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## FUZZY SETS AND SMALL SYSTEMS

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### Abstract

Independently with [7] a corresponding fuzzy approach has been developed in [3–5] with applications in measure theory. One of the results the Egoroff theorem has been proved in an abstract form. In [1] a necessary and sufficient condition for holding the Egoroff theorem was presented in the case of a space with a monotone measure. By the help of [2] and [6] we prove a variant of the Egoroff theorem stated in [4].

### 1. Introduction

In [7] the notion of a fuzzy subset  $A$  of a space  $X$  has been defined as a mapping  $A : X \rightarrow [0, 1]$ . Especially, if  $A : X \rightarrow \{0, 1\}$ , then  $A$  can be identified with a classical set  $B \subset X$  by the help of the equality  $A = \chi_B$ .

Almost at the same time the notion of a set of small measure has been characterized in [3–5] using a sequence  $(\mathcal{N}_n)_{n=1}^{\infty}$  of subfamilies of a  $\sigma$ -algebra  $\mathcal{S} \subset 2^X$  satisfying the following properties:

- (i)  $\emptyset \in \mathcal{N}_n, \mathcal{N}_{n+1} \subset \mathcal{N}_n$  for every  $n \in \mathbb{N}$ ,
- (ii) if  $A \in \mathcal{N}_n, B \in \mathcal{S}$  and  $B \subset A$ , then  $B \in \mathcal{N}_n$ ,
- (iii) if  $A, B, C \in \mathcal{N}_n$ , then  $A \cup B \cup C \in \mathcal{N}_{n-1}$ ,
- (iv) if  $A_i \supset A_{i+1}$  ( $i = 1, 2, \dots$ ) and  $\bigcap_i A_i = \emptyset$ , then to every  $n \in \mathbb{N}$  there is  $i$  such that  $A_i \in \mathcal{N}_n$ .

The classical Egoroff theorem states that if a sequence  $(f_n)_n$  of real measurable functions converges to a measurable function  $f$  almost everywhere, then it converges almost uniformly, i.e.  $\forall \varepsilon > 0 \exists A \in \mathcal{A}$  such that  $\mu(A) < \varepsilon$  and  $(f_n)_n$  converges uniformly to  $f$  on  $X - A$ .

**Definition.** We say that a sequence  $(f_n)_n$  converges to  $f$  almost everywhere, if  $\{x \in X; f_n(x) \text{ does not converge to } f(x)\} \in \mathcal{N}_n$  for every  $n$ . We say that  $(f_n)_n$  converges to  $f$  almost uniformly, if for any  $n \in \mathbb{N}$  there exists  $A \in \mathcal{N}_n$  such that  $(f_n)$  converges uniformly to  $f$  on  $X - A$ .

## 2. Egoroff theorem

**Theorem.** Let  $(\mathcal{N}_n)_n$  be a small system of subfamilies of a measurable space  $(X, \mathcal{S})$ . Let  $(f_n)_n$  converges to  $f$  almost everywhere. Then  $(f_n)_n$  converges to  $f$  almost uniformly.

*Proof.* First we use a result from [6]: If  $(\mathcal{N}_n)_n$  satisfies (i)–(iv), then there exists a monotone continuous function  $\mu : \mathcal{S} \rightarrow [0, 1]$  such that

$$\mathcal{N}_n = \{A \in \mathcal{S}; \mu(A) < 3^{-n}\},$$

$n = 1, 2, 3, \dots$  In [1] the following theorem has been proved: A monotone function  $\mu : \mathcal{S} \rightarrow [0, 1]$  satisfies the Egoroff theorem if and only if it satisfies the following condition (E):

For every double sequence  $\{E_n^{(m)}\}$  of measurable sets which satisfies

$$E_n^{(m)} \searrow E^{(m)} (n \rightarrow \infty), \quad \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0$$

there exist increasing sequences  $\{n_i\}_{i=1}^{\infty}$  and  $\{m_i\}_{i=1}^{\infty}$  of natural numbers such that

$$\lim_{k \rightarrow \infty} \mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) = 0.$$

We are going to prove that the monotone continuous set function  $\mu$  satisfies condition (E). Let  $\{E_n^{(m)}\}$  is double sequence of measurable sets for which

$$E_n^{(m)} \searrow E^{(m)} (n \rightarrow \infty), \quad \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0.$$

From the monotonicity it follows that

$$0 = \mu(\emptyset) \leq \mu(E^{(m_0)}) \leq \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0.$$

We have proven that  $\mu(E^{(m)}) = 0$  for arbitrary  $m$ . From this it follows that there is a natural number  $n_1$  for which

$$\mu(E_{n_1}^{(1)}) \leq \frac{1}{3}.$$

Similarly there is a number  $n_2 > n_1$  for which

$$\mu(E_{n_2}^{(2)}) \leq \frac{1}{3^2},$$

etc. Putting  $m_i = i$ , we get

$$\mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) \leq \sum_{i=k}^{\infty} \frac{1}{3^i} = \frac{\frac{1}{3^k}}{1 - \frac{1}{3}} = \frac{1}{2 \cdot 3^{k-1}}.$$

From this it follows that

$$\lim_{k \rightarrow \infty} \mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) = 0.$$

□

### 3. Conclusion

We presented a new proof of the Egoroff theorem for small systems [4]. It follows from a representation theorem in [6] and the Egoroff theorem for monotone measures in [2].

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