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AN ASYNCHRONOUS THREE-FIELD DOMAIN DECOMPOSITION METHOD FOR FIRST-ORDER EVOLUTION PROBLEMS

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Abstract

We present an asynchronous multi-domain time integration algorithm with a dual domain decomposition method for the initial boundary-value problems for a parabolic equation. For efficient parallel computing, we apply the three-field domain decomposition method with local Lagrange multipliers to ensure the continuity of the primary unknowns at the interface between subdomains. The implicit method for time discretization and the multi-domain spatial decomposition enable us to use different time steps (subcycling) on different parts of a computational domain, and thus efficiently capture the underlying physics with less computational effort. We illustrate the performance of the proposed multi-domain time integrator by means of a simple numerical example.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain split into a finite number of non-overlapping subdomains Ω^k ($k = 1, \dots, N_D$). Let $\Omega = \bigcup_{k=1}^{N_D} \Omega^k$, $\Gamma^k = \partial\Omega^k$, $\Sigma = \bigcup_{k=1}^{N_D} \Gamma^k \setminus \partial\Omega$. We introduce the bilinear form

$$((u, v))_k := \int_{\Omega^k} \sum_{|i| \leq 1} \sum_{|j| \leq 1} (-1)^{|i|} a_{ij}(x) D^j u D^i v \, dx \quad \forall u, v \in H^1(\Omega^k), \quad (1)$$

where $i = (i_1, i_2)$ and $j = (j_1, j_2)$ are two-dimensional vectors, i_1, i_2, j_1, j_2 are nonnegative integers and $|i| = i_1 + i_2$ and $|j| = j_1 + j_2$. The summation in (1) means that summation should be carried out over all i and j , for which $|i| \leq 1$, $|j| \leq 1$ holds. We assume that the coefficient functions a_{ij} belong to $L^\infty(\Omega)$. We assume there exists a positive number ϵ (independent of v) such that $((v, v))_k \geq \epsilon \|v\|_{H^1(\Omega^k)}^2$ for every $v \in H_0^1(\Omega^k)$. Further, for every $u, v \in \prod_k H^1(\Omega^k)$ we set $((u, v)) := \sum_k ((u, v))_k$. From now on we are going to use the following notation: $V := \prod_k H^1(\Omega^k)$ and $M := \prod_k H^{-1/2}(\Gamma^k)$, (\cdot, \cdot) will be the usual inner product in $L^2(\Omega)$, $(\cdot, \cdot)_k$ will be the inner product in $L^2(\Omega^k)$ and $\langle \cdot, \cdot \rangle_k$ will be the duality pairing between $H^{-1/2}(\Gamma^k)$

and $H^{1/2}(\Gamma^k)$. Finally, introduce the space $\Phi := \{\varphi \in L^2(\Sigma); \exists v \in H_0^1(\Omega), \varphi = v|_{\Sigma}\}$ equipped with the norm $\|\varphi\|_{\Phi} = \inf \{\|v\|_{H^1(\Omega)}; v \in H_0^1(\Omega), v|_{\Sigma} = \varphi\}$. We now consider the following two equivalent model problems. Let $T > 0$ be fixed and assume $u_0 \in H_0^1(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$:

(i) find $u \in L^2(0, T; H_0^1(\Omega))$ with $\partial_t u \in L^2(0, T; L^2(\Omega))$, such that

$$(\partial_t u, v) + ((u, v)) = (f, v) \quad \forall v \in H_0^1(\Omega) \quad \text{and} \quad u(x, 0) = u_0(x) \text{ in } \Omega; \quad (2)$$

(ii) find $u^k \in L^2(0, T; H^1(\Omega^k))$ with $\partial_t u^k \in L^2(0, T; L^2(\Omega^k))$, $\lambda^k \in L^2(0, T; H^{-1/2}(\Gamma^k))$ and $w \in L^2(0, T; \Phi)$, such that $u^k(x, 0) = u_0(x)|_{\Omega^k}$ and (for $k = 1, \dots, N_D$)

$$\begin{cases} (\partial_t u^k, v^k)_k + ((u^k, v^k))_k - \langle \lambda^k, v^k \rangle_k = (f, v^k)_k & \forall v^k \in H^1(\Omega^k), \\ \langle u^k, \mu^k \rangle_k = \langle w, \mu^k \rangle_k & \forall \mu^k \in H^{-1/2}(\Gamma^k), \\ \sum_{k=1}^{N_D} \langle \lambda^k, \varphi \rangle_k = 0 & \forall \varphi \in \Phi. \end{cases} \quad (3)$$

Let us mention that problem (3) is well suited for domain decomposition methods. By the standard linear parabolic equation theory [4], both problems (2) and (3) admit the unique solution, such that $u = u^k$ in Ω^k , $\lambda^k = \nabla u \cdot \mathbf{n}_A^k$ on $\partial\Omega^k$ and $w = u$ on Σ . To solve problem (3) numerically, we propose a new numerical scheme which is based on the subcycling algorithm using non-standard asynchronous time discretization amenable for parallel computing.

2. Asynchronous multi-domain discretization in time

Let us fix $p \in \mathbb{N}$ and let $\tau := T/p$ be a time step. Next, we introduce a substep time $\tau^k = \tau/s^k$, which is proportional to the system time step $\tau = t_{n+1} - t_n$, where s^k is the number of substeps for domain k , as shown schematically in Figure 1. Further, we introduce the backward difference quotient $\delta_{\tau^k} \phi_{n,j}^k := (\phi_{n,j}^k - \phi_{n,j-1}^k)/\tau^k$ for $n = 0, \dots, p-1$. In view of the assumed relationships between the discretization steps, the present “method of asynchronous discretization in time” consists in the following: find, successively for $n = 0, 1, 2, \dots, p-1$, functions $u_{n,j}^k \in H^1(\Omega^k)$, $\lambda_{n,j}^k \in H^{-1/2}(\partial\Omega^k)$ and $w_{n+1} \in \Phi$, $k = 1, \dots, N_D$, $j = 1, \dots, s^k$, as solutions of the problems

$$(\delta_{\tau^k} u_{n,j}^k, v_j^k)_k + ((u_{n,j}^k, v_j^k))_k - \langle \lambda_{n,j}^k, v_j^k \rangle_k = (f_{n,j}^k, v_j^k)_k \quad \forall v_j^k \in H^1(\Omega^k), \quad (4)$$

$$\langle u_{n,j}^k, \mu_j^k \rangle_k = \langle w_{n,j}^k, \mu_j^k \rangle_k \quad \forall \mu_j^k \in H^{-1/2}(\Gamma^k), \quad (5)$$

$$\sum_{k=1}^{N_D} \langle \lambda_{n,s^{N_D}}^k, \varphi \rangle_k = 0 \quad \forall \varphi \in \Phi, \quad (6)$$

starting with the functions $u_{0,0}^k(x) = u_0(x)|_{\Omega^k} \in H^1(\Omega^k)$.

In this work, the equation of continuity of fluxes is required only at the final (system) time step, see (6). The unknown $w_{n,j}^k$ on the common interface Σ is linearly interpolated at the intermediate steps by

$$w_{n,j}^k = \left(1 - \frac{j}{s^k}\right) w_n + \left(\frac{j}{s^k}\right) w_{n+1} \quad \forall j = 1, \dots, s^k, \quad k = 1, \dots, N_D.$$

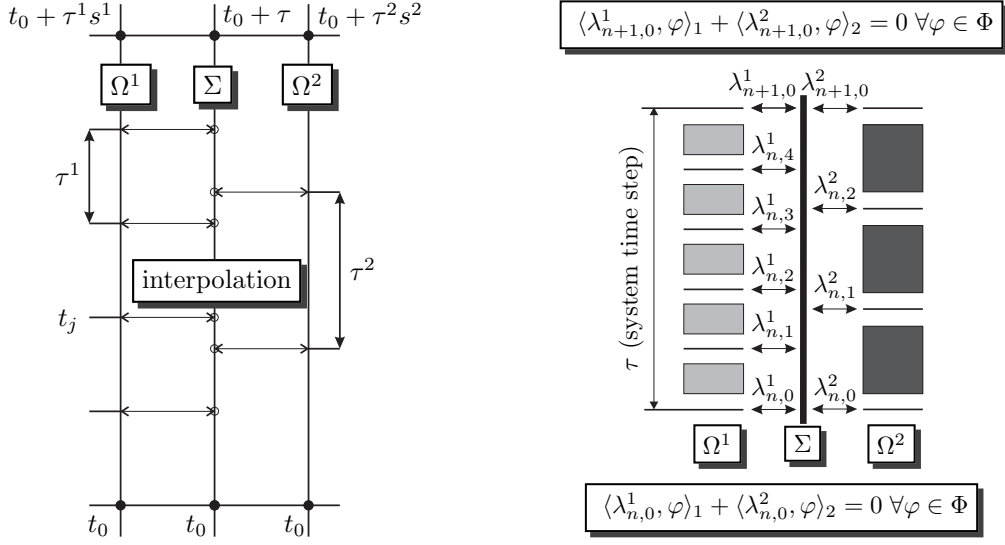


Figure 1: Substeps of the system time step. Example for $N_D = 2$, $s^1 = 5$, $s^2 = 3$.

Theorem 1. *Problem (4)–(6) has a unique solution.*

Proof. Without loss of generality we assume $u_{n,0}^k = 0$. First, we associate with any $\varphi \in \Phi$ a vector function

$$\tilde{\varphi} = (\tilde{u}_{n,1}^1, \tilde{u}_{n,2}^1, \dots, \tilde{u}_{n,s^1}^1, \tilde{u}_{n,1}^2, \tilde{u}_{n,2}^2, \dots, \tilde{u}_{n,s^2}^2, \dots, \tilde{u}_{n,1}^{N_D}, \tilde{u}_{n,2}^{N_D}, \dots, \tilde{u}_{n,s^{N_D}}^{N_D}) \in \prod_k H^1(\Omega^k)^{s^k},$$

components of which are defined as solutions of the following Dirichlet problems

$$\begin{aligned} (\delta_{\tau^k} \tilde{u}_{n,j}^k, v^k)_k + ((\tilde{u}_{n,j}^k, v^k))_k &= 0 & \forall v^k \in H_0^1(\Omega^k), \\ \langle \tilde{u}_{n,j}^k, \mu^k \rangle_k &= \langle (j/s^k) \varphi, \mu^k \rangle_k & \forall \mu^k \in H^{-1/2}(\Gamma^k) \end{aligned}$$

for $k = 1, \dots, N_D$, $j = 1, \dots, s^k$. Note that

$$\|\tilde{\varphi}\|_{\prod_k H^1(\Omega^k)^{s^k}} := \sum_k \sum_j \|\tilde{u}_{n,j}^k\|_{H^1(\Omega^k)} \leq c \|\varphi\|_{\Phi}. \quad (7)$$

Now we assume a given function $\psi \in \Phi$ and set a vector functions

$$\mathbf{u} = (u_{n,1}^1, u_{n,2}^1, \dots, u_{n,s^1}^1, u_{n,1}^2, u_{n,2}^2, \dots, u_{n,s^2}^2, \dots, u_{n,1}^{N_D}, u_{n,2}^{N_D}, \dots, u_{n,s^{N_D}}^{N_D})$$

and

$$\boldsymbol{\lambda} = (\lambda_{n,1}^1, \lambda_{n,2}^1, \dots, \lambda_{n,s^1}^1, \lambda_{n,1}^2, \lambda_{n,2}^2, \dots, \lambda_{n,s^2}^2, \dots, \lambda_{n,1}^{N_D}, \lambda_{n,2}^{N_D}, \dots, \lambda_{n,s^{N_D}}^{N_D})$$

so that $\mathbf{u} = \tilde{\psi}$ and

$$(\delta_{\tau^k} u_{n,j}^k, v^k)_k + ((u_{n,j}^k, v^k))_k - \langle \lambda_{n,j}^k, v^k \rangle_k = 0 \quad \forall v^k \in H^1(\Omega^k), \quad (8)$$

$$\langle u_{n,j}^k - (j/s^k) \psi, \mu^k \rangle_k = 0 \quad \forall \mu^k \in H^{-1/2}(\Gamma^k) \quad (9)$$

for $j = 1, \dots, s^k$, $k = 1, \dots, N_D$. We now define the operator $\mathcal{S} : \Phi \rightarrow \Phi^*$ by

$$\langle \mathcal{S}(\psi), \cdot \rangle_{\Phi^*, \Phi} = \sum_{k=1}^{N_D} \langle \lambda_{n, s^k}^k, \cdot \rangle_k.$$

From (7) and (8) we easily compute (recall $\mathbf{u} = \tilde{\psi}$)

$$\langle \mathcal{S}(\psi), \varphi \rangle_{\Phi^*, \Phi} = \sum_{k=1}^{N_D} \langle \lambda_{n, s^k}^k, \varphi \rangle_k \leq \alpha \|\psi\|_{\Phi} \|\varphi\|_{\Phi}. \quad (10)$$

On the other hand, taking $\varphi = \psi$ we have, combining (8) and (9),

$$\begin{aligned} \langle \mathcal{S}(\psi), \psi \rangle_{\Phi^*, \Phi} &= \sum_{k=1}^{N_D} \langle \lambda_{n, s^k}^k, \psi \rangle_k = \sum_{k=1}^{N_D} (\delta_{\tau^k} u_{n, s^k}^k, u_{n, s^k}^k)_k + \sum_{k=1}^{N_D} ((u_{n, s^k}^k, u_{n, s^k}^k))_k \\ &\geq \gamma \|\psi\|_{\Phi}^2. \end{aligned} \quad (11)$$

In (10) and (11), α and γ are positive constants, independent of ψ and φ . Hence, \mathcal{S} is an isomorphism from Φ onto Φ^* .

We now turn back, for a moment, to (4)–(6) and consider $\check{u}_{n, j}^k \in H_0^1(\Omega^k)$ and $\check{\lambda}_{n, j}^k \in H^{-1/2}(\partial\Omega^k)$ as the solution of the problem

$$(\delta_{\tau^k} \check{u}_{n, j}^k, v^k)_k + ((\check{u}_{n, j}^k, v^k))_k - \langle \check{\lambda}_{n, j}^k, v^k \rangle_k = (f_{n, j}^k, v^k)_k \quad \forall v^k \in H^1(\Omega^k),$$

for $k = 1, \dots, N_D$, $j = 1, \dots, s^k$. The existence of such solutions is ensured by [1]. We now define the functional $g \in \Phi^*$ by

$$\langle g, \cdot \rangle_{\Phi^*, \Phi} = \sum_{k=1}^{N_D} \langle -\check{\lambda}_{n, s^k}^k, \cdot \rangle_k.$$

Problem (4)–(6) can now be reduced to problem

$$\mathcal{S}(\psi) = g.$$

Now with ψ in hand, we determine $u_{n, j}^k \in H^1(\Omega^k)$ and $\lambda_{n, j}^k \in H^{-1/2}(\partial\Omega^k)$ as the solution of decoupled (independent) Dirichlet problems (8) and (9) for $k = 1, \dots, N_D$, $j = 1, \dots, s^k$. It is easy to verify, that $u_{n, j}^k$, $\lambda_{n, j}^k$ and $w_{n, j}^k = \left(\frac{j}{s^k}\right) \psi$ solve uniquely problem (4)–(6). Recall that we considered for simplicity $u_{n, 0}^k = 0$. The proof is complete. \square

Remark 2. *Let us explicitly mention, that \mathcal{S} corresponds to the Poincaré-Steklov operator on Σ , well known in the theory of domain decomposition methods for elliptic problems, see [2, 3].*

3. Numerical example

We approximate the problem (3) in space choosing V_h , M_h and Φ_h finite dimensional subspaces of V , M and Φ and introduce $\mathbf{u}_h(x, t) = \mathbf{N}_u(x)\tilde{\mathbf{u}}(t)$, $\mathbf{w}_h(x, t) = \mathbf{N}_w(x)\tilde{\mathbf{w}}(t)$ and $\boldsymbol{\lambda}_h(x, t) = \mathbf{N}_\Lambda(x)\tilde{\boldsymbol{\Lambda}}(t)$, such that $\mathbf{u}_h(t) \in V_h$, $\mathbf{w}_h(t) \in \Phi_h$ and $\boldsymbol{\lambda}_h(t) \in M_h$ for all $t \in (0, T)$, respectively. Application of FEM-discretization in space leads to the following system of equations ($j = 1, \dots, s^k$, $k = 1, \dots, N_D$):

$$\begin{cases} \mathbf{M}^k \delta_{\tau^k} \mathbf{u}_{n,j}^k + \mathbf{K}^k \mathbf{u}_{n,j}^k + (\mathbf{C}^k)^T \boldsymbol{\Lambda}_{n,j}^k = \mathbf{f}_{n,j}^k, \\ \mathbf{C}^k \mathbf{u}_{n,j}^k - \left(\frac{j}{s^k}\right) \mathbf{B}^k \mathbf{w}_{n+1} - \left(1 - \frac{j}{s^k}\right) \mathbf{B}^k \mathbf{w}_n = \mathbf{0}, \\ \sum_{k=1}^{N_D} (\mathbf{B}^k)^T \boldsymbol{\Lambda}_{n,s^k}^k = \mathbf{0}. \end{cases}$$

Using the common nomenclature of heat conduction, \mathbf{M}^k is the capacitance matrix, \mathbf{K}^k is the conductance matrix and the vector \mathbf{f}^k represents the nodal values of the source corresponding to subdomain Ω^k . Operators \mathbf{C}^k and \mathbf{B}^k are the Boolean matrices extracting the interface degrees of freedom from \mathbf{u} and the corresponding degrees of freedom from \mathbf{w} for a particular subdomain k . The above system can be written in a matrix form, which has a block-bordered structure amenable to parallel computation. In this work, we solve for all unknowns simultaneously by a monolithic method using a direct solver. In order to briefly present the performance of the proposed algorithm, we consider a simple test problem. A square of size 1.0×1.0 is divided into two equal subdomains, and each subdomain is divided into 5×10 square elements, see Figure 2. We consider the analytical solution given by

$$u^*(x_1, x_2, t) = \sin(\pi x_1) \sin(\pi x_2) \sin(t)$$

and assume the coefficient functions as constants: $a_{ij}(x) = \delta_{ij} 10^{-4}$ in Ω . Hence, the right hand side takes the form

$$f^* = [\cos(t) + 2 \times 10^{-4} \pi^2 \sin(t)] \sin(\pi x_1) \sin(\pi x_2).$$

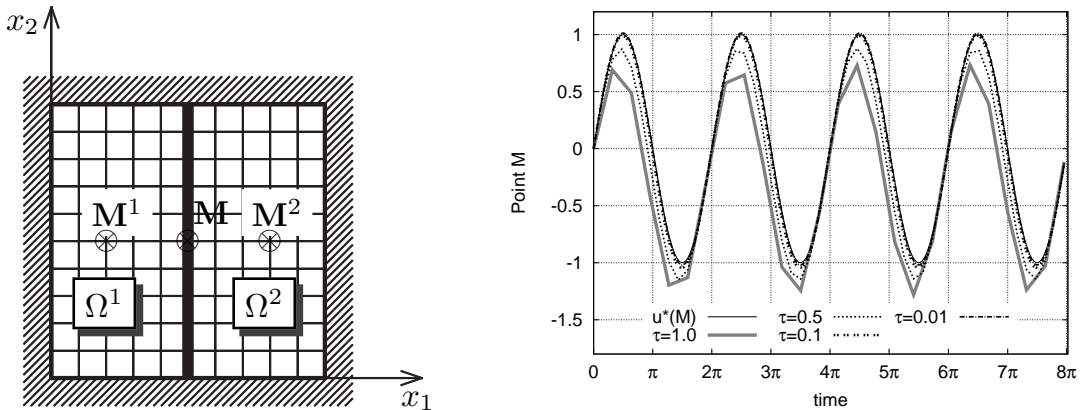


Figure 2: 2D test problem (left). Numerical results at the point M for various system time steps (right).

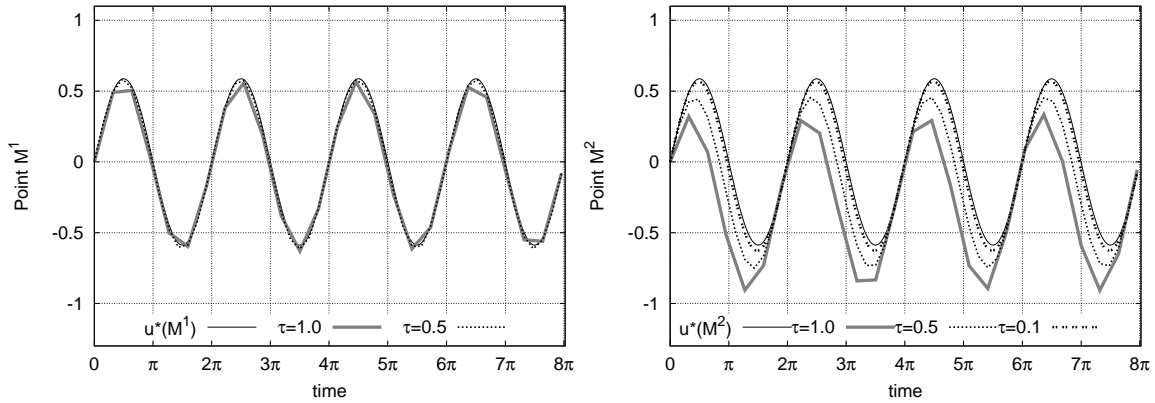


Figure 3: Results at the points M^1 (left) and M^2 (right).

In Figures 2 and 3 we have shown the results at points M , M^1 and M^2 for various system time steps τ , the ratio $s^1 : s^2 = 10 : 1$. As predicted by the theory, the numerical results are stable and match well with the analytical solution for sufficiently small system time step (approx. $\tau \approx 0.1$).

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References

- [1] Babuška, I.: The finite element method with Lagrangian multipliers. *Numer. Math.* **20** (1973), 179–192.
- [2] Brezzi, F. and Marini, L.: Macro hybrid elements and domain decomposition methods. In: (J. A. Désideri, L. Fezoui, B. Larrouturou, B. Rousselet (Eds.), *Optimization et Contrôle Cépadue's-Editions*, pp. 89–96. Toulouse, 1993.
- [3] Brezzi, F. and Marini, L.: A three-field domain decomposition method. In: (A. Quarteroni, J. Periaux, Y. A. Kuznetof, O. Widlund (Eds.), *Domain Decomposition Methods in Science and Engineering*, vol. 157, pp. 27–34. American Mathematical Society, Series CONM, 1994.
- [4] Rektorys, K.: *The method of discretization in time and partial differential equations*. Springer, 1982.