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SUPERCONVERGENCE FOR CONVECTION-DIFFUSION PROBLEMS WITH LOW REGULARITY

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Abstract

The finite element method is applied to a convection-diffusion problem posed on the unite square using a tensor product mesh and bilinear elements. The usual proofs that establish superconvergence for this setting involve a rather high regularity of the exact solution - typically $u \in H^3(\Omega)$, which in many cases cannot be taken for granted. In this paper we derive superconvergence results where the right hand side of our a priori estimate no longer depends on the H^3 norm but merely requires finiteness of some weaker functional measuring the regularity. Moreover, we consider the streamline diffusion stabilization method and how superconvergence is affected by the regularity of the solution. Finally, numerical experiments for both discretizations support and illustrate the theoretical results.

1. Introduction

We consider the scalar convection-diffusion boundary value problem

$$\begin{aligned} Lu := -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f \quad &\text{in } \Omega = (0,1)^2, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, $0 < \varepsilon \ll 1$ is a small parameter. We assume $f \in L_2(\Omega),\ c \in L_\infty(\Omega), \mathbf{b} \in (W^1_\infty(\Omega))^2$ and

$$c - \frac{1}{2}\operatorname{div}\mathbf{b} \geq \omega > 0. \tag{2}$$

As a discretization for (1) we use the finite element method. Introducing the Hilbert space

$$H^1_0(\Omega) := \{v \in H^1(\Omega) \,|\, v_{|\partial\Omega} = 0\}$$

the variational formulation of the given boundary value problem reads:

Find $u \in H^1_0(\Omega)$ such that

$$a(u,v) := \varepsilon\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{b}\cdot\nabla u\, v\,\mathrm{d}\mathbf{x} + \int_\Omega c\,uv\,\mathrm{d}\mathbf{x} = \int_\Omega fv\,\mathrm{d}\mathbf{x}, \quad \text{for all } v \in H^1_0(\Omega). \tag{3}$$

The natural norm in which one shows coercivity of the bilinear form is the ε-weighted H^1 norm

$$\|u\|_{1,\varepsilon}^2 := |u|_{1,\varepsilon}^2 + \|u\|_0^2, \quad \text{with} \quad |u|_{1,\varepsilon}^2 := \varepsilon |u|_1^2.$$

Moreover, on subdomains $D \subseteq \Omega$ we will work with fractional order Sobolev spaces $H^s(D)$ ($s \in \mathbb{R}_+$), the elements of which are finite in the corresponding norm

$$\begin{aligned} \|u\|_{m+\sigma,D}^2 &:= \|u\|_{m,D}^2 + |u|_{\sigma,D}^2 \quad \text{with} \\ |u|_{\sigma,D}^2 &:= \sum_{|\alpha|=m} \iint_{D\times D} \frac{|u^{(\alpha)}(\mathbf{x}) - u^{(\alpha)}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2\sigma}} \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}, \quad m \in \mathbb{N},\, \sigma \in (0,1). \end{aligned} \tag{4}$$

Furthermore, for $D = \Omega$ we drop the set index Ω in the notation of (semi-)norms.

In this paper we will discretize (3) with bilinear finite elements on tensor product meshes $\mathcal{T}_h$ with respective mesh size h_x and h_y. The parameter h is going to denote the maximal element diameter of the current mesh.

From standard finite element analysis it is well known that the error can be bounded by

$$\|u - u_h\|_{1,\varepsilon} \le Ch\, |u|_{2,\Omega}.$$

Also, it is a known phenomenon that superconvergence is achieved for the difference of the interpolant u^I and the FE-solution u_h, measured in the same norm. For example, an analysis as in [9] using Lin-identities yields

$$\|u^I - u_h\|_{1,\varepsilon} \le Ch^2 \|u\|_{3,\Omega}.$$

However, in many cases the regularity requirements for superconvergence are not realistic, not even for the simple model problem

$$-\Delta u = 1, \quad u_{|\delta\Omega} = 0,$$

where due to corner singularities one may assume that at best

$$u \in H^{3-\delta}(\Omega),\ \forall \delta > 0.$$

If the coefficients of (1) themselves have low regularity the situation is even worse. For instance, if the right hand side has some singularity which forces f to lie in some low order Sobolev space then by the lifting property of the solution operator this carries over to the solution u:

$$f \in H^\sigma(\Omega) \Longrightarrow u \in H^{2+\sigma}(\Omega),$$

i.e. it is clear that in the general case we cannot expect $u \in H^3(\Omega)$.

Remark 1. *Also note that for domains with obtuse angles, for instance the case of convex polygons, the effect of corner singularities can be even stronger. If we consider Dirichlet boundary conditions near some vertex P corresponding to the maximal interior angle α, the solution u locally behaves as $r^{\frac{\pi}{\alpha}}$, with r being the distance from P, and hence forces u to lie at best in the Sobolev space $H^{1+\frac{\pi}{\alpha}-\delta}(\Omega)$ for all $\delta > 0$.*

Now naturally the question arises: How does lower regularity of u affect the phenomenon of supercloseness? Moreover, does the rate of convergence depend on the regularity in some "continuous" way? In this paper we address those issues both theoretically and numerically and derive superconvergence results where the right hand side of our a priori estimate no longer depends on the H^3 norm but merely requires finiteness of some weaker functional, *e.g.* the fractional Sobolev seminorm $|\,.\,|_{2+\sigma}$, $\sigma \in (0,1)$, or equivalent norms in interpolation spaces.

The paper is structured as follows. In Section 2 we derive bounds for the standard Galerkin method, followed by corresponding numerical experiments in Section 3. Subsequently, we discuss the extension of the estimates to the streamline diffusion FEM in Section 4 and present numerical results for the SDFEM in Section 5.

Indeed, it is our aim to extend the analysis to stabilized methods on layer-adapted meshes, similarly as in [9]. That analysis is based on a solution decomposition and uses, for instance for the smooth part of the solution S, bounds of the type

$$\|S^I - S_h\|_{1,\varepsilon} \le C(N^{-1}\ln N)^2(|S|_3 + \|S\|_{2,\infty}).$$

In the present paper where we mostly consider isotropic meshes we therefore tried to avoid the application of the theory of interpolation spaces to obtain results in fractional order Sobolev spaces. While we succeeded in the error analysis of the Galerkin part (see Section 2) we had some trouble with the convective term and some stabilization term of the SDFEM (see Section 4). For these terms we, so far, have no alternatives as to apply interpolation spaces. The main ingredients and results of this theory needed in this article are presented in the Appendix.

2. Galerkin error analysis

Let

$$V_h := \{v \in H_0^1(\Omega) : v_{|T} \in Q_1(T),\ \text{for all}\ \ T \in \mathcal{T}_h\}$$

be the bilinear finite element space. Then the Galerkin approximation u_h of u solves

$$a(u_h, v) = \int_\Omega f v \,\mathrm{d}\mathbf{x}, \quad \text{for all } v \in V_h.$$

In the sequel we suppose that $u \in H^2(\Omega)$ and denote by $u^I \in V_h$ the nodal interpolant of the exact solution. To get an estimation of the error $\|u^I - u_h\|_{1,\varepsilon}$ we make use of coercivity of the bilinear form $a(.,.)$ of (3) and apply Galerkin orthogonality:

$$\alpha_{coerc}\|u^I - u_h\|_{1,\varepsilon}^2 \le a(u^I - u_h, u^I - u_h) = a(u^I - u, u^I - u_h). \tag{5}$$

Consequently, for an arbitrary function $v \in V_h$ we will estimate the following three terms:

$$a(u - u^I, v) = \varepsilon \int_\Omega \nabla(u - u^I)\cdot\nabla v \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{b}\cdot\nabla(u - u^I) v \,\mathrm{d}\mathbf{x} + \int_\Omega c\,(u - u^I) v \,\mathrm{d}\mathbf{x}.$$

2.1. The diffusion term

Let us first bound the diffusion term of the Galerkin part given by

$$\varepsilon \int_\Omega \nabla(u - u^I)\cdot\nabla v \,\mathrm{d}\mathbf{x} = \varepsilon \int_\Omega (u - u^I)_x v_x \,\mathrm{d}\mathbf{x} + \varepsilon \int_\Omega (u - u^I)_y v_y \,\mathrm{d}\mathbf{x}.$$

Clearly, it is enough to estimate integrals on an arbitrary element that only involve derivatives with respect to the first argument. In combination with the triangle inequality, summing up all contributions will give an upper bound.
The key observation for this type of integrals is the fact that on every element a similar expression vanishes for quadratic polynomials:

$$\int_T (p - p^I)_x v_x \,\mathrm{d}\mathbf{x} = 0, \text{ for all } p \in \mathcal{P}_2. \tag{6}$$

Hence, we insert additional degrees of freedom by just subtracting zero on every element and subsequently bound the interpolation error of $u - p$:

$$\begin{aligned} \varepsilon |\int_T (u - u^I)_x v_x \,\mathrm{d}\mathbf{x}| &= \varepsilon |\int_T ((u-p) - (u-p)^I)_x v_x \,\mathrm{d}\mathbf{x}| \\ &\le C\varepsilon \left(h_x \|(u-p)_{xx}\|_{0,T} + h_y \|(u-p)_{xy}\|_{0,T}\right) \|v_x\|_{0,T}, \end{aligned} \tag{7}$$

Note that we are still in the position to choose some particular polynomial $p \in \mathcal{P}_2$. The following lemma motivates this choice.

Lemma 1. *Let $D \subset \mathbb{R}^2$ be a bounded domain. Then for all $\sigma \in (0,1)$ and $w \in H^\sigma(D)$*

$$\| w - \Pi w \|_{0,D} \le \frac{\mathrm{diam}(D)^{1+\sigma}}{|D|^{\frac{1}{2}}} |w|_{\sigma,D}, \tag{8}$$

where $\Pi w := \frac{1}{|D|}\int_D w \,\mathrm{d}\mathbf{x}$ denotes the average of w over D.

Proof. By the Cauchy-Schwarz inequality we obtain

$$\int_D \left(w(\mathbf{x}) - \frac{1}{|D|}\int_D w(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \right)^2 \mathrm{d}\mathbf{x} \le \frac{1}{|D|}\int_D \int_D \left(w(\mathbf{x}) - w(\mathbf{x}')\right)^2 \mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}$$

Since the diameter of D is the supremum of all distances of points in D, we have for all $\mathbf{x}, \mathbf{x}' \in D$: $|\mathbf{x} - \mathbf{x}'| \le \mathrm{diam}(D)$ and hence,

$$\int_D \left(w(\mathbf{x}) - \frac{1}{|D|}\int_D w(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \right)^2 \mathrm{d}\mathbf{x} \le \frac{\mathrm{diam}(D)^{2+2\sigma}}{|D|} \iint\limits_{D\times D} \frac{|w(\mathbf{x}) - w(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^{2+2\sigma}}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}.$$

□

We now continue to estimate (7). If we denote by $\overline{u_{xx}}_{|T}$ and $\overline{u_{xy}}_{|T}$ the averages of the two partial derivatives of u on some element T and insert the quadratic polynomial

$$p(x,y) = \frac{1}{2}\overline{u_{xx}}_{|T}\, x^2 + \overline{u_{xy}}_{|T}\, xy$$

into (7), Lemma 1 yields the fractional estimate

$$\varepsilon |\int_T (u - u^I)_x v_x \,\mathrm{d}\mathbf{x}| \le C\varepsilon h^\sigma \left(h_x |u_{xx}|_{\sigma,T} + h_y |u_{xy}|_{\sigma,T}\right) \|v_x\|_{0,T},$$

on every element $T \in \mathcal{T}_h$ and hence globally

$$\varepsilon| \int_\Omega \nabla(u - u^I) \cdot \nabla v \, \mathrm{d}\mathbf{x}| \leq C\varepsilon^{\frac{1}{2}} h^{1+\sigma} |u|_{2+\sigma,\Omega} \|v\|_{1,\varepsilon,\Omega}. \tag{9}$$

After some closer look on the constant in (9) coming from Lemma 1, one easily checks that it unfavourably depends on the element aspect ratio h_x/h_y. However, by working with interpolation spaces, it is possible to improve this bound to obtain anisotropic estimates of the difference of a function and its mean in the L_2 norm. To derive this estimate let us consider rectangular domains in the sequel. We start with the following one dimensional consideration.

Lemma 2. *Let $h > 0$, $I := (0, h)$ and $u \in H^1(I)$. Define $\Pi : H^1(I) \to \mathbb{R}$ to be the averaging operator: $\Pi u := \frac{1}{h} \int_0^h u \, \mathrm{d}x$. Then*

$$\|u - \Pi u\|_{0,I} \leq \frac{\sqrt{3}}{3} h \, |u|_{1,I}.$$

Proof. The estimate is an immediate consequence of the Bramble-Hilbert Lemma. After assuming that u lies in the dense subset $\mathcal{C}^1(I)$ a direct calculation yields the same estimate plus the constant involved. □

Next we will apply this lemma in two dimensions to obtain an anisotropic estimate for rectangular domains.

Lemma 3. *Let h_x, $h_y > 0$, $R := (0, h_x) \times (0, h_y)$, $u \in H^1(R)$. Define $\Pi_R : H^1(R) \to \mathbb{R}$ to be the averaging operator over R,* i.e. $\Pi_R u := \dfrac{1}{h_x h_y} \displaystyle\int_R u \, \mathrm{d}\mathbf{x}$. *Then*

$$\|u - \Pi_R u\|_{0,R} \leq \frac{\sqrt{3}}{3} (h_x \|u_x\|_{0,R} + h_y \|u_y\|_{0,R}).$$

Proof. The idea is to apply Lemma 2 consecutively in the two dimensions. Let us therefor define two operators Π_x and Π_y that act on only one respective dimension:

$$(\Pi_x u)(y) := \frac{1}{h_x} \int_0^{h_x} u(s, y) \mathrm{d}s, \quad (\Pi_y u)(x) := \frac{1}{h_y} \int_0^{h_y} u(x, t) \mathrm{d}t.$$

Also note that since $\mathcal{C}^1(R)$ is dense in $H^1(R)$ the function u is again assumed to be continuously differentiable on R. One easily verifies that $\Pi_R = \Pi_x \circ \Pi_y = \Pi_y \circ \Pi_x$. By the triangle inequality, our quantity of interest will first be split into two parts:

$$\|u - \Pi_R u\|_0 \leq \|u - \Pi_x u\|_0 + \|\Pi_x u - \Pi_R u\|_0. \tag{10}$$

An application of Lemma 2 on the first term yields

$$\begin{aligned} \|u - \Pi_x u\|_0^2 &= \int_0^{h_y} \left[\int_0^{h_x} \left(u(x, y) - \frac{1}{h_x} \int_0^{h_x} u(s, y) \mathrm{d}s \right)^2 \mathrm{d}x \right] \mathrm{d}y \\ &\leq \frac{h_x^2}{3} \int_0^{h_y} \left[\int_0^{h_x} ((\partial_x u)(x, y))^2 \mathrm{d}x \right] \mathrm{d}y = \frac{h_x^2}{3} \|u_x\|_0^2. \end{aligned}$$

Similarly, for the second part an application of Lemma 2 gives

$$\begin{aligned}\|\Pi_R u - \Pi_x u\|_0^2 &= \int_0^{h_x} \left[\int_0^{h_y} \left(\Pi_y(\Pi_x u) - (\Pi_x u)(y)\right)^2 \mathrm{d}y \right] \mathrm{d}x \\ &\le \frac{h_y^2}{3} \int_0^{h_x} \left[\int_0^{h_y} \frac{1}{h_x} \int_0^{h_x} ((\partial_y u)(s,y))^2 \mathrm{d}s \mathrm{d}y \right] \mathrm{d}x \le \frac{h_y^2}{3} \|u_y\|_0^2.\end{aligned}$$

Finally, collecting both estimates in (10) concludes the proof. $\square$

Our next aim is to get equivalent estimates if we assume $u \in H^{2+\sigma}(\Omega)$ for some $\sigma \in (0,1)$. The essential tool will be Theorem 3 (see Appendix) that allows to "interpolate" the known result from Lemma 2 and Lemma 3 to an estimate that is valid in interpolated spaces.

Thus, using the same notation as in Theorem 3, let us define the respective spaces and the action of the operator T as follows

$$\begin{aligned} A_1 &:= L_2(\Omega), \quad A_2 := H^1(\Omega), \\ B &:= L_2(\Omega), \quad Tu := u - \Pi_\Omega u \end{aligned}$$

Indeed, for the rectangle $R := (x_0,\, x_0 + h_x) \times (y_0,\, y_0 + h_y)$ we know from Lemma 3 that

$$\|Tu\|_{0,R} \le \frac{1}{3}\sqrt{3} h_R\, |u|_{1,R} \quad \text{with} \quad h_R := \sqrt{h_x^2 + h_y^2} \tag{11}$$

and therefore $\|T\|_{H^1(R) \to L_2(R)} \le \frac{1}{3}\sqrt{3}\, h_R$.

By Cauchy-Schwarz the projection Π_R is L_2 stable, *i.e.*

$$\|\Pi_R u\|_{0,R} = |\Pi_R u|\, |R|^{\frac{1}{2}} = \frac{1}{|R|^{\frac{1}{2}}} \left| \int_R u \, \mathrm{d}\mathbf{x} \right| \le \|u\|_{0,R}.$$

Hence,

$$\|Tu\|_{0,R} \le \|u\|_{0,R} + \|\Pi_R u\|_{0,R} \le 2\, \|u\|_{0,R} \tag{12}$$

which yields $\|T\|_{L_2(R) \to L_2(R)} \le 2$. Eventually, an application of Theorem 3 (Appendix) yields a bound on the operator norm of T considered on its domain $[L_2(R), H^1(R)]_{2,\sigma}$:

$$\|T\|_{[L_2(R),H^1(R)]_{2,\sigma} \to L_2(R)} \le 2^{1-\sigma} (\frac{1}{3}\sqrt{3} h)^\sigma.$$

Summarizing we derived the following

Lemma 4. *Let $h_x,\, h_y > 0$, $R := (x_0,\, x_0 + h_x) \times (y_0,\, y_0 + h_y)$, $u \in [L_2(R), H^1(R)]_{2,\sigma}$ for some $\sigma \in [0,1]$ and the averaging operator Π_R be defined as above. Then the following estimate holds*

$$\|u - \Pi_R u\|_{0,R} \le C h^\sigma \|u\|_{[L_2(R),H^1(R)]_{2,\sigma}},$$

where $h := \mathrm{diam}(R)$ and the occuring constant C is independent of the element aspect ratio.

Now let us continue to get an anisotropic estimate of the diffusion term. Resuming (7) we proceed in the same fashion as before: Define $\overline{u_{xx}}_{|T} := \Pi_T u_{xx}$ and $\overline{u_{xy}}_{|T} := \Pi_T u_{xy}$ to be the averages of the respective functions on element T and insert the particular polynomial

$$p(x,y) = \frac{1}{2}\overline{u_{xx}}_{|T}\, x^2 + \overline{u_{xy}}_{|T}\, xy$$

into (7). Eventually, Lemma 4 yields the fractional estimate

$$\varepsilon|\int_T (u-u^I)_x v_x \,\mathrm{d}\mathbf{x}| \leq C\varepsilon h^\sigma \left(h_x\|u_{xx}\|_{[L_2(T),H^1(T)]_{2,\sigma}} + h_y\|u_{xy}\|_{[L_2(T),H^1(T)]_{2,\sigma}}\right)\|v_x\|_{0,T},$$

on every element $T \in \mathcal{T}_h$ and hence globally using Lemma 8 of the Appendix:

$$\varepsilon|\int_\Omega \nabla(u-u^I)\cdot\nabla v \,\mathrm{d}\mathbf{x}| \leq C\varepsilon^{\frac{1}{2}} h^{1+\sigma}\left(\|u_{xx}\|_{[L_2(\Omega),H^1(\Omega)]_{2,\sigma}} + \|u_{xy}\|_{[L_2(\Omega),H^1(\Omega)]_{2,\sigma}}\right)|v|_{1,\varepsilon,\Omega}.$$

Finally, by applying norm equivalence of the fractional Sobolev norm and the norm in the interpolation space (cf. Theorem 4 in the Appendix) we can summarize the result in the following

Lemma 5. *Let the function u satisfy the regularity assumption $u \in H^{2+\sigma}(\Omega)$ for some $\sigma \in [0,1]$. Then the following estimate holds with a constant C independent of the mesh:*

$$\varepsilon|\int_\Omega \nabla(u-u^I)\cdot\nabla v \,\mathrm{d}\mathbf{x}| \leq C\varepsilon^{\frac{1}{2}} h^{1+\sigma}\|u\|_{2+\sigma,\Omega}\|v\|_{1,\varepsilon,\Omega}. \tag{13}$$

2.2. The convection term

Let us now continue with the estimation of the convection term

$$\int_\Omega \mathbf{b}\cdot\nabla(u-u^I)v \,\mathrm{d}\mathbf{x}$$

under low regularity assumptions. First note that without loss of generality it is possible to assume that the vector field $\mathbf{b}$ is piecewise constant on every element. This can easily be seen by inserting an elementwise constant interpolant $\hat{\mathbf{b}}$ of $\mathbf{b}$:

$$|\int_\Omega (\mathbf{b}-\hat{\mathbf{b}})\cdot\nabla(u-u^I)v \,\mathrm{d}\mathbf{x}| \leq h\|\mathbf{b}\|_{1,\infty,\Omega}\; |u-u^I|_{1,\Omega}\; \|v\|_{0,\Omega} \leq Ch^2|u|_{2,\Omega}\; \|v\|_{0,\Omega}.$$

Thus, we integrate by parts and obtain

$$\int_\Omega \mathbf{b}\cdot\nabla(u-u^I)v \,\mathrm{d}\mathbf{x} = -\int_\Omega (u-u^I)(\nabla\cdot\mathbf{b})v \,\mathrm{d}\mathbf{x} - \int_\Omega (u-u^I)\mathbf{b}\cdot\nabla v \,\mathrm{d}\mathbf{x}. \tag{14}$$

The first term can be handled by standard interpolation estimates,

$$|\int_\Omega (u-u^I)(\nabla\cdot\mathbf{b})v \,\mathrm{d}\mathbf{x}| \leq Ch^2|u|_{2,\Omega}\; \|v\|_{0,\Omega}.$$

Hence only the second term still makes trouble. Motivated by what has been done in the previous subsection we will again add and subtract the second order polynomials

$$p_{|T}(x,y) = \frac{1}{2}\overline{u_{xx}}_{|T}\, x^2 + \frac{1}{2}\overline{u_{yy}}_{|T}\, y^2$$

on every element. Thus, the second term of (14) can be split into

$$\int_\Omega (u-u^I)\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x} = \sum_T \int_T ((u-p)-(u-p)^I)\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x} \;+ \sum_T \int_T (p-p^I)\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x}. \tag{15}$$

This decomposition allows us to treat the first part similar to the steps applied to the diffusion term. Additionally, an inverse inequality gives

$$\begin{aligned}\left|\sum_T \int_T ((u-p)-(u-p)^I)\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x}\right| &\le Ch^2 \sum_T |u-p|_{2,T}\,\|\mathbf{b}\cdot\nabla v\|_{0,T}\\ &\le Ch^{1+\sigma}\,|u|_{2+\sigma,\Omega}\,\|v\|_{0,\Omega}.\end{aligned}$$

We continue with the second term in (15). Since $p\in\mathcal{P}_2$ and $\mathbf{b}\cdot\nabla v\in\mathcal{P}_1$ one can show directly that

$$\sum_T \int_T (p-p^I)\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x} = -\frac{1}{12}\sum_T\left(h_x^2\int_T p_{xx}\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x} + h_y^2\int_T p_{yy}\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x}\right),$$

and hence estimate

$$\left|\sum_T \int_T (p-p^I)\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x}\right| \le C\left|\sum_T (h_x^2\,\overline{u_{xx}}_{|T} + h_y^2\,\overline{u_{yy}}_{|T})\int_T (b_1 v_x + b_2 v_y)\,\mathrm{d}\mathbf{x}\right| \tag{16}$$

After expanding (16) every single contributions of these four summands can be estimated in the same way. As an example we demonstrate the steps for one occuring term. Hereby, the set $\tilde{\mathcal{T}}_h$ is obtained by discarding from $\mathcal{T}_h$ all elements which share an edge with the east boundary, $\{\mathbf{x}\in\partial\Omega : x=1\}$, of Ω. Moreover, for every element $T\in\tilde{\mathcal{T}}_h$, T^+ shall denote the "east" neighboring element of T. The sets e_T and w_T refer to the "east" or the "west" boundary of T, respectively:

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} h_x^2\,\overline{u_{xx}}_{|T}\int_T b_1 v_x\,\mathrm{d}\mathbf{x} &= h_x^2 \sum_{T\in\mathcal{T}_h}\overline{u_{xx}}_{|T}\,b_{1|T}\left(\int_{e_T} - \int_{w_T}\right) v\,\mathrm{d}y\\ &= h_x^2\sum_{T\in\tilde{\mathcal{T}}_h}(\overline{u_{xx}}_{|T}\,b_{1|T} - \overline{u_{xx}}_{|T^+}\,b_{1|T^+})\int_{e_T} v\,\mathrm{d}y\\ &= h_x^2\sum_{T\in\tilde{\mathcal{T}}_h}(\overline{u_{xx}}_{|T}\,(b_{1|T}-b_{1|T^+}) + b_{1|T^+}(\overline{u_{xx}}_{|T} - \overline{u_{xx}}_{|T^+}))\int_{e_T} v\,\mathrm{d}y.\end{aligned} \tag{17}$$

Before we can continue to estimate the differences of the means on neighboring elements we need the following two lemmas.

Lemma 6. *Let $u\in H^1(I)$ be defined on the interval $I:=(0,2h)$ for some $h>0$ and let $\Pi_{(a,b)}u := \frac{1}{b-a}\int_a^b u(x)\,\mathrm{d}x$ be the mean value operator on the interval $(a,b)\subset I$. Then the difference of the two neighboring means $\Pi_{(0,h)}u$ and $\Pi_{(h,2h)}u$ can be bounded by*

$$|\Pi_{(0,h)}u - \Pi_{(h,2h)}u| \le \frac{2\sqrt{2}}{3}h^{\frac{1}{2}}|u|_{1,I}.$$

Proof. Let us first assume that the function $u \in \mathcal{C}^1(I)$. Essentially, we insert a zero such that we can introduce the derivative of u by means of the fundamental theorem of calculus. The general statement follows from the density of $\mathcal{C}^1(I)$ in $H^1(I)$:

$$
\begin{aligned}
|\Pi_{(0,h)}u - \Pi_{(h,2h)}u| &= \frac{1}{h}\left|\int_0^h (u(x)-u(h))\mathrm{d}x + \int_h^{2h}(u(h)-u(y))\mathrm{d}y\right| \\
&\le \frac{1}{h}\int_0^h \left|\int_h^x u'(s)\mathrm{d}s\right|\mathrm{d}x + \frac{1}{h}\int_h^{2h}\left|\int_y^h u'(s)\mathrm{d}s\right|\mathrm{d}y \\
&\le \frac{2}{3}h^{\frac{1}{2}}\left(|u|_{1,\,(0,h)} + |u|_{1,\,(h,2h)}\right).
\end{aligned}
$$

□

Lemma 7. *Let $D \subset \mathbb{R}^2$ be open such that $\bar{D} := \bar{T}_1 \cup \bar{T}_2$ with the neighboring rectangles*

$$T_1 := (0,\,h_x)\times(0,\,h_y) \quad \textit{and} \quad T_2 := (h_x,\,2h_x)\times(0,\,h_y).$$

Then the neighboring means of some function $u \in H^1(D)$ can be bounded by

$$|\Pi_{T_1}u - \Pi_{T_2}u| \le \frac{2\sqrt{2}}{3}\frac{h_x^{\frac{1}{2}}}{h_y^{\frac{1}{2}}}\,\|u_x\|_{0,\,D}.$$

Proof. We want to apply Lemma 6. Thus, write

$$
\begin{aligned}
|\Pi_{T_1}u - \Pi_{T_2}u| &= \frac{1}{h_y}\left|\int_0^{h_y}\frac{1}{h_x}\left(\int_0^{h_x}u(x,y)\mathrm{d}x - \int_{h_x}^{2h_x}u(x,y)\mathrm{d}x\right)\mathrm{d}y\right| \\
&\le \frac{2\sqrt{2}}{3}\frac{h_x^{\frac{1}{2}}}{h_y}\left|\int_0^{h_y}\left(\int_0^{2h_x}((\partial_x u)(x,y))^2\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}y\right| \\
&\le \frac{2\sqrt{2}}{3}\frac{h_x^{\frac{1}{2}}}{h_y^{\frac{1}{2}}}\|u_x\|_{0,\,D}.
\end{aligned}
$$

□

The fractional version of Lemma 7 is again derived in interpolation spaces.

Corollary 1. *Let the assumptions be as in Lemma 7 and $\sigma \in [0,1]$. Then*

$$|\Pi_{T_1}u - \Pi_{T_2}u| \le C\frac{h_x^{\sigma-\frac{1}{2}}}{h_y^{\frac{1}{2}}}\,\|u\|_{[L_2(D),H^1(D)]_{2,\sigma}}.$$

Proof. Since

$$|\Pi_{T_1}u - \Pi_{T_2}u| \le \frac{\sqrt{2}}{h_x^{\frac{1}{2}}h_y^{\frac{1}{2}}}\,\|u\|_{0,\,D}$$

the statement follows from Lemma 7 using interpolation spaces. □

Eventually, from (17) we move on by estimating

$$
\begin{aligned}
\left|\sum_{T\in\mathcal{T}_h} h_x^2\,\overline{u_{xx}}_{|T}\int\limits_T b_1 v_x \mathrm{d}\mathbf{x}\right| &\le Ch_x^2 \sum_{T\in\tilde{\mathcal{T}}_h}\left(h_x\,|\overline{u_{xx}}_{|T}| + \frac{h_x^{\sigma-\frac{1}{2}}}{h_y^{\frac{1}{2}}}\|u_{xx}\|_{[L_2(T\cup T^+),H^1(T\cup T^+)]_{2,\sigma}}\right)\int\limits_{e_T} v\,\mathrm{d}y\\
&\le Ch_x^2\left(\|u_{xx}\|_{0,\Omega} + h_x^{\sigma-1}\|u_{xx}\|_{[L_2(\Omega),H^1(\Omega)]_{2,\sigma}}\right)\|v\|_{0,\Omega}\\
&\le Ch_x^{1+\sigma}\|u\|_{2+\sigma,\Omega}\|v\|_{0,\Omega}.
\end{aligned}
$$

Altogether, we continue from (16) and obtain

$$
\left|\sum_T\int_T (p-p^I)\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x}\right| \le Ch^{1+\sigma}\,\|u\|_{2+\sigma,\Omega}\|v\|_{0,\Omega}.
$$

Summarizing, the convection term admits the following bound:

$$
\int_\Omega \mathbf{b}\cdot\nabla(u-u^I)v\,\mathrm{d}\mathbf{x} \le Ch^{1+\sigma}\,\|u\|_{2+\sigma,\Omega}\|v\|_{0,\Omega}. \tag{18}
$$

Remark 2. *Note that by using standard interpolation estimates, one obtains a bound of order $\mathcal{O}(h^2)$ for the reaction term.*

Finally, the collection of every single contribution together with (5) allows to summarize the results in the following

Theorem 1. *Let $\sigma \in [0,1]$ and assume $u \in H^{2+\sigma}(\Omega)$ for the exact solution of (3). Furthermore, let u^I be the nodal interpolant of u and u_h its bilinear finite element approximation. Then the following bound holds:*

$$
\|u^I - u_h\|_{1,\varepsilon} \le C(\varepsilon^{\frac{1}{2}}h^{1+\sigma} + h^{1+\sigma} + h^2)\|u\|_{2+\sigma}. \tag{19}
$$

3. Numerical experiment for the Galerkin method

Let us now have a look on how numerical experiments reflect the order of convergence suggested by the theoretical estimates. Because the Galerkin method is not the adequate method for $\varepsilon \ll 1$ we only present results for $\varepsilon = 1$ and consider the following convection-diffusion-reaction boundary value problem

$$
-\Delta u - 0.5u_x - u_y + u = 1,\ \ u_{|\partial\Omega} = 0
$$

in domains Ω_a that are parallelograms spanned by the two vectors $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-x_0\\1\end{pmatrix}$, $x_0 \ge 0$. It is well known that the solution exhibits corner singularities in dependence of the obtuse angle at the origin. Hence, the parameter x_0 controls the strength of the singularity at the origin and thereby the regularity of the solution. A closer investigation using regularity theory in non-smooth domains (cf. [8], [3] and [6]) reveals that the solution has the following regularity in dependence of the parameter x_0:

$$
u \in H^{s-\delta}(\Omega), \quad \forall\delta > 0,\ \text{with } s = 1 + \frac{1}{0.5 + \arctan(x_0)/\pi}.
$$

x_0	0.75	0.5	0.25
TOC	1.4188	1.5442	1.7302
EOC($l-2$)	1.4308	1.5627	1.7555
EOC($l-1$)	1.4241	1.5534	1.74865
EOC(l)	1.4230	1.5504	1.7430

Table 1: Rates of convergence for $\|u^I - u_h\|_{1,\varepsilon}$.

For three different values of x_0 we compare in Table 1 the theoretical and experimental orders of convergence (TOC/EOC) for the quantity $\|u^I - u_h\|_{1,\varepsilon}$. The EOCs are displayed for the last three levels of uniform grid refinement. Since for this example we cannot access the exact solution, a reference solution is computed on level $l+1$ using biquadratic Q_2 elements that substitutes for the exact solution. One observes that the numerical rates are astonishingly close to the theoretical orders of supercloseness.

4. SDFEM error analysis

Let us now turn to the streamline-diffusion finite element method (SDFEM) as a discretization for the boundary value problem (1). Using the same finite element space V_h as in Section 2, the discrete problem related to the SDFEM reads as follows:

Find $u \in V_h$ such that for all $v \in V_h$

$$a_{SD}(u,v) := a(u,v) + \sum_{T\in\mathcal{T}_h} \delta_T(-\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu,\, \mathbf{b}\cdot\nabla v)_T = (f,v) + \sum_{T\in\mathcal{T}_h} \delta_T(f, \mathbf{b}\cdot\nabla v)_T, \tag{20}$$

where $(.,.)$ and $(.,.)_T$ denote the standard L_2 scalar products on Ω or T respectively. The parameters δ_T have to be chosen for all elements T. If for an arbitrary element we define the local Péclet number by

$$\mathrm{Pe}_T := \frac{\|\mathbf{b}\|_{\infty,T} h_T}{2\varepsilon}$$

then the analysis of the SDFEM (cf. [7]) suggests on isotropic meshes to choose the following values for δ_T:

$$\delta_T = \begin{cases} \delta_0 h_T / \|\mathbf{b}\|_{\infty,T}, & \text{if } Pe_T > 1 \\ \delta_1 h_T^2/\varepsilon, & \text{if } Pe_T \le 1, \end{cases}$$

with appropriate user chosen constants δ_0 and δ_1. Next we introduce the streamline-diffusion norm

$$\|v\|_{SD}^2 := \varepsilon|v|_1^2 + \|v\|_0^2 + \sum_{T\in\mathcal{T}_h} \delta_T \|\mathbf{b}\cdot\nabla v\|_{0,T}^2, \tag{21}$$

in which one shows coercivity of the bilinear form, cf. [7]. Since the SDFEM still preserves consistency of the discretization we have, similar to (5),

$$\alpha_{SD}\|u^I - u_h\|_{SD}^2 \le a_{SD}(u^I - u_h, u^I - u_h) = a_{SD}(u^I - u, u^I - u_h). \tag{22}$$

The streamline-diffusion finite element method was proposed by Hughes et al. in [4] and first analyzed by Johnson and Nävert [5] in order to handle the known instabilities of the Galerkin method. It was proved in the SD-norm that for linear or bilinear elements

$$\|u - u_h\|_{SD} \leq C(\varepsilon^{\frac{1}{2}} + h^{\frac{1}{2}})h|u|_2, \tag{23}$$

which carries over to a bound for $\|u^I - u_h\|_{SD}$ and also implies convergence in L_2 with order $\mathcal{O}(h^{\frac{3}{2}})$ (assuming $\varepsilon \leq Ch$, the convection-dominated case). Remark that for problem (1) the application of the Nitsche-Trick for proving optimal L_2 convergence is not possible.

In several papers the optimal accuracy of the SDFEM in L_2 was discussed depending on the geometry of the mesh (see [12]). Using Lin identities (cf. [10], [9] and [11]) one can prove a supercloseness result on tensor product meshes or uniform triangular meshes of the type

$$\|u^I - u_h\|_{SD} \leq C(\varepsilon^{\frac{1}{2}}h^2 + h^2)(|u|_3 + |u|_{2,\infty}) \tag{24}$$

which implies L_2 convergence of optimal order $\mathcal{O}(h^2)$ in the convection-dominated case.

By imitating the techniques for the Galerkin bilinear form in Section 2 on the two essential additional stabilizing terms in (20) we derive the following bounds

$$\sum_T \delta_T \int_T \mathbf{b} \cdot \nabla(u - u^I)\mathbf{b} \cdot \nabla v \,\mathrm{d}\mathbf{x} \leq Ch^{1+\sigma}\|u\|_{2+\sigma,\Omega}\|v\|_{SD,\Omega},$$
$$\varepsilon \sum_T \delta_T \int_T \Delta u \,\mathbf{b} \cdot \nabla v \,\mathrm{d}\mathbf{x} \leq C\varepsilon h^{\sigma}\|u\|_{2+\sigma,\Omega}\|v\|_{SD}.$$

Together with the estimates for the Galerkin bilinear form, inequalitites (22) and (23) the above bounds yield the result

$$\begin{aligned}\|u^I - u_h\|_{SD} &\leq C(\varepsilon^{\frac{1}{2}}h^{1+\sigma} + h^{\max\{\frac{3}{2},1+\sigma\}} + \varepsilon^{\frac{1}{2}}\min\{h, \varepsilon^{\frac{1}{2}}h^{\sigma}\})\|u\|_{2+\sigma,\Omega} \\ &\leq Ch^{\max\{\frac{3}{2},1+\sigma\}}\|u\|_{2+\sigma,\Omega}, \quad \text{for } \varepsilon \leq Ch.\end{aligned}$$

For $\sigma < \frac{1}{2}$ these estimates seem to be suboptimal since the regularity does not have any impact on the rate of convergence in this case. In fact, it is possible to improve these bounds by pursuing a different strategy. The idea is to take the known bounds for H^2- and H^3-regularity and apply Theorem 3 of the Appendix to obtain fractional estimates. The bounds (24) obtained via Lin identities, however, are not practicable for the interpolation theorem, since they involve norms in two different spaces to measure the regularity. However, using the techniques from the Galerkin estimates above, it is possible to get rid of the $|\,.\,|_{2,\infty}$ norm at the right hand side of (24):

Let us first consider the convection term. From the derivation of (18) and the analysis leading to the standard estimate (23) (cf. [7]) one recalls that

$$\left|(\mathbf{b} \cdot \nabla(u - u^I), v)\right| \leq Ch^2\|u\|_3\|v\|_{SD}, \;\; \left|(\mathbf{b} \cdot \nabla(u - u^I), v)\right| \leq Ch^{\frac{3}{2}}|u|_2\|v\|_{SD,\Omega}. \tag{25}$$

For every $v \in V_h$ let us define the operator $T_v : H^2(\Omega) \to \mathbb{R} : u \mapsto \int_\Omega \mathbf{b} \cdot \nabla(u - u^I)v \,\mathrm{d}\mathbf{x}$. Since from (25) we know upper bounds for $\|T_v\|_{H^2(\Omega)\to\mathbb{R}}$ and $\|T_v\|_{H^3(\Omega)\to\mathbb{R}}$ an application of Theorem 3 yields the interpolated estimate

$$\left|\int_\Omega \mathbf{b} \cdot \nabla(u - u^I)v \,\mathrm{d}\mathbf{x}\right| \leq Ch^{\frac{3}{2}+\frac{\sigma}{2}}\|u\|_{2+\sigma}\|v\|_{SD}. \tag{26}$$

	$\|u\|_2$	$\|u\|_3$	$\|u\|_{2+\sigma}$
$\int \mathbf{b}\cdot\nabla(u-u^I)v$	$h^{\frac{3}{2}}$	h^2	$h^{\frac{3}{2}+\frac{\sigma}{2}}$
$\sum \delta_T \int \mathbf{b}\cdot\nabla(u-u^I)\mathbf{b}\cdot\nabla v$	$h^{\frac{3}{2}}$	h^2	$h^{\frac{3}{2}+\frac{\sigma}{2}}$
$\varepsilon \sum \delta_T \int \Delta(u-u^I)\mathbf{b}\cdot\nabla v$	$\varepsilon^{\frac{1}{2}}h$	εh	$\varepsilon^{\frac{1}{2}+\frac{\sigma}{2}}h$

Table 2: Interpolation results.

Analogously, a similar definition of operators $\{T_v\}_{v\in V_h}$ adapted to the remaining terms in the stabilized bilinear form $a_{SD}(.,.)$ together with Theorem 3 of the Appendix yields their respective fractional bounds. Table 2 shows the known error bounds for H^2- and H^3-regularity and the interpolated result. A collection of every single contribution and the representation of the error (22) yields the final estimate that only requires $H^{2+\sigma}$-regularity:

Theorem 2. *Let $\sigma \in [0,1]$ and assume $u \in H^{2+\sigma}(\Omega)$ for the exact solution of* (3). *Furthermore, let u^I be the nodal interpolant of u and u_h its bilinear finite element approximation using streamline diffusion stabilization. Then the following bound holds:*

$$\|u^I - u_h\|_{SD} \leq C(\varepsilon^{\frac{1}{2}}h^{1+\sigma} + h^{\frac{3}{2}+\frac{\sigma}{2}} + h^2 + \varepsilon^{\frac{1}{2}+\frac{\sigma}{2}}h)\,\|u\|_{2+\sigma}. \tag{27}$$

5. Numerical Experiments for the SDFEM

The folowing numerical experiment shall illustrate the dependency of the rate of supercloseness on the regularity of the solution in comparison with (27). Thus, consider the following homogeneous Dirichlet boundary value problem in the domain $\Omega = (0,1)^2$ with $\varepsilon = 10^{-3}$:

$$-\varepsilon\Delta u - \begin{pmatrix}1+x\\1+y\end{pmatrix}\cdot\nabla u + (2+x^2)u = f. \tag{28}$$

We assume the exact solution of (28) to be

$$u_{\text{ex}}(\mathbf{x}) = |\mathbf{x}|^{-\alpha}xy(1-x)(1-y)$$

and determine the source term f such that u_{ex} satisfies the differential equation (28). A closer investigation shows that the parameter $\alpha > 0$ controls the regularity in the following way

$$u_{\text{ex}} \in H^{3-\alpha-\delta}(\Omega), \quad \forall\delta > 0.$$

For the computations we choose several values for α. The corresponding regularity of u_{ex} together with the experimental orders of supercloseness of the last three uniform refinements are displayed in Table 3. Concerning the decay of $\|u^I - u_h\|_{SD}$ one observes that indeed the order of convergence depends on the regularity of the solution. However, the observed rates reflect slightly better convergence properties than the theory predicts.

$s : u_{\text{ex}} \in H^s(\Omega)$	2.1	2.2	2.3	2.7
EOC($l-2$)	1.6251	1.7398	1.8549	2.0416
EOC($l-1$)	1.6176	1.7294	1.8442	2.0260
EOC(l)	1.6140	1.7229	1.8366	2.0162

Table 3: EOC for $\|u^I - u_h\|_{SD}$.

A. Appendix - Interpolation Spaces

Here we give a short survey of some facts about interpolation spaces as far as their properties have been used in the article. The interpolation spaces are to be understood in the sense of the "real interpolation method". For more information and a proof of the main theorems the reader is referred to [2] and [1].

First, for two Banach spaces A_1 and A_2 with $A_2 \subset A_1$ we give a definition of the interpolation space $[A_1, A_2]_{2,\sigma}$ (which is also a Banach space).

Definition 1. *Let A_1, A_2 with $A_2 \subset A_1$ be two Banach spaces and $\sigma \in (0,1)$. The Banach space $[A_1, A_2]_{2,\sigma}$ consists of all $u \in A_1$ that are finite in the following norm*

$$\|u\|_{[A_1,A_2]_{2,\sigma}} := \left(\int_0^\infty t^{-2\sigma-1} \left(\inf_{v \in A_2} (\|u-v\|_{A_1} + t\|v\|_{A_2}) \right)^2 \mathrm{d}t \right)^{\frac{1}{2}}. \tag{29}$$

Moreover we agree on the convention that $[A_1, A_2]_{2,0} := A_1$ and $[A_1, A_2]_{2,1} := A_2$.

Concerning interpolation spaces, the main tool used in this article is the following Theorem formulated in terms of operators on these very spaces.

Theorem 3. *Let A_1, A_2 with $A_2 \subset A_1$ and B be three Banach spaces and let T be a linear operator that maps A_1 to B. Furthermore let A_{12} denote the interpolation space $[A_1, A_2]_{2,\sigma}$ for some $\sigma \in (0,1)$. Then T can be considered as a linear operator from A_{12} to B. Moreover, the corresponding operator norm satisfies*

$$\|T\|_{A_{12} \to B} := \sup_{u \in A_{12} \setminus \{0\}} \frac{\|Tu\|_B}{\|u\|_{A_{12}}} \leq \|T\|_{A_1 \to B}^{1-\sigma} \|T\|_{A_2 \to B}^{\sigma}.$$

Furthermore, it is possible to characterize the following particular interpolation spaces as fractional order Sobolev spaces. A proof of the subsequent theorem can be found, *e.g.*, in [2].

Theorem 4. *Let $\sigma \in [0,1]$. For all domains Ω with Lipschitz boundary one has*

$$H^\sigma(\Omega) = [L_2(\Omega), H^1(\Omega)]_{\sigma,2}$$

and the norms are equivalent.

Also note that in order to sum up estimates that were derived locally one needs the following summation property for (29) which follows from a direct calculation.

Lemma 8. *Let Ω be a domain and $\mathcal{T}$ a partition on Ω. Then*

$$\sum_{T \in \mathcal{T}_h} \|u\|^2_{[L_2(T), H^1(T)]_{\sigma,2}} \leq 2\|u\|^2_{[L_2(\Omega), H^1(\Omega)]_{\sigma,2}}.$$

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