in the author's paper [14]. Calderon and Zygmund in their proof used the partition of unity constructed by Whitney [5], while the proof in [14] is based on the partition of unity given in Lemma 1.2, which makes it possible to improve the result obtained in the former paper. The function $\Delta(x)$

has the form $\Delta(x) = (E_\varepsilon \rho)(x)$, where $\nu_m = 2^{-m-2}$.

2. Approximation by infinitely differentiable functions preserving the boundary values for general function spaces

In this chapter we use nonlinear mollifiers with a variable step to deal with the problem of density of infinitely differentiable functions in general function spaces. The major part of results of this chapter are included in the author's paper [16].

2.1. Conditions imposed on the function spaces

Let $\Omega \subset E_n$ be an open set and let a complete normed (or semi-normed) function space $Z(\Omega)$ with a norm $||\cdot||_{Z(\Omega)}$ satisfy the following conditions:

(1) (a) $C_0^\infty(\Omega) \subset Z(\Omega) \subset L^{loc}(\Omega)$.

(b) (Minkowski inequality.) If $A \subset E_m$ is a measurable set and $\phi(x,y)$ a function measurable on $\Omega \times A$, then

(2) $$||\int_A \phi(x,y)dy||_{Z(\Omega)} \leq \int_A ||\phi(x,y)||_{Z(\Omega)} dy .$$

(c) If $f \in Z(\Omega)$ and $\phi \in C_0^\infty(\Omega)$, then $f\phi \in Z(\Omega)$.

Moreover, we will frequently assume that the following conditions are fulfilled as well:

(d) If $f \in Z(\Omega)$ and $\phi \in C_0^\infty(\Omega)$, then $f\phi \in Z(\Omega)$ and

(3) $$||f\phi||_{Z(\Omega)} \leq c_\phi ||f||_{Z(\Omega)}$$

with c_ϕ independent of f .

Further, we shall assume that $||\cdot||_{Z(G)}$ has sense for any open subset $G \subset \Omega$.

(e) (Monotonicity of the norm.) If $f \in Z(\Omega)$ and $G_1 \subset G_2 \subset \subset \Omega$, then

$$||f||_{Z(G_1)} \leq ||f||_{Z(G_2)} .$$

(f) (Additivity of the norm.) If open bounded sets G_1 , G_2

satisfy $G_1 \subset \bar{G}_2 \subset G_2 \subset \bar{G}_2 \subset \Omega$, then for an arbitrary $f \in Z(\Omega)$,

$$||f||_{Z(\Omega)} \leq c_1(||f||_{Z(G_2)} + ||f||_{Z(\Omega \setminus \bar{G}_1)})$$

with c_1 independent of f.

(g) If $f \in Z(\Omega)$, then

$$||f(x+h)||_{Z(\Omega_{|h|})} \leq c_2 ||f(x)||_{Z(\Omega)} \qquad (4)$$

with c_2 independent of f.

Actually we shall employ a weaker condition:

(g′) The inequality (4) is fulfilled for functions $f \in Z_0(\Omega)$ *) with a constant c_2 depending on supp f.

A constant (multiplicative) factor independent of the function may be added on the right-hand sides of the conditions (b) and (e) without affecting the result given below.

The conditions (a) - (g) are fulfilled for the majority of function spaces usually considered in the theory of differentiable functions of several variables.

If $Z(\Omega) = W_p^{\ell}(\Omega)$, then the conditions (a), (e), (f), (g) are obvious, the conditions (c), (d) follow from the Leibniz formula; the Minkowski inequality is also valid in the spaces $W_p^{\ell}(\Omega)$ (see [16]). The conditions (a) - (g) are also fulfilled for the Sobolev weight spaces $W_{p,a(x)}^{\ell}(\Omega)$ with an arbitrary continuous, positive in Ω function $\left(||f||_{W_{p,a(x)}^{\ell}(\Omega)} = \sum_{|\alpha| \leq \ell} ||a(x)D^{\alpha}f(x)||_{L_p(\Omega)}\right)$. In the case of spaces $\hat{W}_p^{\ell}(\Omega)$ with a finite seminorm $||f||_{\hat{W}_p^{\ell}(\Omega)} =$ $= \sum_{|\alpha|=\ell} ||D^{\alpha}f||_{L_p(\Omega)}$ all the conditions are fulfilled except (d).

2.2. Approximation theorems

THEOREM 2.1. If the conditions (a) - (c) *are fulfilled, then a sufficient condition for the set* $C^{\infty}(\Omega)$ *to be dense in* $Z(\Omega)$ *is*

$$\forall f \in Z_0(\Omega) \quad \lim_{h \to 0} ||f(x+h) - f(x)||_{Z(\Omega)} = 0 \qquad (1)$$

---

*) $f \in Z_0(\Omega) \Leftrightarrow f \in Z(\Omega)$ and the support supp f is compact and included in Ω.

(outside Ω *the function* f *is supposed to be equal to zero). Moreover, if also* (d) - (g) *are fulfilled, then the condition* (1) *is necessary as well.*

<u>REMARK.</u> If the conditions (e), (f) are fulfilled, then the condition (1) is equivalent to

$$(2) \qquad \forall f \in Z_0(\Omega) \quad \lim_{h\to 0} ||f(x+h) - f(x)||_{Z(\Omega_{|h|})} = 0 .$$

P r o o f . <u>Necessity</u>: First we prove that C_0^∞ is dense in $Z_0(\Omega)$. Indeed, let $f \in Z_0(\Omega)$, $\psi \in C^\infty(\Omega) \cap Z(\Omega)$ and let G_1 , G_2 be open bounded sets such that $\operatorname{supp} f \subset G_1 \subset \bar{G}_1 \subset G_2 \subset \bar{G}_2 \subset \Omega$. Put $\phi = \psi\chi$, where $\chi \in C_0^\infty(\Omega)$, $\chi(x) = 1$ for $x \in G_1$, $\chi(x) = 0$ for $x \in \Omega \setminus G_2$. Then $\phi \in C_0^\infty(\Omega)$ and, by virtue of the properties (f), (e) and (d),

$$\begin{aligned}||f - \phi||_{Z(\Omega)} &\leqq c_1\big(||f - \phi||_{Z(G_2)} + ||f - \phi||_{Z(\Omega\setminus\bar{G}_1)}\big) = \\ &= c_1\big(||f - \psi||_{Z(G_2)} + ||(f - \psi)\chi||_{Z(\Omega\setminus\bar{G}_1)}\big) \leqq \\ &\leqq c_1\big(||f - \psi||_{Z(\Omega)} + ||(f - \psi)\chi||_{Z(\Omega)}\big) \leqq \\ &\leqq c_1 \, ||f - \psi||_{Z(\Omega)} + c_2||f - \psi||_{Z(\Omega)} = \\ &= c_3||f - \psi||_{Z(\Omega)} .\end{aligned}$$

Let us note that the constant c_3 depends on f (since c_1 and c_2 depend on the sets $\operatorname{supp} f$, G_1 , G_2 , Ω) but not on ψ . Thus the inequality obtained implies our assertion.

Further, for $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}||f(x+h) - f(x)||_{Z(\Omega_{|h|})} &\leqq ||f(x+h) - \phi(x+h)||_{Z(\Omega_{|h|})} + \\ &+ ||\phi(x+h) - \phi(x)||_{Z(\Omega_{|h|})} + ||f(x) - \phi(x)||_{Z(\Omega_{|h|})} .\end{aligned}$$

In virtue of the property (g) we have

$$||f(x+h) - \phi(x+h)||_{Z(\Omega_{|h|})} \leqq c_4||f(x) - \phi(x)||_{Z(\Omega)}$$

with c_4 independent of f and ϕ . If we employ the property (g′) instead of (g), then $c_4 = c_4\big(\operatorname{supp}(f - \phi)\big) = c_4(\bar{G}_2)$, that is, c_4 depends on f but not on ϕ .

Let us continue the function $\phi(x)$ outside of Ω by zero. Then $\phi(x) \in C_0^\infty(E_n)$. For h sufficiently small (such that $\text{supp}\, \phi(x + h) \subset \Omega$) the function $\phi(x + h)$ belongs to $C_0^\infty(\Omega) \subset Z(\Omega)$. Since

$$\phi(x + h) - \phi(x) = \sum_{i=1}^{n} h_i \int_0^1 \frac{\partial \phi}{\partial x_i}(x + th)dt ,$$

we have

$$||\phi(x + h) - \phi(x)||_{Z(\Omega_{|h|})} \leq ||\phi(x + h) - \phi(x)||_{Z(\Omega)} \leq$$

$$\leq |h| \sum_{i=1}^{n} \int_0^1 ||\frac{\partial \phi}{\partial x_i}(x + th)||_{Z(\Omega)} dt$$

by the properties (e) and (b).

Let the open bounded sets G_1, G_2 satisfy $\text{supp}\, \phi \subset G_1 \subset \overline{G}_1 \subset \subset G_2 \subset \overline{G}_2 \subset \Omega$. We choose h so small that $\text{supp}\, \phi(x + th) \subset G_1$ for $0 \leq t \leq 1$ and that $G_2 \subset \Omega_{|h|}$. The function $\frac{\partial \phi}{\partial x_i}(x + th) \in \in C_0^\infty(\Omega) \subset Z(\Omega)$ and the properties (f) and (g) (or (g')) yield

$$||\frac{\partial \phi}{\partial x_i}(x + th)||_{Z(\Omega)} \leq c_5 \Big(||\frac{\partial \phi}{\partial x_i}(x + th)||_{Z(G_2)} +$$

$$+ ||\frac{\partial \phi}{\partial x_i}(x + th)||_{Z(\Omega \setminus \overline{G}_1)} \Big) \leq$$

$$\leq c_5 ||\frac{\partial \phi}{\partial x_i}(x + th)||_{Z(\Omega_{|th|})} \leq c_6 ||\frac{\partial \phi}{\partial x_i}(x)||_{Z(\Omega)} .$$

Notice that the constant c_6 depends on ϕ (since it depends on the sets $\text{supp}\, \phi$, G_1, G_2, Ω) but not on h.

Hence

$$||f(x + h) - f(x)||_{Z(\Omega_{|h|})} \leq (1 + c_4)||f(x) - \phi(x)||_{Z(\Omega)} +$$

$$+ c_6 |h| \sum_{i=1}^{n} ||\frac{\partial \phi}{\partial x_i}||_{Z(\Omega)} .$$

Given $\varepsilon > 0$, we first choose $\phi \in C_0^\infty(\Omega)$ such that the first summand on the right-hand side of this inequality is less than $\frac{1}{2}\varepsilon$, and then for this ϕ we take δ so small that $c_6 \delta \sum_{i=1}^{n} ||\frac{\partial \phi}{\partial x_i}||_{Z(\Omega)} < \frac{1}{2}\varepsilon$. Then for $|h| < \delta$ the inequality $||f(x + h) - f(x)||_{Z(\Omega_{|h|})} < \varepsilon$ holds, which implies (2) and hence also (1) (see Remark), thus completing the proof.

<u>Sufficiency</u>: Let us consider the mollifiers

$$(B_{\bar{\delta}}f)(x) = \sum_{m=1}^{\infty} \int_{E_n} \psi_m(x - \delta_m z)f(x - \delta_m z)\omega(z)dz .$$

Since $\sum_{m=1}^{\infty} \psi_m(x) = 1$, $x \in \Omega$; $\int_{E_n} \omega(z)dz = 1$; $\omega(z) = 0$, $|z| \geq 1$,

we have

(3) $$(B_{\bar{\delta}}f)(x) - f(x) =$$

$$= \sum_{m=1}^{\infty} \int_{|z|\leq 1} [\psi_m(x - \delta_m z)f(x - \delta_m z) - \psi_m(x)f(x)]\omega(z)dz .$$

According to the Minkowski inequality (2) from Sec. 2.1 we find

$$||(B_{\bar{\delta}}f)(x) - f(x)||_{Z(\Omega)} \leq$$

$$\leq \sum_{m=1}^{\infty} \left|\left| \int_{|z|\leq 1} [\psi_m(x - \delta_m z)f(x - \delta_m z) - \psi_m(x)f(x)]\omega(z)dz \right|\right|_{Z(\Omega)} \leq$$

$$\leq \sum_{m=1}^{\infty} \int_{|z|\leq 1} ||\psi_m(x - \delta_m z)f(x - \delta_m z) - \psi_m(x)f(x)||_{Z(\Omega)} |\omega(z)|dz .$$

By virtue of the property (c) the function $f_m(x) = \psi_m(x)f(x)$ with compact support $\operatorname{supp} f \subset \Omega$ belongs to $Z(\Omega)$. Now for the given $\varepsilon > 0$ and fixed m we find (making use of the property (1)) δ_m so small that for $|z| \leq 1$,

(4) $$||f_m(x - \delta_m z) - f_m(x)||_{Z(\Omega)} < 2^{-m}\left(\int_{|y|\leq 1} |\omega(y)|dy \right)^{-1} \varepsilon$$

holds. Hence

(5) $$||B_{\bar{\delta}}f - f||_{Z(\Omega)} < \sum_{m=1}^{\infty} 2^{-m}\varepsilon = \varepsilon ,$$

which was to be proved.

<u>*COROLLARY.*</u> *Infinitely differentiable functions are dense in the spaces* $W_p^{\ell}(\Omega)$, $S_p^r(\Omega)$, $W_{p,a(x)}^{\ell}(\Omega)$, $\hat{W}_p^{\ell}(\Omega)$ *with* $1 \leq p < \infty$, *in the space* $W_{p_1,\ldots,p_n}^{\ell_1,\ldots,\ell_n}(\Omega)$ *with* $1 \leq p_i < \infty$, *in the spaces*

$B_{p_1,\ldots,p_n,\theta_1,\ldots,\theta_n}^{r_1,\ldots,r_n}(\Omega)$, $b_{p_1,\ldots,p_n,\theta_1,\ldots,\theta_n}^{r_1,\ldots,r_n}(\Omega)$ *with* $1 \leq p_i < \infty$, $1 \leq \theta_i < \infty$.

For the definitions and properties of the above spaces we refer the reader to the books [15], [3] and [17].

Theorem 2.1 asserts that, if the space $Z(\Omega)$ satisfies the conditions (a) - (c) and the condition (1), then there is a sequence $\phi_s \in C^{\infty}(\Omega)$ such that

(6) $$||f - \phi_s||_{Z(\Omega)} \to 0 , \quad s \to \infty .$$

Now we shall show that ϕ_s can be chosen in such a way that these functions have in a certain sense the same boundary values as f .

THEOREM 2.2. Let Ω *be an open set in* E_n *and let a complete normed (or seminormed) space* $Z(\Omega)$ *satisfy the conditions* (a) - (d) *and* (1). *For any functions* $\mu(x) \in C^{\infty}(\Omega)$ *and* $f \in Z(\Omega)$ *there is a sequence* $\phi_s(x) \in C^{\infty}(\Omega)$ *satisfying* (6) *and*

(7) $$||(f - \phi_s)\mu||_{Z(\Omega)} \to 0 , \quad s \to \infty .$$

P r o o f . We will consider the same mollifiers $B_{\bar{\delta}}f$ as in the proof of Theorem 2.1 but, given $\varepsilon > 0$, we will choose δ_m in a little different way. In virtue of (3), we have

$$[(B_{\bar{\delta}}f)(x) - f(x)]\mu(x) =$$

$$= \sum_{m=1}^{\infty} \mu(x) \int_{|z|\leq 1} [\psi_m(x - \delta_m z)f(x - \delta_m z) - \psi_m(x)f(x)]\omega(z)dz$$

and

$$||(B_{\bar{\delta}}f - f)\mu||_{Z(\Omega)} \leq \sum_{m=1}^{\infty} ||\mu(x)F_m(x)||_{Z(\Omega)} ,$$

where the support of the infinitely differentiable function

$$F_m(x) = \int_{|z|\leq 1} [\psi_m(x - \delta_m z)f(x - \delta_m z) - \psi_m(x)f(x)]\omega(z)dz$$

belongs to $V_{m+1} \setminus V_{m-2}$ in virtue of the condition (12) from Sec.1.2. Denote by $\chi_m(x)$ a function from $C_0^{\infty}(\Omega)$ equal to 1 on $V_{m+1} \setminus V_{m-2}$. Then

$$\mu(x)F_m(x) = \mu(x)\chi_m(x)F_m(x) \equiv \mu_m(x)F_m(x) ,$$

where $\mu_m(x) \in C_0^{\infty}(\Omega)$, $F_m(x) \in Z(\Omega)$ (since $F_m(x) \in C_0^{\infty}(\Omega)$). Consequently,

$$||\mu F_m||_{Z(\Omega)} = ||\mu_m F_m||_{Z(\Omega)} \leq c_m||F_m||_{Z(\Omega)}$$

in virtue of the property (d) and without loss of generality we may assume that $c_m \geq 1$.

Hence

$$(8) \qquad ||(B_{\bar\delta}f - f)\mu||_{Z(\Omega)} \leq$$

$$\leq \sum_{m=1}^{\infty} c_m \int_{|z|\leq 1} ||\psi_m(x - \delta_m z)f(x - \delta_m z) - \psi_m(x)f(x)||_{Z(\Omega)} |\omega(z)|dz \ .$$

Now we choose δ_m so small as to guarantee the validity of an inequality stronger than (4), namely,

$$(9) \qquad ||f_m(x - \delta_m z) - f_m(x)||_{Z(\Omega)} < 2^{-m}c_m^{-1}\Big(\int_{|y|\leq 1} |\omega(y)|dy\Big)^{-1} \varepsilon$$

for $|z| \leq 1$. Then in addition to (5) we obtain from (8) and (9) that

$$||(B_{\bar\delta}f - f)\mu||_{Z(\Omega)} < \sum_{m=1}^{\infty} 2^{-m}\varepsilon = \varepsilon \ ,$$

which was to be proved.

Let us specify Theorem 2.2 for the case $Z(\Omega) = W_p^{\ell}(\Omega)$.

THEOREM 2.3. For any continuous, positive in Ω *function* $\mu(x)$ *and any function* $f \in W_p^{\ell}(\Omega)$, $1 \leq p < \infty$, *there is a sequence* $\phi_s(x) \in C^{\infty}(\Omega)$ *such that*

$$(10) \qquad ||f - \phi_s||_{W_p^{\ell}(\Omega)} \to 0 \ , \quad s \to \infty$$

and

$$(11) \qquad ||(D^{\alpha}f - D^{\alpha}\phi_s)\mu(x)||_{L_p(\Omega)} \to 0 \ , \quad s \to \infty \ , \quad |\alpha| \leq \ell \ .$$

P r o o f . Given a function μ , we consider such an infinitely differentiable function $\tilde\mu$ that $|D^{\alpha}\mu(x)| \leq \tilde\mu(x)$, $|\alpha| \leq \ell$. It follows from the proof of Theorem 2.2 that there are such $\phi_s(x) \in C^{\infty}(\Omega)$ that (10) is fulfilled and

$$(12) \qquad \sum_{|\alpha|\leq \ell} ||D^{\alpha}[(f - \phi_s)D^{\alpha}\tilde\mu]||_{L_p(\Omega)} \to 0 \ , \quad s \to \infty \ , \quad |\gamma| \leq \ell$$

holds. To this end we have to take $D^{\gamma}\tilde\mu$, $|\gamma| \leq \ell$, instead of μ in the proof of Theorem 2.2, establish the inequality (8) with $c_m^{(\gamma)}$ instead of c_m and choose δ_m so as to guarantee the validity of (9) with $\max_{|\gamma|\leq \ell} c_m^{(\gamma)}$ instead of c_m . Then the inequality

$$||(B_{\bar\delta}f - f)D^{\gamma}\tilde\mu||_{W_p^{\ell}(\Omega)} < \varepsilon \quad \text{holds for} \quad |\gamma| \leq \ell \ .$$

Now we shall prove

(13) $$||(D^\alpha f - D^\alpha \phi_s)D^\gamma \tilde{\mu}||_{L_p(\Omega)} \to 0 \ , \quad s \to \infty \ , \quad |\alpha| \leq \ell \ , \quad |\gamma| \leq \ell - |\alpha| \ .$$

For $\alpha = 0$ this is a consequence of (12). Proceeding by induction, we assume that (13) holds for $|\alpha| \leq k$, and consider α with $|\alpha| = k + 1$. The Leibniz formula together with the Minkowski inequality yields

(14) $$||(D^\alpha f - D^\alpha \phi_s)D^\gamma \tilde{\mu}||_{L_p(\Omega)} \leq ||D^\alpha[(f - \phi_s)D^\gamma \tilde{\mu}]||_{L_p(\Omega)} + \sum_{\substack{0 \leq \beta \leq \alpha \\ |\beta| > 0}} \frac{\alpha!}{\beta!(\alpha - \beta)!} ||D^{\alpha-\beta}(f - \phi_s)D^{\beta+\gamma} \tilde{\mu}||_{L_p(\Omega)} \ .$$

Now (13) for $|\alpha| = k + 1$ follows from (14), (12) and the induction hypothesis (in the second term we have $|\alpha - \beta| \leq k$, $|\beta + \gamma| \leq \ell - |\alpha - \beta|$). This completes the proof.

Notice that, generally speaking, the sequence ϕ_s depends non-linearly on f and moreover, depends on the parameters of the space W_p^ℓ : on p and ℓ .

Since the rate of growth of the weight $\mu(x)$ may be arbitrary when μ approaches the boundary $\Gamma(\Omega)$, the condition (11) expresses the fact that, when approaching the boundary, the behavior of the functions $D^\alpha f$ and $D^\alpha \phi_s$ differs only little in the above described sense.

3. Approximation by infinitely differentiable functions in Sobolev spaces

In this chapter we prove a theorem that in several points completes Theorem 2.3. We shall construct such a sequence of functions $\phi_s(x) \in C^\infty(\Omega)$, $\phi_s \to f$ in $W_p^\ell(\Omega)$ that ϕ_s linearly depends on f , ϕ_s is independent of p and the growth of the derivatives $D^\alpha \phi_s$, $|\alpha| > \ell$, when approaching the boundary is in a certain sense bounded from above. We will consider weight functions $\mu(x) = \lambda(x)^{|\alpha|-\ell}$, depending on the order of the derivatives involved. The case $\lambda(x) = \rho(x)$, where $\rho(x)$ is the distance from $x \in \Omega$ to the boundary $\Gamma(\Omega)$, was treated and the result given in this section obtained in the author's paper [6].

3.1. Auxiliary inequalities

We shall use the following two simple but useful inequalities.

LEMMA 3.1. If for every $x \in X$ *(X being a measurable set in* E_n *) the finite or countable sum* $\sum_i a_i(x)$ *consists of at most* N *nonzero summands, then*

(1) $$\|\sum_i a_i(x)\|_{L_p(X)} \leq N^{1-1/p}\Big(\sum_i \|a_i(x)\|^p_{L_p(X)}\Big)^{1/p}, \quad 1 \leq p \leq \infty .$$

LEMMA 3.2. If X *is either a finite or a countable union of measurable sets:* $X = \bigcup_i X_i$ *, the multiplicity of the covering* $\{X_i\}$ *being equal to* κ *, then*

(2) $$\Big(\sum_i \|f\|^p_{L_p(X_i)}\Big)^{1/p} \leq \kappa^{1/p}\|f\|_{L_p(X)}, \quad 1 \leq p \leq \infty .$$

3.2. Properties of linear mollifiers with a variable step

We shall show that the linear mollifiers with a variable step

(1) $$(E_\delta f)(x) = \sum_{m=-\infty}^{\infty} \psi_m(x) \int_{E_n} f(x - \delta\nu_m z)\omega(z)dz \equiv \sum_{m=-\infty}^{\infty} \psi_m(x) f_m(x) ,$$

where

(2) $$f_m(x) = \int_{E_n} f(x - \delta\nu_m z)\omega(z)dz = (A_{\delta\nu_m} f)(x) ,$$

which were introduced in Sec. 1.3, possess the same properties (6) - (9) from Sec. 1.2 as the usual mollifiers. Let us recall that

(3) $$(E_\delta f)(x) = \sum_{m=s-1}^{s+1} \psi_m(x) f_m(x) ,$$

where $s = s(x)$ is determined from the condition $a^{-s-1} < \rho(x) \leq a^{-s}$, the identity being valid even for the points y belonging to a certain neighbourhood of the point x .

In the present section we assume that $\omega(x)$ satisfies only the conditions (1) - (3) from Sec. 1.2.

LEMMA 3.3. If $f \in L^{loc}(\Omega)$ *, then*

$$E_\delta f \in C^\infty(\Omega)$$

for $0 < \delta \leq \delta_0$ $(\delta_0 = a^{-3}(a - 1))$, *and*

(4) $$D^{\alpha}(E_\delta f)(x) = \sum_{0\leq\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \sum_{m=-\infty}^{\infty} D^{\alpha-\beta} \psi_m(x) D^{\beta} f_m(x) \equiv$$

$$\equiv \sum_{0\leq\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} J^{(\delta)}_{\alpha\beta}(x)$$

holds with

(5) $$J^{(\delta)}_{\alpha\beta}(x) = \sum_{m=-\infty}^{\infty} D^{\alpha-\beta} \psi_m(x) D^{\beta} f_m(x) .$$

LEMMA 3.4. *If* $f \in L^{loc}(\Omega)$ *, then for almost all* $x \in \Omega$ *,*

$$\lim_{\delta\to 0} (E_\delta f)(x) = f(x) .$$

Lemmas 3.3 and 3.4 follow from the formula (2) and from the respective properties of the usual mollifiers.

LEMMA 3.5. *If* $f \in L_p(\Omega)$, $1 \leq p \leq \infty$ *, then*

(6) $$||E_\delta f||_{L_p(\Omega)} \leq c||f||_{L_p(\Omega)}$$

holds for $0 < \delta \leq \delta_0$ *with* $c = 5\int_{E_n} |\omega(x)|dx$.

P r o o f . According to Lemma 1.3 the multiplicity of the covering $\{\text{supp } \psi_m\}$ equals 2 , hence

$$||E_\delta f||_{L_p(\Omega)} \leq 2^{1-\frac{1}{p}}\Big(\sum_{m=-\infty}^{\infty} \int_\Omega |\psi_m(x)\int_{E_n} f(x - \delta\nu_m z)\omega(z)dz|^p dx\Big)^{1/p}$$

in virtue of Lemma 3.1.

Taking into account that $\text{supp } \psi_m \subset \tilde{\Omega}_m \equiv \Omega_{m-1} \cup \Omega_m \cup \Omega_{m+1}$, we obtain by applying the Minkowski inequality

(7) $$||E_\delta f||_{L_p(\Omega)} \leq$$

$$\leq 2^{1-\frac{1}{p}}\left[\sum_{m=-\infty}^{\infty} \int_{\tilde{\Omega}_m}\left| \int_{|z|\leq 1} f(x - \delta\nu_m z)\omega(z)dz\right|^p dx\right]^{1/p} \leq$$

$$\leq 2^{1-\frac{1}{p}}\left[\sum_{m=-\infty}^{\infty} \Big(\int_{|z|\leq 1} |\omega(z)|\ ||f(x - \delta\nu_m z)||_{L_p(\tilde{\Omega}_m)} dz\Big)^p\right]^{1/p} \leq$$

$$\leq 2^{1-\frac{1}{p}}\left[\sum_{m=-\infty}^{\infty}\left(\int_{|z|\leq 1}|\omega(z)|\ ||f(y)||_{L_p[(\tilde{\Omega}_m)^{\delta\nu_m}]}dz\right)^p\right]^{1/p} =$$

$$= 2^{1-\frac{1}{p}}\int_{|z|\leq 1}|\omega(z)|dz\left(\sum_{m=-\infty}^{\infty}||f||^p_{L_p[(\tilde{\Omega}_m)^{\delta\nu_m}]}\right)^{1/p} .$$

Since $\nu_m \leq a^{-m}$ and $0 < \delta \leq \delta_0$, we have

$$(8) \quad (\Omega_m)^{\delta\nu_m} \subset \left(\Omega_{(a^{-m-2})} \setminus \Omega_{(a^{-m+1})}\right)^{a^{-m-3}(a-1)} =$$

$$= \Omega_{(a^{-m-2} - a^{-m-3}(a-1))} \setminus \Omega_{(a^{-m+1} + a^{-m-3}(a-1))} \subset$$

$$\subset \Omega_{(a^{-m-3})} \setminus \Omega_{(a^{-m+2})} = \bigcup_{s=m-2}^{m+2} \Omega_s \equiv \tilde{\Omega}_m .$$

As the multiplicity of the covering $\{\tilde{\Omega}_m\}$ equals 5 , we conclude by Lemma 3.2 that

$$(9) \quad ||E_\delta f||_{L_p(\Omega)} \leq 2^{1-\frac{1}{p}}\int_{|z|\leq 1}|\omega(z)|dz\left(\sum_{m=-\infty}^{\infty}||f||^p_{L_p(\tilde{\Omega})}\right)^{1/p} \leq$$

$$\leq 2\left(\tfrac{5}{2}\right)^{1/p}\int_{|z|\leq 1}|\omega(z)|dz\,||f||_{L_p(\Omega)} \leq$$

$$\leq 5\int_{|z|\leq 1}|\omega(z)|dz\,||f||_{L_p(\Omega)} ,$$

which completes the proof.

<u>*COROLLARY.*</u> *Under the assumptions of Lemma* 3.5 *we have*

$$(10) \quad r_s = ||\sum_{|m|>s}\psi_m(x)f_m(x)||_{L_p(\Omega)} \leq$$

$$\leq c\left(||f||_{L_p(\Omega_{(a^{s-2})})} + ||f||_{L_p(\Omega\setminus\Omega_{(a^{-s+1})})}\right) .$$

P r o o f . By the same argument as above we obtain instead of (9) the inequality

$$r_s \leq 2^{1-\frac{1}{p}}\int_{|z|\leq 1}|\omega(z)|dz\left(\sum_{|m|>s}||f||^p_{L_p(\tilde{\Omega}_m)}\right)^{1/p} ,$$

which implies (10) in virtue of

$$\bigcup_{m=s+1}^{\infty} \tilde{\Omega}_m \subset \Omega \setminus \Omega_{(a^{-s+1})} \ , \quad \bigcup_{m=-\infty}^{-s+1} \tilde{\Omega}_m \subset \Omega_{(a^{s-2})} \ .$$

<u>*LEMMA 3.6.*</u> *If* $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, *then*

$$\lim_{\delta\to 0} ||E_\delta f - f||_{L_p(\Omega)} = 0 \ .$$

P r o o f . Consider the difference

$$(11) \qquad (E_\delta f)(x) - f(x) =$$

$$= \sum_{m=-\infty}^{\infty} \psi_m(x) \int_{E_n} [f(x - \delta\nu_m z) - f(x)]\omega(z)dz \equiv \sum_{m=-\infty}^{\infty} a_m(x,\delta) \ .$$

We shall prove that the series (11) converges uniformly with respect to δ in the norm $||\cdot||_{L_p(\Omega)}$. Indeed, taking into account

$$\Big| \sum_{|m|>s} a_m(x,\delta) \Big| \leq$$

$$\leq \Big| \sum_{|m|>s} \psi_m(x) \int_{E_n} f(x - \delta\nu_m z)\omega(z)dz \Big| + |f(x)| \sum_{|m|>s} \psi_m(x)$$

as well as the inequality (10), we find

$$|| \sum_{|m|>s} a_m(x,\delta) ||_{L_p(\Omega)} \leq r_s + ||f||_{L_p(\bigcup_{|m|>s} \mathrm{supp}\ \psi_m)} \leq$$

$$\leq (c + 1)\big(||f||_{L_p(\Omega_{(a^{s-2})})} + ||f||_{L_p(\Omega \setminus \Omega_{(a^{-s+1})})} \big) ,$$

which implies the uniform convergence.

Since with $\delta \to 0$ each summand $a_m(x,\delta) \to 0$ in $L_p(\Omega)$ (see the property (9) from Sec. 1.2), we have (the limit in $L_p(\Omega)$)

$$\lim_{\delta\to 0} [(E_\delta f)(x) - f(x)] = \sum_{m=-\infty}^{\infty} \lim_{\delta\to 0} a_m(x,\delta) = 0 \ .$$

<u>3.3. Approximation theorem for Sobolev spaces</u>

In this section we shall assume that the mollifier kernel $\omega(x)$ possesses in addition to the properties (1) - (3) from Sec. 1.2 also the property

$$(1) \qquad \int_{E_n} \omega(x)x^k dx = 0 \ , \quad 0 < |k| < \ell \ ,$$

where $x^k = x_1^{k_1} \dots x_n^{k_n}$.

The following lemma concerning the "usual" mollifiers $A_\delta f$ will be useful in the sequel. It essentially employs the condition (1). Denote

$$||f||_{\hat{W}_p^\ell(\Omega)} = \sum_{|\alpha|=\ell} ||D^\alpha f||_{L_p(\Omega)} .$$

LEMMA 3.7. Let Ω *be an open set in* E_n , $1 \leq p \leq \infty$, $f \in W_p^\ell(\Omega)$, $G \subset \Omega$ *a measurable set and* $\overline{G}^\delta \subset \Omega$. *Then*

$$(2) \qquad ||D^\beta(A_\delta f) - D^\beta f||_{L_p(G)} \leq c_1 \delta^{\ell-|\beta|} ||f||_{\hat{W}_p^\ell(G^\delta)}$$

for $|\beta| < \ell$ *and*

$$(3) \qquad ||D^\beta(A_\delta f)||_{L_p(G)} \leq c_2 \delta^{\ell-|\beta|} ||f||_{\hat{W}_p^\ell(G^\delta)}$$

for $|\beta| \geq \ell$ *with* c_1 *and* c_2 *independent of* f *and* δ .

P r o o f . First we prove the inequality (3). To this aim we shall differentiate inside the integral for $x \in G$ (since $\overline{G}^\delta \subset \Omega$, the function $\tilde{f}(y) = f(y - \delta z)$ has a generalized derivative $D^\gamma \tilde{f}(y) = D^\gamma f(y - \delta z)$ in an open ball with its center at the point x):

$$D^\beta(A_\delta f)(x) = D^\beta \int_{|z|\leq 1} f(x - \delta z)\omega(z)dz =$$

$$= D^{\beta-\gamma} \int_{|z|\leq 1} (D^\gamma f)(x - \delta z)\omega(z)dz =$$

$$= D^{\beta-\gamma}\Big(\delta^{-n} \int_{|x-y|\leq\delta} \omega\Big(\frac{x-y}{\delta}\Big) D^\gamma f(y)dy\Big) =$$

$$= \delta^{-|\beta|+|\gamma|}\Big(\delta^{-n} \int_{|x-y|\leq\delta} \big(D^{\beta-\gamma}\omega\big)\Big(\frac{x-y}{\delta}\Big) D^\gamma f(y)dy\Big) =$$

$$= \delta^{\ell-|\beta|} \int_{|z|\leq 1} (D^\gamma f)(x - \delta z) D^{\beta-\gamma}\omega(z)dz .$$

(Here the vector γ satisfies the conditions $\gamma \leq \beta$, $|\gamma| = \ell$.) Now it suffices to use the inequality (8) from Sec. 1.2 (with $D^{\beta-\gamma}\omega$ instead of ω): we obtain (3) with $c_2 =$

$$= \int_{|z|\le 1} |D^{\beta-\gamma}\omega(z)|dz .$$

In order to prove the inequality (2) it is sufficient to establish it for $\beta = 0$ and then use this particular case applying it to $D^{\beta}f$ instead of f, with $\ell - |\beta|$ instead of ℓ (taking into account that, due to $\overline{G}^{\delta} \subset \Omega$, we have $[D^{\beta}(A_{\delta}f)](x) = [A_{\delta}(D^{\beta}f)](x)$ for $x \in G$).

Thus, let $\beta = 0$ and let at first $f \in C^{\infty}(\Omega)$. Using the Taylor formula and taking into account (1), we obtain

$$(A_{\delta}f)(x) - f(x) = \int_{|z|\le 1} [f(x - \delta z) - f(x)]\omega(z)dz =$$

$$= \int_{|z|\le 1} \Big[\sum_{0<|\alpha|<\ell} \frac{D^{\alpha}f(x)}{\alpha!}(-\delta z)^{\alpha}\Big]\omega(z)dz +$$

$$+ \ell \int_{|z|\le 1} \Big[\sum_{|\alpha|=\ell} \frac{(-\delta z)^{\alpha}}{\alpha!} \int_0^1 (1 - t)^{\ell-1} D^{\alpha}f(x - t\delta z)dt\Big]\omega(z)dz =$$

$$= (-1)^{\ell}\ell\delta^{\ell} \int_{|z|\le 1} \Big[\sum_{|\alpha|=\ell} \frac{z^{\alpha}}{\alpha!} \int_0^1 (1 - t)^{\ell-1} D^{\alpha}f(x - t\delta z)dt\Big]\omega(z)dz .$$

Applying the Minkowski inequality, we conclude that

$$||A_{\delta}f - f||_{L_p(G)} \le$$

$$\le \ell\delta^{\ell} \int_{|z|\le 1} \Big(\sum_{|\alpha|=\ell} \frac{1}{\alpha!} \int_0^1 (1 - t)^{\ell-1} ||D^{\alpha}f(x - t\delta z)||_{L_p(G)}dt\Big)|\omega(z)|dz \le$$

$$\le \ell\delta^{\ell} \int_{|z|\le 1} \Big(\sum_{|\alpha|=\ell} \frac{1}{\alpha!} \int_0^1 (1 - t)^{\ell-1} ||D^{\alpha}f(y)||_{L_p(G^{t\delta|z|})}dt\Big)|\omega(z)|dz \le$$

$$\le \delta^{\ell} \int_{|z|\le 1} |\omega(z)|dz \sum_{|\alpha|=\ell} \frac{1}{\alpha!} ||D^{\alpha}f||_{L_p(G^{\delta})} \le$$

$$\le \delta^{\ell} \int_{|z|\le 1} |\omega(z)|dz \big(\min_{|\alpha|=\ell} \alpha!\big)^{-1} ||f||_{\hat{W}_p^{\ell}(G^{\delta})} ,$$

which implies (2) (for instance with $c_1 = \int_{|z|\le 1} |\omega(z)|dz$).

If $f \in W_p^{\ell}(\Omega)$, then we consider a sequence of functions $\phi_s(x) \in C^{\infty}(\Omega)$, $\phi_s \to f$ in $W_p^{\ell}(\Omega)$ (see Chap. 2), write down the inequality (2) for ϕ_s and, passing to the limit, obtain (2) for f (taking into account that due to the boundedness of the operator A_δ, we have $||A_\delta\phi_s \to A_\delta f||_{L_p(G)} \to 0$ with $s \to \infty$).

COROLLARY. *Let* $f \in W_p^\ell(\Omega)$, $1 \leq p \leq \infty$, $0 < \delta \leq \delta_0$. *Then*

$$(4) \qquad ||D^\beta f_m - D^\beta f||_{L_p(\tilde{\Omega}_m)} \leq c_1 \delta^{\ell-|\beta|} \nu_m^{\ell-|\beta|} ||f||_{\hat{w}_p^\ell(\tilde{\tilde{\Omega}}_m)}$$

holds for $|\beta| < \ell$ *and*

$$(5) \qquad ||D^\beta f_m||_{L_p(\tilde{\Omega}_m)} \leq c_2 \delta^{\ell-|\beta|} \nu_m^{\ell-|\beta|} ||f||_{\hat{w}_p^\ell(\tilde{\tilde{\Omega}}_m)}$$

for $|\beta| \geq \ell$ *, where*

$$\tilde{\Omega}_m = \bigcup_{s=m-1}^{m+1} \Omega_s , \quad \tilde{\tilde{\Omega}}_m = \bigcup_{s=m-2}^{m+2} \Omega_s .$$

P r o o f . It suffices to apply Lemma 3.7 with $G = \tilde{\Omega}_m$ and with $\delta\nu_m$ instead of δ and to take into account that $(\tilde{\Omega}_m)^{\delta\nu_m} \subset \subset \tilde{\tilde{\Omega}}_m$ for $\delta \leq \delta_0$ (see (8) from Sec. 3.2).

Let $\lambda(x)$, $x \in \Omega$, be a continuous positive function. Put $\lambda_1(x) = \min\{\lambda(x),\rho(x)\}$, where $\rho(x)$ is the distance of $x \in \Omega$ from the boundary $\Gamma(\Omega)$. Let us consider an arbitrary continuous positive function $\Lambda(x) \equiv \Lambda(x,\varepsilon)$, $x \in \Omega$, satisfying the following condition:

$$(6) \qquad \sup_{t<\rho(x)\leq t(1+\varepsilon)} \Lambda(x) \leq c_3 \inf_{t<\rho(x)\leq t(1+\varepsilon)} \lambda_1(x)$$

for all $t > 0$.

Let us discuss some examples. If $\lambda(x) = \rho(x)$, then we can set $\Lambda(x) = \lambda(x)$. If $\lambda(x) = g(\rho(x))$ with $g(u)$ - a continuous function monotonously increasing on $(0,\infty)$, $g(u) \leq u$, then we can set $\Lambda(x) = g((1-\varepsilon)\rho(x))$. If $\lambda(x) = g(\rho(x))$ with $g(u)$ - a continuous positive function, monotone in a neighbourhood of the origin as well as in a neighbourhood of infinity, then we can set

$$\Lambda(x) = \min_{1-\varepsilon\leq\gamma\leq1+\varepsilon} g_1(\gamma\rho(x)) , \quad g_1(u) = \min\{g(u),u\} .$$

Finally, let us note that (6) implies

$$\Lambda(x) \leq c_3\rho(x) , \quad x \in \Omega .$$

THEOREM 3.1. Let Ω *be an open set in* E_n , $1 \leq p < \infty$ *and* $f \in W_p^{\ell}(\Omega)$. *Then there is such a sequence of functions* $\phi_s(x) \in C^{\infty}(\Omega)$ ($\phi_s(x)$ *linearly depends on* f *and is independent of* p) *that*

(7) $$\lim_{s\to\infty} ||f - \phi_s||_{W_p^{\ell}(\Omega)} = 0 ,$$

and

(8) $$\lim_{s\to\infty} ||(D^{\alpha}f - D^{\alpha}\phi_s)\lambda(x)^{|\alpha|-\ell}||_{L_p(\Omega)} = 0$$

for $|\alpha| \leq \ell$, *while*

(9) $$||D^{\alpha}\phi_s\Lambda(x)^{|\alpha|-\ell}||_{L_p(\Omega)} \leq c_{\alpha,s}||f||_{W_p^{\ell}(\Omega)}$$

for $|\alpha| > \ell$, *with* $c_{\alpha,s}$ *independent of* f *and* Ω .

P r o o f . Consider a mollifier E_δ , where the number a involved in the construction of the partition of unity is chosen so that

(10) $$a^3 < 1 + \varepsilon .$$

The sequence of numbers ν_m is introduced in the following way:

(11) $$\nu_m = c_4 \sup_{a^{-m-2}<\rho(x)\leq a^{-m+1}} \Lambda(x) ,$$

where the constant c_4 independent of m is chosen so that $\nu_m \leq a^{-m}$ (condition (7) from Sec. 1.3). Since

$$\nu_m \leq c_4 c_3 \inf_{a^{-m-2}<\rho(x)\leq a^{-m+1}} \lambda_1(x) \leq c_4 c_3 a^{-m-2}$$

in virtue of (6) with $t = a^{-m-2}$ and of (11), it is sufficient to put $c_4 = a^2 c_3^{-1}$.

Using the formula (4) from Sec. 3.2 and taking into account that $\int_{E_n} \omega(x)dx = 1$, $\sum_{-\infty}^{\infty} \psi_m(x) = 1$ provided $\beta = \alpha$ and $\sum_{-\infty}^{\infty} D^{\alpha-\beta}\psi_m(x) = D^{\alpha-\beta}\sum_{-\infty}^{\infty}\psi_m(x)$ provided $|\beta| < |\alpha|$, we obtain for $x \in \Omega$ and $|\alpha| \leq \ell$

(12) $$D^{\alpha}(E_\delta f)(x) - D^{\alpha}f(x) = \sum_{0\leq\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \mathfrak{J}_{\alpha\beta}^{(\delta)}(x) ,$$

where

(13) $$\mathfrak{J}^{(\delta)}_{\alpha\beta}(x) = \sum_{m=-\infty}^{\infty} D^{\alpha-\beta}\psi_m(x)\left(D^\beta f_m(x) - D^\beta f(x)\right)$$

($\mathfrak{J}^{(\delta)}_{\alpha\beta}(x) \equiv J^{(\delta)}_{\alpha\beta}(x)$ provided $|\beta| < |\alpha|$).

Let $\mu_\alpha(x)$ be a continuous, positive in Ω function. Then

(14) $$||\mu_\alpha(x)[(D^\alpha E_\delta f)(x) - D^\alpha f(x)]||_{L_p(\Omega)} \leq$$
$$\leq \sum_{0\leq\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} ||\mu_\alpha(x)\mathfrak{J}^{(\delta)}_{\alpha\beta}(x)||_{L_p(\Omega)} .$$

If $\beta = \alpha$, $|\alpha| = \ell$ and $\sup_{x\in\Omega} \mu_\alpha(x) < +\infty$, then according to Lemma 3.6,

(15) $$\lim_{\delta\to 0} ||\mathfrak{J}^{(\delta)}_{\alpha\alpha}||_{L_p(\Omega)} = 0 , \quad |\alpha| = \ell .$$

Now let $|\beta| < \ell$. The multiplicity of the covering of the set Ω by the sets $\operatorname{supp}\psi_m$ is equal to 2 , hence by Lemma 3.1 we have

$$||\mu_\alpha\mathfrak{J}^{(\delta)}_{\alpha\beta}||_{L_p(\Omega)} \leq$$
$$\leq 2^{1-\frac{1}{p}}\left(\sum_{m=-\infty}^{\infty} \int_\Omega \mu^p_\alpha(x)\left|D^{\alpha-\beta}\psi_m(x)\left(D^\beta f_m(x) - D^\beta f(x)\right)\right|^p dx\right)^{1/p} .$$

Since $\operatorname{supp} D^{\alpha-\beta}\psi_m \subset \tilde{\Omega}_m = \Omega_{m-1}\cup\Omega_m\cup\Omega_{m+1}$ and the condition (iv) from Lemma 1.2 is fulfilled, we conclude, taking into account the inequality (4),

$$||\mu_\alpha\mathfrak{J}^{(\delta)}_{\alpha\beta}||_{L_p(\Omega)} \leq$$
$$\leq c_5\left(\sum_{m=-\infty}^{\infty}\left(\sup_{x\in\tilde{\Omega}_m}\mu_\alpha(x)\right)^p a^{m|\alpha-\beta|p}\int_{\tilde{\Omega}_m}|D^\beta f_m(x) - D^\beta f(x)|^p dx\right)^{1/p} \leq$$
$$\leq c_6\delta^{\ell-|\beta|}\left(\sum_{m=-\infty}^{\infty}\left(\sup_{x\in\tilde{\Omega}_m}\mu_\alpha(x)a^{m|\alpha-\beta|}\nu_m^{\ell-|\beta|}\right)^p ||f||^p_{\hat{W}^\ell_p(\tilde{\Omega}_m)}\right)^{1/p} .$$

Assume that the function $\mu_\alpha(x)$ and the numbers $\nu_m \leq a^{-m}$ are chosen so that

(16) $$\theta_m \equiv \sup_{x\in\tilde{\Omega}_m}\mu_\alpha(x)a^{m|\alpha-\beta|}\nu_m^{\ell-|\beta|} \leq c_7 , \quad 0\leq\beta\leq\alpha ,$$
$$|\alpha| = 0,1,\ldots , \quad m = 0,\pm 1,\pm 2,\ldots ,$$

with c_7 independent of m . Then according to Lemma 3.2,

$$(17)\qquad ||\mu_\alpha \mathfrak{J}^{(\delta)}_{\alpha\beta}||_{L_p(\Omega)} \leq c_8 \delta^{\ell-|\beta|} ||f||_{\hat{W}^{\ell}_p(\Omega)} , \quad |\beta| < \ell .$$

In the case $|\alpha| > \ell$ we again apply the formula (4) from Sec. 3.2. If $|\beta| < \ell$, then $\sum_{m=-\infty}^{\infty} D^{\alpha-\beta}\psi_m(x) \equiv 0$ and $J^{(\delta)}_{\alpha\beta}(x) \equiv \mathfrak{J}^{(\delta)}_{\alpha\beta}(x)$ and we can employ the inequality (17). If $|\beta| \geq \ell$, then analogously to the above argument we shall immediately estimate the norm including $J^{(\delta)}_{\alpha\beta}$, applying the inequality (5) instead of (4). Eventually, we obtain the inequality

$$(18)\qquad ||\mu_\alpha J^{(\delta)}_{\alpha\beta}||_{L_p(\Omega)} \leq c_9 \delta^{\ell-|\beta|} ||f||_{\hat{W}^{\ell}_p(\Omega)} , \quad |\alpha| > \ell, \quad 0 \leq \beta \leq \alpha .$$

Now (15), (17) and (18) imply

$$(19)\qquad \lim_{\delta\to 0} ||(D^\alpha(E_\delta f) - D^\alpha f)\mu_\alpha||_{L_p(\Omega)} = 0 , \quad |\alpha| \leq \ell$$

provided (16) holds ((16) implies that $\sup_{x\in \mathfrak{Q}_m} \mu_\alpha(x) < \infty$ provided $|\alpha| = \ell$) and

$$(20)\qquad ||D^\alpha(E_\delta f)\mu_\alpha||_{L_p(\Omega)} \leq c_{\alpha,\delta} ||f||_{W^{\ell}_p(\Omega)} , \quad |\alpha| > \ell$$

with $c_{\alpha,\delta}$ independent of f and Ω .

Let us discuss the inequality (16). First of all, if $|\alpha| = \ell$, then (16) is equivalent to the inequalities

$$(21)\qquad \sup_{x\in\Omega} \mu_\alpha(x) < \infty$$

and

$$(22)\qquad \nu_m \leq c_8 a^{-m} .$$

Indeed, (16) for $|\alpha| = \ell$ and $\beta = \alpha$ implies that $\sup_{x\in\Omega} \mu_\alpha(x) \leq c_7$, and for $|\beta| < |\alpha|$, (22) follows. Reversely, (21) and (22) imply (16) provided $|\alpha| = \ell$.

Further, let $|\alpha| < \ell$ and $\mu_\alpha(x) = \lambda(x)^{|\alpha|-\ell}$. Then (16) assumes the form

$$(23)\qquad \theta_m = \left[\inf_{x\in\mathfrak{Q}_m} \lambda(x)\right]^{|\alpha|-\ell} a^{m(|\alpha|-|\beta|)} \nu_m^{\ell-|\beta|} \leq c_7 , \quad 0 \leq \beta \leq \alpha ,$$
$$|\alpha| < \ell , \quad m = 0,\pm 1,\pm 2,\dots .$$

The inequality (23) under the condition (22) is equivalent to the inequality

$$(24) \qquad \nu_m \leq c_9 \inf_{x \in \mathfrak{F}_m} \lambda(x) .$$

Indeed, (23) implies (24) provided $\beta = \alpha$. Conversely, if $\inf_{x \in \mathfrak{F}_m} \lambda(x) \leq a^{-m}$, then $\theta_m \leq c_7'(\inf_{x \in \mathfrak{F}_m} \lambda(x))^{|\alpha|-|\beta|} a^{m(|\alpha|-|\beta|)} \leq c_7'$, and if the converse inequality is valid, then $\theta_m \leq$ $\leq c_7''(\inf_{x \in \mathfrak{F}_m} \lambda(x))^{|\alpha|-\ell} a^{m(|\alpha|-\ell)} \leq c_7''$.

Now let $|\alpha| > \ell$ and $\mu_\alpha(x) = \Lambda(x)^{|\alpha|-\ell}$. Then the inequality (16) assumes the form

$$(25) \qquad \theta_m = \left[\sup_{x \in \mathfrak{F}_m} \Lambda(x)\right]^{|\alpha|-\ell} a^{m(|\alpha|-|\beta|)} \nu_m^{\ell-|\beta|} \leq c_7 ,$$

$$0 \leq \beta \leq \alpha , \quad |\alpha| > \ell , \quad m = 0,\pm 1,\pm 2,\ldots .$$

Under the condition (22) the inequality (25) is equivalent to the inequality

$$(26) \qquad \nu_m \geq c_{10} \sup_{x \in \mathfrak{F}_m} \Lambda(x) ,$$

which can be verified as above.

Combining (22), (24) and (26) we conclude that for the functions $\mu_\alpha(x)$ considered, the inequality (16) is equivalent to the inequality

$$(27) \qquad c_{11} \sup_{x \in \mathfrak{F}_m} \Lambda(x) \leq \nu_m \leq c_{12} \inf_{x \in \mathfrak{F}_m} \lambda_1(x) .$$

The conditions (6), (10) and (11) imply the validity of the inequality (27) and also (16). Consequently, the inequalities (19) with $\mu_\alpha(x) = \lambda(x)^{|\alpha|-\ell}$ and (20) with $\mu_\alpha(x) = \Lambda(x)^{|\alpha|-\ell}$ are valid. The inequality (19) also holds with $\mu_\alpha(x) \equiv 1$.

In this way, the sequence $\phi_s = (E_\delta f)$, where $\delta_s \to 0$, $s \to \infty$, meets the requirements of Theorem 3.1.

Let us deal in more detail with the particular case of $\lambda(x) = \Lambda(x) = \rho(x)$. In this case,

(28) $$\lim_{s\to\infty} ||(D^\alpha f - D^\alpha \phi_s)\rho(x)^{|\alpha|-\ell}||_{L_p(\Omega)} = 0$$

holds provided $|\alpha| \leq \ell$ while

(29) $$||D^\alpha \phi_s \rho(x)^{|\alpha|-\ell}||_{L_p(\Omega)} \leq c_{\alpha,s}||f||_{W_p^\ell(\Omega)}$$

provided $|\alpha| > \ell$.

We shall show that the factor $\rho(x)^{|\alpha|-\ell}$ in (29) cannot be generally replaced by $\rho(x)^{|\alpha|-\ell-\varepsilon}$, $\varepsilon > 0$. Let $x = (\bar{x}, x_n)$, $\bar{x} = (x_1, \ldots, x_{n-1})$ and $\Omega = E_n^+ = \{x : x_n > 0\}$. Consider a function $f \in W_p^\ell(E_n^+)$, such that its trace $f(\bar{x},0) \in B_p^{\ell-1/p}(E_{n-1})$, while $f(\bar{x},0) \notin B_p^{\ell-1/p+\varepsilon}(E_{n-1})$, $1 < p < \infty$.

(The definition of the spaces B_p^ℓ as well as the proof of the embedding theorem formulated can be found for example in the book by S. M. Nikol'skiĭ [15].) Notice that in virtue of (28) the functions $\phi_s(\bar{x},0) = f(\bar{x},0)$ almost everywhere in E_{n-1} . Indeed,

$$x_n^{-1}(f(\bar{x},0) - \phi_s(\bar{x},0)) = x_n^{-1}(f(\bar{x},0) - f(\bar{x},x_n)) +$$
$$+ x_n^{-1}(f(\bar{x},x_n) - \phi_s(\bar{x},x_n)) + x_n^{-1}(\phi_s(\bar{x},x_n) - \phi_s(\bar{x},0)) =$$
$$= x_n^{-1}\int_0^{x_n} \frac{\partial f}{\partial x_n}(\bar{x},\xi_n)d\xi_n + x_n^{-1}(f(x) - \phi_s(x)) + x_n^{-1}\int_0^{x_n} \frac{\partial \phi_s}{\partial x_n}(\bar{x},\xi_n)d\xi_n .$$

Applying the Minkowski and Hardy inequalities and (28) with $\alpha = 0$, we obtain

(30) $$\left(\int_0^1 x_n^{-p}\int_{E_{n-1}} |f(\bar{x},0) - \phi_s(\bar{x},0)|^p d\bar{x}\, dx_n\right)^{1/p} \leq$$
$$\leq c_{13}\Bigg(\left|\left|\frac{\partial f}{\partial x_n}\right|\right|_{L_p(E_{n-1}\times(0,1))} +$$
$$+ ||x_n^{-\ell}(f(x) - \phi_s(x))||_{L_p(E_n\times(0,1))} +$$
$$+ \left|\left|\frac{\partial \phi_s}{\partial x_n}\right|\right|_{L_p(E_{n-1}\times(0,1))}\Bigg) < \infty ,$$

since f , $\phi_s \in W_p^\ell(E_n^+)$ and (28) with $\alpha = 0$ holds. Now (30) implies that $||f(\bar{x},0) - \phi_s(\bar{x},0)||_{L_p(E_{n-1})} = 0$ (since in the opposite case the left hand side of the inequality is infinite), that is, $f(\bar{x},0) \sim \sim \phi_s(\bar{x},0)$ on E_{n-1} .

If (29) holds with $\rho(x)^{|\alpha|-\ell-\varepsilon}$, $\varepsilon > 0$, then according to a theorem by S. V. Uspenskiĭ [17] and by (29),

$$||f(\bar{x},0)||_{B_p^{\ell-1/p+\varepsilon}(E_{n-1})} = ||\phi_s(\bar{x},0)||_{B_p^{\ell-1/p+\varepsilon}(E_{n-1})} \leq$$

$$\leq c_{14}(||\phi_s||_{L_p(E_n^+)} + \sum_{|\alpha|=m} ||x_n^{|\alpha|-\ell-\varepsilon} D^\alpha \phi_s||_{L_p(E_n^+)}) \leq$$

$$\leq c_{15}||f||_{W_p^\ell(E_n^+)} < \infty ,$$

which is not possible.

R E F E R E N C E S

[1] S. L. SOBOLEV: *Some applications of functional analysis in mathematical physics.* (Russian) Izdat. Leningrad. Gos. Univ., Leningrad 1950.

[2] S. L. SOBOLEV: *Introduction to the theory of cubature formulae.* (Russian) Izdat. "Nauka", Moscow 1974.

[3] O. V. BESOV, V. P. IL'IN , S. M. NIKOL'SKIĬ: *Integral representations of functions and imbedding theorems.* (Russian) Izdat. "Nauka", Moscow 1975.

[4] A. A. DEZIN: *On imbedding theorems and the problem of continuation of functions.* (Russian) Dokl. Akad. Nauk SSSR 88 (1953), 741-743.

[5] H. WHITNEY: *Analytic extensions of differentiable functions defined in closed sets.* Trans. Amer. Math. Soc. 36 (1934), 63-89.

[6] J. DENY, J. L. LIONS: *Les espaces du type de Beppo-Levi.* Ann. Inst. Fourier, Grenoble 5 (1953 - 1954), 305-370.

[7] V. I. BURENKOV: *The density of infinitely differentiable functions in Sobolev spaces for an arbitrary open set.* (Russian) Studies in the theory of differentiable functions of several variables and its applications, V. Trudy Mat. Inst. Steklov 131 (1974), 39-50, 244-245.

[8] L. D. KUDRYAVCEV: *On extension of functions and imbedding of classes of functions.* (Russian) Dokl. Akad. Nauk SSSR 107 (1956), 501-505.

[9] L. D. KUDRYAVCEV: *Direct and inverse imbedding theorems. Applications to the solution of elliptic equations by variational method.* (Russian) Trudy Mat. Inst. Steklov 55 (1959), 1-181.

[10] V. I. BURENKOV: *Partitions of unity.* (Russian) Studies in the theory of differentiable functions of several variables and its applications, VII. Trudy Mat. Inst. Steklov 150 (1979).

[11] N. G. MEYERS, J. SERRIN: H ≡ W . Proc. Mat. Acad. Sci. U.S.A. 51 (1964), 1055-1056.

[12] A. P. CALDERON, A. ZYGMUND: *Local properties of solutions of elliptic partial differential equations.* Studia Math. 20 (1961), 171-225.

[13] E. M. STEIN: *Singular integrals and differentiability properties of functions.* Princeton Univ. Press, Princeton, New Jersey 1970.

[14] V. I. BURENKOV: *Regularized distance.* (Russian) Trudy Moscow. Inst. Radiotehn. Elektron. i Avtomat., Vyp. 67 Mat. (1973), 113-117, 152.

[15] S. M. NIKOL'SKIĬ: *Approximation of functions of several variables and imbedding theorems.* (Russian) Izdat. "Nauka",Moscow 1969.

[16] V. I. BURENKOV: *The density of infinitely differentiable functions in spaces of functions specified on an arbitrary open set.* (Russian) Theory of cubature formulas and applications of functional analysis to some problems in math. physics. Inst. Mat. Akad. Nauk SSSR, Sibirsk. otdel., Novosibirsk 1975.

[17] A. KUFNER, O. JOHN, S. FUČÍK: *Function spaces.* Academia Prague and Noordhoff Leyden 1977.

[18] S. V. USPENSKIĬ: *Imbedding theorems for classes with weights.* (Russian) Trudy Mat. Inst. Steklov 60 (1961), 282-303.