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A NOTE ON THE OSCILLATION OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'' = \lambda\, q(t)\, y$ WITH A PERIODIC COEFFICIENT

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Dedicated to Academician O. BORŮVKA *on the occasion of the 80th anniversary of his birthday*

1. INTRODUCTION

We consider two differential equations:

$$\text{(q)} \qquad y'' = q(t)\, y\,,$$

where $q \in C^{\circ}(\mathbf{R})$ ($\mathbf{R} = (-\infty, \infty)$), $q(t + \pi) = q(t)$ for $t \in \mathbf{R}$, $q(t) \not\equiv 0$ and

$$(\lambda \text{q}) \qquad y'' = \lambda\, q(t)\, y\,,$$

where λ represents a real parameter.

It is well known from [7] that (q) is either disconjugate (on $\mathbf{R}$), in other words every nontrivial solution of (q) has at most one zero on $\mathbf{R}$, or it is both-sided oscillatory (on $\mathbf{R}$), in other words $\pm\infty$ are cluster points of zeros of any nontrivial solution of (q). In case of $\lambda = 0$, the equation (λq) is disconjugate. It is equally well known from [7] that the set of all numbers λ for which the equation (λq) is disconjugate, is closed and convex.

Our object now is to obtain necessary and sufficient conditions on the function q for the equation (λq) to be oscillatory for every $\lambda \in \mathbf{R} - \{0\}$. The result is given in the following

Theorem. *Let $q \in C^{\circ}(\mathbf{R})$, $q(t + \pi) = q(t)$ for $t \in \mathbf{R}$ and let $q(t) \not\equiv 0$. Then the equation (λq) is oscillatory for every $\lambda \in \mathbf{R} - \{0\}$ iff*

$$\int_0^{\pi} q(t)\, \mathrm{d}t = 0\,.$$

In fact we shall prove more: *Let the assumptions of the Theorem be satisfied. Then the equation (λq) is oscillatory for every $\lambda \in \mathbf{R} - \{0\}$ iff there exists $\mu > 0$ such that the equation (λq) is oscillatory for every $\lambda \in (-\mu, 0) \cup (0, \mu)$.*

Corollary 1. *Let the assumptions of the Theorem be satisfied. Then the equation* $y'' = \lambda(a + q(t))\, y$, *where* a *is a constant, is oscillatory for every* $\lambda \in \mathbf{R} - \{0\}$ *iff*

$$a = -\frac{1}{\pi}\int_0^\pi q(t)\,\mathrm{d}t\,.$$

2. PRELIMINARY RESULTS

In this paragraph and through the rest of this paper we shall assume that $q \in C^0(\mathbf{R})$, $q(t + \pi) = q(t)$ for $t \in \mathbf{R}$ and $q(t) \not\equiv 0$. In [6, p. 487] and [7, p. 103] we find the following criterion for oscillation of (q).

Lemma 1. *In case of*

$$\int_0^\pi q(t)\,\mathrm{d}t \leqq 0$$

the equation (q) *is oscillatory.*

Remark 1. Some estimates of

$$\int_0^\pi q(t)\,\mathrm{d}t$$

in the case of (q) having only periodic or half-periodic solutions with period π, are presented in [9].

Corollary 2. *At least one of the two equations* $y'' = q(t)\, y$, $y'' = -q(t)\, y$ *is oscillatory.*

The proof follows immediately from Lemma 1 and from the inequality

$$\int_0^\pi q(t)\,\mathrm{d}t \int_0^\pi (-q(t))\,\mathrm{d}t \leqq 0\,.$$

Remark 2. Corollary 2 is available only for equations having a periodic coefficient. If the coefficient is not periodic, Corollary 2 does not hold.

Example. Let

$$p(t) := \begin{cases} -\dfrac{1}{4t^2} & \text{for} \quad t \in (-\infty, -1\rangle \cup \langle 1, \infty)\,; \\[2mm] \dfrac{36t^2 - 28}{3t^4 - 14t^2 - 21} & \text{for} \quad t \in (-1, 1)\,. \end{cases}$$

Then $p \in C^{\circ}(\mathbf{R})$ and

$$y(t) := \begin{cases} \sqrt{(t \,.\, \operatorname{sign} t)} & \text{for} \quad t \in (-\infty, -1\rangle \cup \langle 1, \infty) ; \\ \dfrac{1}{32}(-3t^4 + 14t^2 + 21) & \text{for} \quad t \in (-1, 1) \end{cases}$$

is a solution of $y'' = p(t)\, y$ on $\mathbf{R}$. Evidently the equation $y'' = p(t)\, y$ is non-oscillatory on $\mathbf{R}$. The equation $y'' = -p(t)\, y$ is also non-oscillatory on $\mathbf{R}$, because $-p(t) > 0$ for $t \in (-\infty, -1\rangle \cup \langle 1, \infty)$.

Corollary 3. *Let the function q change its sign on $\mathbf{R}$ and let the equation* (q) *be disconjugate. Then there exists a number μ ($\geqq 1$) such that the equation* (λq) *is disconjugate for $\lambda \in \langle 0, \mu\rangle$ and oscillatory for $\lambda \in (-\infty, 0) \cup (\mu, \infty)$.*

Proof. Let (q) be a disconjugate equation. Then by Lemma 1

$$\int_0^{\pi} q(t) \,\mathrm{d}t > 0$$

and consequently

$$\lambda \int_0^{\pi} q(t) \,\mathrm{d}t < 0$$

for $\lambda \in (-\infty, 0)$. Hence the equation (λq) is oscillatory for $\lambda \in (-\infty, 0)$. By assumption the function q changes its sign on $\mathbf{R}$. Therefore there exists an interval (t_0, t_1) such that $q(t) < 0$ for $t \in (t_0, t_1)$. If λ_0 (> 0) is a number large enough, then the equation (λ_0q) has a nontrivial solution having at least two zeros in (t_0, t_1), which implies that the equation (λ_0q) is oscillatory. The set of all numbers λ, for which the equation (λq) is disconjugate, is closed, convex and upper bounded. Consequently there exists a number μ ($\geqq 1$) such that the equation (λq) is disconjugate precisely for $\lambda \in \langle 0, \mu\rangle$.

Corollary 4. *Let $q(t) \geqq 0$ for $t \in \mathbf{R}$. Then the equation* (λq) *is oscillatory for $\lambda \in (-\infty, 0)$ and disconjugate for $\lambda \in \langle 0, \infty)$.*

Proof. It holds

$$\lambda \int_0^{\pi} q(t) \,\mathrm{d}t < 0$$

for $\lambda \in (-\infty, 0)$ and by Lemma 1 the equation (λq) is oscillatory for this λ. The Sturm comparison theorem yields the disconjugacy of the equation (λq) for $\lambda \geqq 0$.

Remark 3. The oscillation of the equation (λq) with an arbitrary function q has been investigated in [4].

Lemma 2. ([5, p. 408]). *Let v be a nontrivial solution of* (q) *having two zeros in the interval $\langle a, b \rangle$. Then*

$$\int_a^b q^-(t)\,\mathrm{d}t < -\frac{4}{b-a},$$

where $q^-(t) = \min(q(t), 0)$.

In the sense of the Floquet theory we can associate with every equation (q) (having a π-periodic coefficient q) a certain algebraic equation, the so-called characteristic equation of (q),

$$\varrho^2 - 2A\varrho + 1 = 0\,, \tag{1}$$

where A is a constant. The roots of (1) are called *the characteristic multipliers* of (q).

Lemma 3 ([1, 2]). *The oscillatory equation* (q) *has real characteristic multipliers exactly if there exists a number x and a nontrivial solution v of* (q) *such that* $v(x) = v(x + \pi) = 0$.

3. PROOF OF THE THEOREM

($\Leftarrow$). Let

$$\int_0^\pi q(t)\,\mathrm{d}t = 0\,.$$

Then

$$\lambda \int_0^\pi q(t)\,\mathrm{d}t = 0$$

and $\lambda\, q(t) \not\equiv 0$ for $\lambda \in \mathbf{R} - \{0\}$. Consequently, by Lemma 1, the equation (λq) is oscillatory for $\lambda \in \mathbf{R} - \{0\}$.

($\Rightarrow$). Let (λq) be an oscillatory equation for every $\lambda \in \mathbf{R} - \{0\}$ and let

$$\varrho^2 - 2A(\lambda)\,\varrho + 1 = 0$$

be the characteristic equation of (λq). Then

$$A(\lambda) = 1 + \tfrac{1}{2}\sum_{n=1}^{\infty} [f_n(\pi) + \varphi_n'(\pi)]\,\lambda^n \tag{2}$$

with

$$f_0(t) = 1\,, \quad \varphi_0(t) = t\,,$$

$$f_n(t) = \int_0^t \int_0^s q(x)\, f_{n-1}(x)\,\mathrm{d}x\,\mathrm{d}s\,,$$

$$\varphi_n(t) = \int_0^t \int_0^s q(x)\, \varphi_{n-1}(x)\, dx\, ds\,,$$

$$(n = 1, 2, \ldots;\ t \in \mathbf{R})\,,$$

whereby the series on the right-hand side of (2) converges for every λ (cf. [3, p. 177] and [8, p. 232]), and $A(0) = 1$.

Let there be a sequence $\{\lambda_n\}$, $\lambda_n \neq 0$, $\lim_{n\to\infty} \lambda_n = 0$ such that $A(\lambda_n) \geqq 1$. Then the equations $(\lambda_n q)$: $y'' = \lambda_n\, q(t)\, y$ have real characteristic multipliers and for any $x \in \mathbf{R}$ we have

$$\lim_{n\to\infty} \int_x^{x+\pi} (\lambda_n\, q(t))^-\, dt = 0\,,$$

where $(\lambda_n\, q(t))^- = \min(\lambda_n\, q(t), 0)$. By Lemma 2 the equations $(\lambda_n q)$ for which

$$\int_x^{x+\pi} (\lambda_n\, q(t))^-\, dt > -\frac{4}{\pi}$$

have no nontrivial solutions with at least two zeros in the interval $\langle x, x + \pi\rangle$ and therefore, by Lemma 3, these equations have no real characteristic multipliers, which is a contradiction.

Thus, there exists a number $\mu > 0$ such that $A(\lambda) < 1$ for $\lambda \in (-\mu, \mu) - \{0\}$. Since $A(0) = 1$, the function $A(\lambda)$ has a local extreme at the point $\lambda = 0$ and $A'(0) = 0$. Now (2) implies

$$A'(0) = \tfrac{1}{2}(f_1(\pi) + \varphi_1'(\pi))$$

and after some evident modifications we obtain

$$A'(0) = \frac{1}{2}\left[\int_0^\pi \int_0^t q(x)\, dx\, dt + \int_0^\pi t q(t)\, dt\right] = \frac{\pi}{2}\int_0^\pi q(t)\, dt$$

(see [3, p. 178] and [6, p. 472]). Hence

$$\int_0^\pi q(t)\, dt = 0\,, \qquad \text{q.e.d.}$$

By the Theorem the equation $y'' = \lambda(a + q(t))\, y$, where a is a constant, is oscillatory for every $\lambda \in \mathbf{R} - \{0\}$ iff

$$\int_0^\pi (a + q(t))\, dt = 0\,,$$

that is iff

$$a = -\frac{1}{\pi} \int_0^\pi q(t) \, dt .$$

We have thus proved Corollary 1.

References

[1] *Борувка О.:* Теория глобальных свойств обыкновенных линейных дифференциальных уравнений второго порядка. Дифференциальные уравнения, № 8, т. *XII*, 1976, 1347—1383.

[2] *Borůvka O.:* Sur les blocs des équations différentielles $y'' = q(t) y$ aux coefficients périodiques. Rend. Mat. (2), *8*, S. VI, 1975, 519—532.

[3] *Еругин Н. П.:* Линейные системы обыкновенных дифференциальных уравнений с периодическими и квазипериодичексими коэффициентами. Минск 1963.

[4] *Fink A. M.* and *Mary D. F. St.:* A generalized Sturm comparison theorem and oscillation coefficients. Monatsh. Math., *73* (1969), 207—212.

[5] *Hartman P.:* Ordinary differential equations (in Russian), Moscow 1970.

[6] *Якубович В. А., Старжинский В. М.:* Линейные дифференциальные уравнения с периодическими коэффициентами и их приложения. Издательство „Наука", Москва 1970.

[7] *Markus L.* and *Moore R. A.:* Oscillation and disconjugacy for linear differential equations with almost periodic coefficients. Acta Math., *96* (1956), 99—123.

[8] *Малкин И. Г.:* Теория устойчивости движения. Издательство „Наука", Москва 1966.

[9] *Neuman F.:* Linear differential equations of the seond order and their applications. Rend. Mat. *4*, 1971, 559—617.

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