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IDENTIFICATION PROBLEM FOR NONLINEAR BEAM — EXTENSION FOR DIFFERENT TYPES OF BOUNDARY CONDITIONS

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Abstract: Identification problem is a framework of mathematical problems dealing with the search for optimal values of the unknown coefficients of the considered model. Using experimentally measured data, the aim of this work is to determine the coefficients of the given differential equation. This paper deals with the extension of the continuous dependence results for the Gao beam identification problem with different types of boundary conditions by using appropriate analytical inequalities with a special attention given to the Wirtinger's inequality and its modification. On the basis of these results for the different types of the boundary conditions the existence theorem for the identification problem can be proven.

Keywords: identification problem, nonlinear Gao beam, Wirtinger's inequality, Wirtinger-Poincaré-Almansi inequality

MSC: 26D20, 49J15, 65L09, 74K10

1. Introduction

Beams are commonly used in engineering constructions and there are many practical applications for parameter identification problem. This paper deals with a nonlinear Gao beam model. A problem of identifying coefficients in the Gao beam is presented in a recent paper [11], where the aim is to find unknown material parameters for this beam by using an optimal control approach. The existence of at least one solution of the optimal control problem is proven by using continuous dependence of the solution on the material parameters. But the results are proven only for one type of physically relevant boundary conditions. In this paper, in Section 3, we prove the continuous dependence for other types of boundary conditions. The proof is based on analytical inequalities presented in Section 2.

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First, let us start with the nonlinear Gao beam model, which was firstly introduced in [5]. With respect to a small correction of this model which was proposed in [9], the nonlinear Gao beam is given by the fourth order equation:

$$E\,I\,w^{IV} - E\,\alpha\,(w')^2\,w'' + P\,(1-\nu^2)\,w'' = f \qquad \text{in } (0,L), \tag{1}$$

where

$$I \;=\; \frac{2}{3}t^3 b, \qquad \alpha \;=\; 3\,t\,b\,(1-\nu^2), \qquad f \;=\; (1-\nu^2)\,q.$$

Here, E denotes Young's elastic modulus of the material, I is the constant area moment of inertia, w is the deflection of the beam, $2t$ and b represents the thickness and width of the beam, respectively. The Poisson's ratio is represented by the symbol ν, q is the applied transverse load and P stands for the constant axial force acting at the end point of the beam $x = L$. We distinguish two types of acting axial force: $P > 0$ and $P < 0$ causing a compression and a tension, respectively. The beam model needs to be completed by one of the following boundary conditions:

(B1) simply supported beam: $w(0) = w(L) = w''(0) = w''(L) = 0$;

(B2) clamped beam: $w(0) = w'(0) = w(L) = w'(L) = 0$;

(B3) propped cantilever beam: $w(0) = w'(0) = w(L) = w''(L) = 0$;

(B4) cantilever beam: $w(0) = w'(0) = 0$,

$$w''(L) = \; E\,I\,w'''(L) - \tfrac{1}{3}E\,\alpha\,(w'(L))^3 + P\,(1-\nu^2)\,w'(L) = 0.$$

The spaces of admissible displacements are denoted as V_i, $i = 1,\dots,4$, and defined by the corresponding stable boundary conditions contained in (B1),…,(B4):

$$\begin{aligned}
V_1 =& \{v \in H^2((0,L))\colon v(0) = v(L) = 0\},\\
V_2 =& \{v \in H^2((0,L))\colon v(0) = v'(0) = v(L) = v'(L) = 0\},\\
V_3 =& \{v \in H^2((0,L))\colon v(0) = v'(0) = v(L) = 0\},\\
V_4 =& \{v \in H^2((0,L))\colon v(0) = v'(0) = 0\},
\end{aligned}$$

where $H^2((0,L))$ is the Sobolev space which consists of those square integrable functions for which all generalized partial derivatives up to the order two are also square integrable on the interval $(0,L)$. In the following, V will be one of above $V_1,\dots,V_4$.

The variational formulation of the problem (1) reads as follows:

$$\begin{cases} \text{Find } w \in V \text{ such that} \\ a(w,v) + \pi(w,v) = \mathcal{L}(v), \qquad\qquad \forall v \in V, \end{cases} \tag{2}$$

where

$$\begin{aligned}
a(w,v) &= \int_0^L EIw''v''\mathrm{d}x - \int_0^L P(1-\nu^2)w'v'\mathrm{d}x,\\
\pi(w,v) &= \int_0^L E\,t\,b\,(1-\nu^2)(w')^3 v'\mathrm{d}x, \qquad \mathcal{L}(v) = \int_0^L (1-\nu^2)\,q\,v\,\mathrm{d}x.
\end{aligned}$$

The following theorem provides the essential assumptions to the existence of a unique solution to problem (2), see [8].

Theorem 1. *Let E, t, b be positive constants, $\nu \in (0, 0.5\rangle$, $q \in L^2(0, L)$ and $P < \overline{P}$, where $\overline{P} = \dfrac{1}{1-\nu^2} P_{cr}^E$. Then the problem (2) has a unique solution.*

Remark 1. *Since it is not possible to find analytical expression for the critical force for the Gao beam, we use a lower bound $\overline{P}$ which can be expressed by Euler's critical load P_{cr}^E as*

$$\overline{P} = \frac{1}{1-\nu^2} P_{cr}^E = \frac{1}{1-\nu^2} \frac{\pi^2 EI}{(\mathcal{K} \cdot L)^2}, \tag{3}$$

see [4], [11]. *The constant $\mathcal{K}$ depends on the boundary conditions as follows:*

	(B1)	(B2)	(B3)	(B4)
$\mathcal{K}$	1	0.5	0.7	2

The existence and uniqueness of a solution to (2) *can be established under stronger assumptions on physical data. In section* 3 *we will consider the piecewise constant material parameters E and ν. In this case, the assumptions of Theorem 1 have to be modified as follows: let E, ν are positive, piecewise constant functions over a finite, fixed partition of $\langle 0, L\rangle$, b, t are positive constants in $\langle 0, L\rangle$ and*

$$P < \overline{P}_{\min}, \quad \overline{P}_{\min} = \frac{\pi^2 \, \overline{E} \, I}{(1-\overline{\nu}^2)(\mathcal{K} \cdot L)^2},$$

where $\overline{E}$, $\overline{\nu}$ are the minimal values of E, and ν, respectively. The proof of Theorem 1 *generalized for piecewise constant material parameters can be done in a similar way as in* [8].

2. Analytical inequalities

In this section we introduce several analytical inequalities that will be used in the next section for extension of results for the identification problem. Let us start with *Wirtinger's inequality* in its original version, see [10].

Theorem 2. *Let $y(x) \in L^2(\mathbb{R})$ be a periodic function with period 2π and let $y'(x) \in L^2(\mathbb{R})$. If $\int_0^{2\pi} y(x)\, \mathrm{d}x = 0$, then the following inequality holds:*

$$\int_0^{2\pi} (y(x))^2 \, \mathrm{d}x \leq \int_0^{2\pi} (y'(x))^2 \, \mathrm{d}x. \tag{4}$$

The proof of the inequality is based on the Fourier expansions of f and f', see [2]. To better suit our needs, let the Theorem 2 be interpreted for $f \in H^1((0, 2\pi))$: if $\int_0^{2\pi} f(x)\, \mathrm{d}x = 0$, then (4) holds.

The inequality (4) from Theorem 2 can be generalized for a function defined on the interval $\langle 0, L\rangle$. Let us assume that $y(x)$ is a periodic function with period L and let $y'(x) \in L^2(0, L)$. Substituting $t = \frac{L}{2\pi}x$ we obtain a modification of (4):

$$\int_0^L (\widehat{y}(t))^2 \,\mathrm{d}t \le \left(\frac{L}{2\pi}\right)^2 \int_0^L (\widehat{y}'(t))^2 \,\mathrm{d}t, \tag{5}$$

where $\widehat{y}(t) = y(x(t)) = y(\frac{2\pi t}{L})$. If we consider the nonlinear Gao beam with boundary conditions (B2) and set $y(x) = w'(x)$, the assumptions of (5) are satisfied and we get:

$$\int_0^L (w'(x))^2 \,\mathrm{d}x \le \left(\frac{L}{2\pi}\right)^2 \int_0^L (w''(x))^2 \,\mathrm{d}x. \tag{6}$$

The following inequality is known as the *Wirtinger-Poincaré-Almansi inequality*, see [7].

Theorem 3. *Let $y(x)$ be a function defined on the interval $\langle 0, \pi\rangle$ such that $y(0) = y(\pi) = 0$ and $y'(x) \in L^2(0, \pi)$. Then*

$$\int_0^\pi (y(x))^2 \,\mathrm{d}x \le \int_0^\pi (y'(x))^2 \,\mathrm{d}x. \tag{7}$$

The inequality (7) can be generalized for a function y on $\langle 0, L\rangle$. If we have $y(0) = y(L) = 0$ and $y'(x) \in L^2(0, L)$ than

$$\int_0^L (y(x))^2 \,\mathrm{d}x \le \left(\frac{L}{\pi}\right)^2 \int_0^L (y'(x))^2 \,\mathrm{d}x. \tag{8}$$

Similar inequality can be defined on $\langle 0, L\rangle$ for functions satisfying only a single condition $y(0) = 0$, for details see [6].

The key idea is to symmetrize the problem by defining the function $y(x)$ on interval $\langle 0, 2L\rangle$, i.e. for any $x \in \langle L, 2L\rangle$ we define $y(x) = y(L + \xi) = y(L - \xi) = y(2L - x)$, where $\xi = x - L$. Thus, from $y(0) = y(2L)$, $y \in L^2(0, 2L)$ and (8) we have:

$$\int_0^{2L} (y(x))^2 \,\mathrm{d}x \le \left(\frac{2L}{\pi}\right)^2 \int_0^{2L} (y'(x))^2 \,\mathrm{d}x\,.$$

Due to the symmetry on $\langle 0, 2L\rangle$ we get:

$$\int_0^L (y(x))^2 \,\mathrm{d}x \le \left(\frac{2L}{\pi}\right)^2 \int_0^L (y'(x))^2 \,\mathrm{d}x. \tag{9}$$

This idea could be used for the cantilever beam, i.e. the nonlinear beam with the boundary conditions (B4). We can symmetrize the deflection w on the interval $\langle 0, 2L\rangle$, by setting $y(x) = w'(x)$ and using (9) we get:

$$\int_0^L (w'(x))^2 \,\mathrm{d}x \le \left(\frac{2L}{\pi}\right)^2 \int_0^L (w''(x))^2 \,\mathrm{d}x. \tag{10}$$

If we consider the nonlinear Gao beam with the boundary conditions (B1) we can use the same idea as for boundary conditions (B2). The function w' satisfies the assumptions of Theorem 2 modified on interval $\langle 0, L\rangle$, so with respect to (5) we get

$$\int_0^L (w'(x))^2\,\mathrm{d}x \le \left(\frac{L}{2\pi}\right)^2 \int_0^L (w''(x))^2\,\mathrm{d}x. \tag{11}$$

It is obvious that

$$\int_0^L (w'(x))^2\,\mathrm{d}x \le \left(\frac{L}{2\pi}\right)^2 \int_0^L (w''(x))^2\,\mathrm{d}x \le \left(\frac{L}{\pi}\right)^2 \int_0^L (w''(x))^2\,\mathrm{d}x. \tag{12}$$

Using Theorem 3, its generalization (8) and (6) we get

$$\int_0^L (w(x))^2\,\mathrm{d}x \le \left(\frac{L}{\pi}\right)^2 \int_0^L (w'(x))^2\,\mathrm{d}x \le \frac{1}{4}\left(\frac{L}{\pi}\right)^4 \int_0^L (w''(x))^2\,\mathrm{d}x. \tag{13}$$

Finally, for the propped cantilever beam, i.e. for the nonlinear beam with the boundary conditions (B3), we can use the same idea as for cantilever beam which leads to the inequality (10).

3. Identification problem - extension for other types of boundary conditions

In this section we extend the results presented in [11], where the identification of the material parameters given by the Young modulus E and Poisson ratio ν in the Gao beam equation (1) is studied by using an optimal control approach. We suppose that the beam is piecewise homogeneous, i.e. the parameters E, ν are piecewise constant. For this reason let the interval $(0, L)$ be decomposed into mutually disjoint open intervals K_i, called material elements, $i = 1, \cdots, r$, i.e. $K_i \cap K_j = \emptyset$, $\forall i \neq j$ and $\langle 0, L\rangle = \bigcup_{i=1}^{r} \bar{K}_i$. The material parameters are chosen from an admissible set U_{ad} :

$$\begin{aligned} U_{ad} = \{&(E,\nu) \in (L^\infty(0,L))^2 : 0 < E_{\min} \le E \le E_{\max} < \infty \text{ in } (0,L), \\ &0 < \nu \le 0.5 \text{ in } (0,L), (E,\nu)|_{K_i} \in (P_0(K_i))^2, i = 1,\dots,r\}, \end{aligned} \tag{14}$$

where $E_{\min}$, $E_{\max}$ are given constants and $P_0(K_i)$ is the set of constant functions on K_i. Therefore, the admissible set U_{ad} is the closed, convex subset of couples of piecewise constant functions on the partition of $(0, L)$.

The variational formulation of the state problem with respect to the corresponding boundary conditions (B1)–(B4), see [11], reads as follows:

$$\begin{cases} \text{For given } (E,\nu) \in U_{ad} \\ \text{find } w := w(E,\nu) \in V \text{ such that} \\ a(w,v) + \pi(w,v) = \mathcal{L}(v), \qquad \forall v \in V, \end{cases} \tag{$\mathcal{P}(E,\nu)$}$$

where the forms a, π and $\mathcal{L}$ have the same meaning as above. According to Remark 1, to have a unique solution to the problem $(\mathcal{P}(E,\nu))$ for any $(E,\nu) \in U_{ad}$, let t and b be positive constants, $q \in L^2(0,L)$ and

$$P \ < \ \widehat{P}_{\min}, \text{ where } \widehat{P}_{\min} \ = \ \frac{\pi^2 E_{\min} I}{(\mathcal{K} \cdot L)^2} \ \leq \ \frac{\pi^2 \, E \, I}{(1-\nu^2)(\mathcal{K} \cdot L)^2}, \tag{15}$$

$E_{\min}$ is the lower bound of E in (14) and constant $\mathcal{K}$ is given in Remark 1. The inequality (15) is obvious with respect to Theorem 1 and Remark 1.

The parameter identification problem reads as follows:

$$\begin{cases} \text{Find } (E^*,\nu^*) \in U_{ad}, \text{ such that} \\ J(w(E^*,\nu^*)) \ = \ \min\limits_{(E,\nu)\in U_{ad}} J(w(E,\nu)), \\ \text{where } w(E,\nu) \text{ solves } (\mathcal{P}(E,\nu)) \\ \text{and } J : V \longrightarrow \mathbb{R} \text{ is a cost functional.} \end{cases} \tag{$\mathbb{P}$}$$

Continuous dependence of the solution $w(E,\nu)$ on the material parameters (E,ν) is stated in the following theorem which was published in [11] but only for the boundary conditions (B1) which correspond to the space V_1. Here, we will present the extension of the previous results for the remaining boundary conditions (B2), (B3) and (B4) and the spaces V_2, V_3 and V_4. Unless distinguished the space V will be one of above $V_1, \ldots, V_4$. In the previous section we introduced the inequalities which will be used in a proof of the following Theorem. In case of the boundary conditions (B3) we have to consider a stronger assumption for axial force with respect to (15) and (10). Thus let

$$P \ < \ \widehat{P}^i_{\min}, \text{ where } \widehat{P}^i_{\min} \ = \ \frac{\pi^2 E_{\min} I}{(\mathcal{K}_i \cdot L)^2} \ \leq \ \frac{\pi^2 \, E \, I}{(1-\nu^2)(\mathcal{K}_i \cdot L)^2}, \tag{16}$$

where $\mathcal{K}_i, i = 1,2,3$, is given with respect to the inequalities from Section 2 and the boundary conditions. It means that in the following we will be working under the assumptions: let $\mathcal{K}_1 = 1$ for the boundary conditions (B1), $\mathcal{K}_2 = 0.5$ for (B2) and $\mathcal{K}_3 = 2$ for the boundary conditions (B3) and (B4).

Theorem 4. *Let $(E_n,\nu_n) \in U_{ad}$, $n = 1,2,\ldots$ and $(E,\nu) \in U_{ad}$, such that*

$$E_n \underset{n\to\infty}{\longrightarrow} E \text{ in } L^\infty(0,L) \quad \text{and} \quad \nu_n \underset{n\to\infty}{\longrightarrow} \nu \text{ in } L^\infty(0,L)$$

and $w_n := w(E_n,\nu_n) \in V$ be the solution to $(\mathcal{P}(E_n,\nu_n))$. Then

$$w_n \underset{n\to\infty}{\longrightarrow} w(E,\nu) \in V,$$

and $w(E,\nu)$ solves $(\mathcal{P}(E,\nu))$.

Proof. The proof consists of three steps. First, we show that the sequence $\{w_n\}$ is bounded in V.

Step 1. Let $w_n \in V$ solve $(\mathcal{P}(E_n, \nu_n))$:

$$\int_0^L E_n I w_n'' v'' \,\mathrm{d}x + tb \int_0^L E_n (1-\nu_n^2)(w_n')^3 v' \,\mathrm{d}x \\ - \int_0^L P(1-\nu_n^2)\, w_n' \, v' \,\mathrm{d}x = \int_0^L (1-\nu_n^2)\, q\, v \,\mathrm{d}x, \quad \forall v \in V.$$

We set $v := w_n$ and get

$$\int_0^L E_n I (w_n'')^2 \,\mathrm{d}x + tb \int_0^L E_n (1-\nu_n^2)(w_n')^4 \,\mathrm{d}x - \int_0^L P(1-\nu_n^2)\,(w_n')^2 \,\mathrm{d}x = \int_0^L (1-\nu_n^2)\, q\, w_n \,\mathrm{d}x. \tag{17}$$

From (14) it is clear that $1-\nu_n^2 > 0$, since $0 < \nu_n \le 0.5$. Therefore,

$$tb \int_0^L E_n (1-\nu_n^2)\,(w_n')^4 \,\mathrm{d}x \;\ge\; 0, \quad \forall (E_n, \nu_n) \in U_{ad}. \tag{18}$$

To estimate the term with the axial force P, we will apply the inequalities and their modifications presented in Section 2 according to the boundary conditions (B1)–(B4). It is clear that for the space V of admissible displacements $H_0^2((0,L)) \subset V \subset H^2((0,L))$ holds. In the following, we will use the fact that the space $H^k((0,L))$, $k = 1, 2, \dots$, can be continuously embedded into $C^{k-1}(\langle 0, L\rangle)$, see [1]. Especially, we have

$$\exists\, c > 0\colon \max_{x \in \langle 0, L\rangle} |v'(x)| \;\le\; c\|v\|_2 \quad \forall v \in H^2((0,L)),$$

where $\|\cdot\|_k$, $k = 0, 1, \dots$, denotes the norm in $H^k((0,L))$. We will also use the following inequality

$$\exists\, \bar{c} > 0\colon \|v''(x)\|_0^2 \;\ge\; \bar{c}\|v\|_2^2 \quad \forall v \in V, \tag{19}$$

which holds for any V defined by the boundary conditions (B1), (B2), (B3), or (B4). For functions v from V_2, V_3 and V_4 we have $v(0) = v'(0) = 0$, thus we can use twice the generalization (9) of Theorem 3, first for $y = v$ and then for $y = v'$. So we can write

$$\int_0^L (v(x))^2 \,\mathrm{d}x \;\le\; \left(\frac{2L}{\pi}\right)^2 \int_0^L (v'(x))^2 \,\mathrm{d}x \;\le\; \left(\frac{2L}{\pi}\right)^4 \int_0^L (v''(x))^2 \,\mathrm{d}x,$$

which gives us the inequality (19) with $\bar{c} = \left(\frac{\pi}{2L}\right)^4$. For functions from V_1 the inequality (19) holds with constant $\bar{c} = 4\left(\frac{\pi}{L}\right)^4$, which follows from (13).

First, we consider the simply supported beam with boundary conditions (B1). Since $0 < \nu_n \le 0.5$, we have $1 - \nu_n^2 < 1$ and using the inequality (12), we get that

$$\int_0^L P(1-\nu_n^2)(w_n'(x))^2 \,\mathrm{d}x \;\le\; \left(\frac{L}{\pi}\right)^2 \int_0^L P\,(w_n''(x))^2 \,\mathrm{d}x \tag{20}$$

holds for any $P \geq 0$. Then we can write

$$\begin{aligned}\int_0^L E_n\, I\, (w_n'')^2\,\mathrm{d}x + tb\int_0^L E_n(1-\nu_n^2)(w_n')^4\,\mathrm{d}x \;-\; \int_0^L P(1-\nu_n^2)(w_n')^2\,\mathrm{d}x \\ \geq \int_0^L E_n\, I(w_n'')^2\,\mathrm{d}x \;-\; P\left(\frac{L}{\pi}\right)^2 \int_0^L (w_n'')^2\,\mathrm{d}x \\ \geq \int_0^L E_{min}\, I(w_n'')^2\,\mathrm{d}x \;-\; P\left(\frac{L}{\pi}\right)^2 \int_0^L (w_n'')^2\,\mathrm{d}x \\ = \; c_1\, \|w_n''\|_0^2 \;\geq\; \bar{c}\, c_1\, \|w_n\|_2^2,\end{aligned} \tag{21}$$

where we used (18), (20), (19) and the notation $c_1 := E_{\min} I - P\left(\frac{L}{\pi}\right)^2$. The constant c_1 is positive due to assumption (16). If $P < 0$ then (21) trivially holds with $c_1 = E_{\min} I$.

For the clamped beam with the boundary conditions (B2), i.e. $w(0) = w'(0) = w(L) = w'(L) = 0$, we estimate the term with the axial force by the inequality (6) and by using $1 - \nu_n^2 < 1$. Therefore, we have

$$\int_0^L P(1-\nu_n^2)(w_n'(x))^2\,\mathrm{d}x \;\leq\; \left(\frac{L}{2\pi}\right)^2 \int_0^L P\;\left(w_n''(x)\right)^2\,\mathrm{d}x. \tag{22}$$

For the propped cantilever and cantilever beam with the boundary conditions (B3) and (B4), respectively, the inequality (10) together with $1 - \nu_n^2 < 1$ can be used, i.e.

$$\int_0^L P(1-\nu_n^2)(w_n'(x))^2\,\mathrm{d}x \;\leq\; \left(\frac{2L}{\pi}\right)^2 \int_0^L P\,(w_n''(x))^2\,\mathrm{d}x. \tag{23}$$

Similarly as for (B1) now we can get by using the inequalities (22), (23) that

$$\begin{aligned}\int_0^L E_n\, I\, (w_n'')^2\,\mathrm{d}x + tb\int_0^L E_n(1-\nu_n^2)(w_n')^4\,\mathrm{d}x \;-\; \int_0^L P(1-\nu_n^2)(w_n')^2\,\mathrm{d}x \\ \geq\; c_i\, \|w_n''\|_0^2 \;\geq\; \bar{c}\, c_i\, \|w_n\|_2^2,\end{aligned} \tag{24}$$

where $i = 2, 3$, $c_2 := E_{\min} I - P\left(\frac{L}{2\pi}\right)^2 > 0$ and $c_3 := E_{\min} I - P\left(\frac{2L}{\pi}\right)^2 > 0$. If $P < 0$ then (24) is trivially valid with $c_2 = c_3 = E_{\min} I$.

For the right hand side in (17) we get

$$\int_0^L (1-\nu_n^2) q\, w_n\,\mathrm{d}x \;\leq\; \|q\|_{L^2((0,L))} \|w_n\|_2, \tag{25}$$

where Hölder's inequality and (14) were used. Finally, from (17), (21), (24) and (25) we see that $\{w_n\}$ is bounded in V. Therefore, there exists its subsequence, for simplicity we denote it as $\{w_n\}$ again, such that

$$w_n \underset{n\to\infty}{\rightharpoonup} w \quad \text{(weakly)} \;\text{ in }\; V.$$

Step 2. Now we show that w solves (2). Similarly as in [11], it can be proven for each $v \in V$ that

$$\int_0^L E_n\, I\, w_n''\, v''\,\mathrm{d}x \;+\; tb \int_0^L E_n(1-\nu_n^2)(w_n')^3\, v'\,\mathrm{d}x \;-\; \int_0^L P(1-\nu_n^2) w_n'\, v'\,\mathrm{d}x$$
$$= \int_0^L (1-\nu_n^2) q\, v\,\mathrm{d}x \;\underset{n\to\infty}{\longrightarrow}\; \int_0^L EIw''\, v''\,\mathrm{d}x \;+\; tb\int_0^L E(1-\nu^2)(w')^3\, v'\,\mathrm{d}x$$
$$-\; \int_0^L P(1-\nu^2) w' v' \mathrm{d}x \;=\; \int_0^L (1-\nu^2) qv\,\mathrm{d}x.$$

Step 3. To prove the strong convergence, it is sufficient to show that $[[w_n]] \to [[w]]$ for $n \to \infty$ in V, where

$$[[w]]^2 \;:=\; \int_0^L EI(w''(x))^2\,\mathrm{d}x.$$

For more details, see [11], [3]. □

To prove the existence of at least one solution of the identification problem $(\mathbb{P})$, see [11], we suppose that the cost functional J is continuous in V, i.e.

$$v_n \underset{n\to\infty}{\longrightarrow} v \quad \Longrightarrow \quad J(v_n) \underset{n\to\infty}{\longrightarrow} J(v). \tag{26}$$

Theorem 5. *Let U_{ad} be given by* (14) *and let J satisfy* (26)*. Then the identification problem* $(\mathbb{P})$ *has a solution.*

4. Conclusion

In this paper, we discuss the extension of the results presented in [11]. Several analytical inequalities and their modifications were used to prove the continuous dependence of the solution to the state problem on the material parameters for different types of boundary conditions for the nonlinear Gao beam.

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