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A CHARACTERIZATION OF THE INTERVAL FUNCTION OF A CONNECTED GRAPH

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0. By a graph we mean a finite undirected graph with no loop or multiple edge (i.e. a graph in the sense of [1] or [2], for example). Throughout the paper we assume, that a connected graph G is given. Let V and E denote its vertex set and its edge set, respectively. Moreover, we denote by $d(u,v)$ the distance between u and v in G, for any u, $v \in V$. Note that $d(u,v)$ is equal to the length of an arbitrary shortest $u-v$ path in G, for any u, $v \in V$. Clearly, the vertex set V and the distance function d form a finite metric space. (Kay and Chartrand [2] found a necessary and sufficient condition for a finite metric space to be generated by the vertex set and the distance function of a connected graph).

Similarly as in [3], by the *interval function* I of G we mean the mapping of $V \times V$ into the set of all subsets of V defined as follows (for every $(u,v) \in V \times V$):

$$I(u,v) = \{x \in V;\ x \text{ belongs to a } u-v \text{ path of length } d(u,v) \text{ in } G\}.$$

The interval function of a connected graph, which was defined and intensively studied in Mulder [3], is an important tool for the study of metric properties of graphs.

The definition of the interval function I of G depends on the notion of the distance in G (or on the notion of shortest paths in G). We are going to derive an essentially different characterization of the interval function.

1. Denote by $\mathbf{J}$ the set of all mappings J of $V \times V$ into the set of all subsets of V such that J fulfils the following Axioms I–VI (for arbitrary u, v, w, $x \in V$):

I $\quad |J(u,v)| = 2$ if and only if $\{u,v\} \in E$;

II $\quad u \in J(u,v)$;

III $\quad$ if $w \in J(u,v)$, then $|J(u,w) \cap J(w,v)| = 1$;

IV if $w \in J(u,v)$, then $J(w,v) \subseteq J(u,v)$;

V if $w \in J(u,v)$ and $x \in J(w,v)$, then $w \in J(u,x)$;

VI $J(v,u) = J(u,v)$.

Put $J = I$; it is clear that J fulfils Axiom I; using 1.1.2 from [3] we easily get

$$I \in \mathbf{J}.$$

We now make several observations concerning $\mathbf{J}$.

Using Axioms II and III we obtain $J(u,u) = \{u\}$ for $J \in \mathbf{J}$ and $u \in V$.

Let $J \in \mathbf{J}$. For u, $v \in V$ we define the set $\Sigma_J(u,v)$ as follows:

$$\Sigma_J(u,v) = \{(u)\} \quad \text{if } u = v;$$
$$\Sigma_J(u,v) = \Big\{(x_1,\dots,x_k,v); k \geqslant 1, x_k \in J(u,v), \{x_k,v\} \in E \text{ and } (x_1,\dots,x_k) \in \Sigma_J(u,x_k)\Big\} \quad \text{if } u \neq v.$$

Lemma. *Let $J \in \mathbf{J}$ and u, $v \in V$. Assume that $u \neq v$. Then*

(1) $\{u,v\} \subseteq J(u,v)$;

(2) *if $w \in J(u,v) - \{u\}$, then $J(w,v) \subseteq J(u,v) - \{u\}$;*

(3) *there exists $x \in J(u,v)$ such that $\{x,v\} \in E$;*

(4) $J(u,v) - \{v\} = \bigcup_{\substack{x \in J(u,v) \\ \{x,v\} \in E}} J(u,x)$;

(5) *if $(w_1, \dots, w_m) \in \Sigma_J(u,v)$, then $w_1, \dots, w_m \in J(u,v)$ and $(w_1,\dots,w_m)$ is a $u-v$ path in G (i.e. a $u-v$ path considered as a sequence of vertices);*

(6) $\Sigma_J(u,v) \neq \emptyset$.

Proof. (1) follows from Axioms II and VI.

Let $w \in J(u,v) - \{u\}$. According to Axiom IV, $J(w,v) \subseteq J(u,v)$. Suppose $u \in J(w,v)$. Obviously, $u \neq w$. As follows from Axioms IV and VI, $J(w,u) = J(u,w) \subseteq J(v,w) = J(w,v) \subseteq J(u,v)$. Axiom III implies that $|J(w,u)| = 1$, which contradicts (1). Thus $u \notin J(w,v)$ and we get (2).

(3) follows from (1), (2), and Axiom I.

First, let $w \in J(u,v) - \{v\}$. Since $w \neq v$, (3) implies that there exists $x \in J(w,v)$ such that $\{x,v\} \in E$. According to Axiom V, $w \in J(u,x)$. Using (2) and Axiom VI, we get (4).

(5) follows from the definition of $\Sigma_J(u,v)$, (2), and Axiom VI.

Combining (2), (3) and Axiom VI with the definition of $\Sigma_J(u,v)$, we get (6), which completes the proof. □

2. Let J, $J' \in \mathbf{J}$, let $n \geqslant 0$ be an integer. We write $\mathrm{P}_n(J, J')$ to express the fact that

$$J(u, v) \subseteq J'(u, v) \quad \text{for each pair of } u \text{ and } v \text{ in } V \text{ such that } d(u, v) = n.$$

We now give a characterization of the interval function of G, which is the main result of present paper.

Theorem. *Let $J \in \mathbf{J}$. Then $J = I$ if and only if J fulfils the following Axioms VII and VIII (for arbitrary $u, v, x, y \in V$):*

VII *if $\{u, x\}, \{v, y\} \in E$, $x \in J(u, v)$, $y \in J(u, v)$ and $u \in J(x, y)$, then $v \in J(x, y)$;*

VIII *if $\{u, x\}, \{v, y\} \in E$, $x \in J(u, v)$, $y \notin J(u, v)$ and $x \notin J(u, y)$, then $v \in J(x, y)$.*

Proof. (A) Assume that $J = I$. We shall prove that J fulfils Axioms VII and VIII. Consider arbitrary u, v, x, $y \in V$ such that $\{u, x\}$, $\{v, y\} \in E$ and $x \in J(u, v)$. Put $n = d(u, v)$. Then $d(x, v) = n - 1$.

(Axiom VII) Assume that $y \in J(u, v)$ and $u \in J(x, y)$. We want to prove that $v \in J(x, y)$. Since $\{v, y\} \in E$ and $y \in J(u, v)$, we have $d(u, y) = n - 1$. Certainly, $d(x, y) \leqslant n$. Since $u \in J(x, y)$, we get $d(x, y) = n$. Thus $v \in J(x, y)$.

(Axiom VIII) Assume that $y \notin J(u, v)$ and $x \notin J(u, y)$. We want to prove that $v \in J(x, y)$. Since $y \notin J(u, v)$, we have $d(u, y) \geqslant n$. Since $x \notin J(u, y)$, we have $d(x, y) \geqslant d(u, y) \geqslant n$. Since $d(x, v) = n - 1$ and $d(v, y) = 1$, we get $v \in J(x, y)$.

(B) Conversely, let us now assume that J fulfils Axioms VII and VIII. We shall prove that $\mathrm{P}_n(I, J)$ and $\mathrm{P}_n(J, I)$ for each integer n such that $0 \leqslant n \leqslant D$, where D denotes the diameter of G. We proceed by induction on n. It is clear that $\mathrm{P}_n(I, J)$ and $\mathrm{P}_n(J, I)$ for $n = 0$ and 1. Therefore, let us assume that $2 \leqslant n \leqslant D$ and

$$\mathrm{P}_k(I, J) \text{ and } \mathrm{P}_k(J, I) \text{ for each } k \in \{0, \dots, n-1\}. \tag{7}$$

The rest of the proof will be divided into two steps.

Step 1. We shall prove that $\mathrm{P}_n(I, J)$. Consider arbitrary u, $v \in V$ such that $d(u, v) = n$. We want to prove that $I(u, v) \subseteq J(u, v)$. Suppose, to the contrary, $I(u, v) - J(u, v) \neq \emptyset$. Consider $w \in I(u, v) - J(u, v)$. Since $w \in I(u, v)$, there exist a $v - u$ path $(y_0, \dots, y_n)$ in G and an integer i such that $0 \leqslant i \leqslant n$ and $w = y_i$. Clearly, $y_0 = v$ and $y_n = u$. Since $w \notin J(u, v)$, we have $0 < i < n$. Consider an arbitrary $j \in \{1, \dots, n-1\}$. It follows from (7) that $I(v, y_j) = J(v, y_j)$ and $I(y_j, u) = J(y_j, u)$. If $y_j \in J(u, v)$, then Axioms IV and VI imply that $I(v, y_j) \subseteq J(u, v)$ and $I(y_j, u) \subseteq J(u, v)$, and thus $w \in J(u, v)$, which is a contradiction. We conclude that $y_1, \dots, y_{n-1} \notin J(u, v)$.

As follows from (6), there exist $x_0, \ldots, x_m \in V$ $(m \geqslant 1)$ such that $(x_0, \ldots, x_m) \in \Sigma_J(u, v)$. According to (5), $(x_0, \ldots, x_m)$ is a $u - v$ path in G. Thus $x_0 = u$ and $x_m = v$. Since $n = d(u, v)$, $m \geqslant n$. Since $(x_0, \ldots, x_{i+1}) \in \Sigma_J(x_0, x_{i+1})$, it follows from (5) and Axioms V and VI that

$$(8_i) \qquad x_{i+1} \in J(x_i, v)$$

for each $i \in \{0, \ldots, m-1\}$. Since $(y_0, \ldots, y_n)$ is a $v - u$ path in G and $y_1, \ldots, y_{n-1} \notin J(u, v)$, we see that

$$(9_i) \qquad (y_i, \ldots, y_n = x_0, \ldots, x_i) \text{ is a path in } G$$

for each $i \in \{0, \ldots, n\}$.

Put $x_{-1} = y_{n-1}$. Certainly, the following statements (10_i), (11_i) and (12_i) hold for $i = 0$:

$$(10_i) \qquad d(x_i, y_i) = n;$$

$$(11_i) \qquad v \in J(x_i, y_i);$$

$$(12_i) \qquad x_{i-1} \notin J(x_i, y_i).$$

Clearly, $x_{n-1} \in J(x_0, x_n)$. Since $y_n = x_0$, $x_{n-1} \in J(x_n, y_n)$. Thus (12_n) does not hold. This means that there exists $h \in \{0, \ldots, n-1\}$ such that each of the statements (10_h), (11_h) and (12_h) holds but at least one of the statements (10_{h+1}), (11_{h+1}) and (12_{h+1}) does not.

Combining (8_h) and (11_h) with Axioms IV–VI, we get

$$(13) \qquad x_{h+1} \in J(x_h, y_h);$$

$$(14) \qquad v \in J(x_{h+1}, y_h).$$

It follows from (9_h) and (10_h) that $d(x_h, y_{h+1}) = n - 1$. According to (7), $J(x_h, y_{h+1}) = I(x_h, y_{h+1})$. Obviously, $x_{h-1} \in I(x_h, y_{h+1})$. Thus $x_{h-1} \in J(x_h, y_{h+1})$. If $y_{h+1} \in J(x_h, y_h)$, then it follows from Axioms IV and VI that $x_{h-1} \in J(x_h, y_h)$, which contradicts (12_h). Therefore,

$$(15) \qquad y_{h+1} \notin J(x_h, y_h).$$

We now want to show that $x_{h+1} \notin J(x_h, y_{h+1})$. Suppose, to the contrary, $x_{h+1} \in J(x_h, y_{h+1})$. Since $d(x_h, y_{h+1}) = n - 1$, it follows from (7) that $x_{h+1} \in I(x_h, y_{h+1})$. Thus $d(x_{h+1}, y_{h+1}) = n - 2$. It follows from (10_h) that $d(x_{h+1}, y_h) = n - 1$ and

$y_{h+1} \in I(x_{h+1}, y_h)$. According to (7), $y_{h+1} \in J(x_{h+1}, y_h)$. Combining this fact with (13) and Axiom IV, we get $y_{h+1} \in J(x_h, y_h)$, which contradicts (15). Therefore, $x_{h+1} \notin J(x_h, y_{h+1})$.

Since $x_{h+1} \in J(x_h, y_h)$ and $y_{h+1} \notin J(x_h, y_h)$, Axioms VIII implies that

$$(16) \qquad y_h \in J(x_{h+1}, y_{h+1}).$$

Combining (14) and (16) with Axioms IV and VI, we get (11_{h+1}).

As follows from (9_{h+1}), $d(x_{h+1}, y_{h+1}) \leqslant n$. Suppose $d(x_{h+1}, y_{h+1}) \leqslant n - 1$. According to (7), $J(x_{h+1}, y_{h+1}) = I(x_{h+1}, y_{h+1})$. It follows from (16) that $y_h \in I(x_{h+1}, y_{h+1})$. This implies that $d(x_{h+1}, y_h) \leqslant n - 2$. Hence, $d(x_h, y_h) \leqslant n - 1$, which is a contradiction. Thus we have (10_{h+1}).

Since (10_{h+1}) and (11_{h+1}) hold, it follows from the definition of h that (12_{h+1}) does not hold. Thus we have $x_h \in J(x_{h+1}, y_{h+1})$. Combining this fact with (13), (16) and Axiom VII, we get $y_{h+1} \in J(x_h, y_h)$, which contradicts (15).

Thus $I(u, v) \subseteq J(u, v)$ and we have

$$(17) \qquad \mathrm{P}_n(I, J).$$

Step 2. We shall prove that $\mathrm{P}_n(J, I)$. Consider arbitrary u, $v \in V$ such that $d(u, v) = n$. We want to prove that $J(u, v) \subseteq I(u, v)$. Suppose, to the contrary, $J(u, v) - I(u, v) \neq \emptyset$. It follows form (4) that there exists $w \in J(u, v)$ such that $\{w, v\} \in E$ and $J(u, w) - I(u, v) \neq \emptyset$. Assume that there exists $w' \in J(u, w) - \{u\}$ such that $w' \in I(u, v)$. Since $d(w', v) < n$, $J(w', v) = I(w', v)$. According to Axioms V and VI, $w \in J(w', v)$. Thus $w \in I(w', v)$. Since $w' \in I(u, v)$, $w \in I(u, v)$. This means that $d(u, w) = n - 1$. As follows from (7), $J(u, w) = I(u, w)$. We get $J(u, w) \subseteq I(u, v)$, which is a contradiction. Thus we have obtained that $(J(u, w) - \{u\}) \cap I(u, v) = \emptyset$. According to (6), $\Sigma_J(u, w) \neq \emptyset$. There exist x_0, ..., $x_{m-1} \in V$ ($m \geqslant 2$) such that $(x_0, \ldots, x_{m-1}) \in \Sigma_J(u, w)$. Clearly, $x_0 = u$, $x_{m-1} = w$, and x_1, ..., $x_{m-1} \notin I(u, v)$. Put $x_m = v$. Certainly, $(x_0, \ldots, x_m) \in \Sigma_J(u, v)$. According to (5), $(x_0, \ldots, x_m)$ is a $u - v$ path in G. Since $x_{m-1} \notin I(u, v)$, we see that $m > n$. Moreover, we have (8_i) for each $i \in \{0, \ldots, m - 1\}$.

Since $d(u, v) = n$, there exist y_0, ..., $y_n \in V$ such that $y_0 = v$, $y_n = u$, and $(y_0, \ldots, y_n)$ is a $u - v$ path of length n in G. Clearly, y_0, ..., $y_n \in I(u, v)$. We get (9_i) for each $i \in \{0, \ldots, n\}$.

Obviously, both (10_0) and (11_0) hold. Since $m > n$, $x_n \neq v$. Since $y_n = u$, (2) implies that $v \notin J(x_n, y_n)$. Thus (11_n) does not hold. This means there exists $h \in \{0, \ldots, n - 1\}$ such that both (10_h) and (11_h) hold but at least one of the statements (10_{h+1}) and (11_{h+1}) does not.

Similarly as in Step 1, we have (13) and (14).

We want to show that $d(x_{h+1}, y_h) \geqslant n$. Suppose to the contrary $d(x_{h+1}, y_h) \leqslant n-1$. Since $d(x_h, y_h) = n$, $d(x_{h+1}, y_h) = n-1$. According to (7), $J(x_{h+1}, y_h) = I(x_{h+1}, y_h)$. Since $v \in J(x_{h+1}, y_h)$, we have $v \in I(x_{h+1}, y_h)$. Obviously, $d(v, y_h) = h$. Thus $d(x_{h+1}, v) = n-h-1$. According to (7), $J(x_{h+1}, v) = I(x_{h+1}, v)$. Combining (8_{k-1}) and (7), we see that $d(x_k, v) = n-k$ and $J(x_k, v) = I(x_k, v)$ for each integer k such that $h+1 < k \leqslant n$. This means that $d(x_n, v) = 0$ and therefore $m = n$, which is a contradiction. Thus we have $d(x_{h+1}, y_h) \geqslant n$.

As follows from (9_{h+1}), $d(x_{h+1}, y_{h+1}) \leqslant n$. We want to show that (10_{h+1}). To the contrary, let $d(x_{h+1}, y_{h+1}) < n$. Since $d(x_{h+1}, y_h) \geqslant n$, we have $d(x_{h+1}, y_h) = n$ and $d(x_{h+1}, y_{h+1}) = n-1$. Then $y_{h+1} \in I(x_{h+1}, y_h)$. It follows from (17) that $y_{h+1} \in J(x_{h+1}, y_h)$. Combining this fact and (13) with Axioms V and VI, we get $x_{h+1} \in J(x_h, y_{h+1})$. Since $d(x_h, y_h) = n$, we see that $d(x_h, y_{h+1}) = n-1$. It follows from (7) that $x_{h+1} \in I(x_h, y_{h+1})$. Hence $d(x_{h+1}, y_{h+1}) = n-2$, which is a contradiction. Thus we have (10_{h+1}).

Combining (9_h) and (10_h), we see that $y_{h+1} \in I(x_h, y_h)$. As follows from (17), $y_{h+1} \in J(x_h, y_h)$. According to (10_{h+1}), $d(x_{h+1}, y_{h+1}) = n$. Therefore, $x_h \in I(x_{h+1}, y_{h+1})$. As follows from (17), $x_h \in J(x_{h+1}, y_{h+1})$. According to (13), $x_{h+1} \in J(x_h, y_h)$. Since $x_h \in J(x_{h+1}, y_{h+1})$ and $y_{h+1} \in J(x_h, y_h)$, Axiom VII implies that $y_h \in J(x_{h+1}, y_{h+1})$. Combining this fact and (14) with Axioms IV and VI, we have (11_{h+1}), which contradicts the definition of h.

Thus $J(u,v) \subseteq I(u,v)$, hence $P_n(J, I)$, which completes the proof of the theorem. $\square$

R e m a r k. There is a connection between the interval function of G and the set of all shortest paths in G. A characterization of the set of all shortest paths in G was given by the present author in Theorem 1 of [4].

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