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# A CONTRIBUTION TO KHINTCHINE'S (ХИНЧИН) PRINCIPLE OF TRANSFER.

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Let $\vartheta_{ij}$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$ be real numbers. Put

$$S_i(x) = \vartheta_{i1}x_1 + \dots + \vartheta_{im}x_m + x_{m+i} \ (i = 1, \dots, n),$$

$$T_j(y) = \vartheta_{1j}y_1 + \dots + \vartheta_{nj}y_n + y_{n+j} \ (j = 1, \dots, \dots, m).$$

It is possible to introduce a number $\alpha$ (the "index" of the system $\vartheta_{ij}$) indicating to what degree of precision the system of equations $S_i(x) = 0$ can be solved by means of integers $x_k$. Let $\beta$ have an analogous meaning for the system $T_j(y) = 0$. The author investigates the relations between $\alpha$ and $\beta$.

## § 1. Introduction.

Let $\vartheta$ be a real number. Then there exists an infinite set of pairs of integers $[p_i, q_i]$, $q_i > 0$, $i = 1, 2, 3, \dots$, so that $\lim\limits_{i \to \infty} q_i = +\infty$ and $\left|\vartheta - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2}$ or otherwise $|\vartheta q_i - p_i| < \frac{1}{q_i}$.

The preceding problem may be generalized as follows: Let $m$ and $n$ be two positive integers and let $\vartheta_{ij}$ $(1 \leqq i \leqq n;\ 1 \leqq j \leqq m)$ be a system of $mn$ real numbers; $m$, $n$ and the numbers $\vartheta_{ij}$ are given. The question is to solve the system of inequalities

$$|\vartheta_{i1}x_1 + \vartheta_{i2}x_1 + \dots + \vartheta_{im}x_m + x_{m+i}| < \frac{1}{t} \ (1 \leqq i \leqq n)$$

by means of integers $x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{m+n}$ in such a way that $\max\limits_{1 \leqq j \leqq m} (|x_j|) > 0$.

The solution of this problem is contained in the following theorem:

*Let $\vartheta_{ij}$ $(1 \leqq i \leqq n;\ 1 \leqq j \leqq m)$ be real numbers, $m \geqq 1, n \geqq 1$; further let $t \geqq 1$. Then there exists at least one lattice point $[x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n}]$ so that the inequalities*

$$|\vartheta_{i1}x_1 + \vartheta_{i2}x_2 + \ldots + \vartheta_{im}x_m + x_{m+i}| < \frac{1}{t^{\frac{m}{n}}} \quad (1 \leqq i \leqq n),$$

$$0 < \max_{1 \leqq j \leqq m} (|x_j|) \leqq t$$

*are fulfilled simultaneously.*

Let $\varphi(t)$ indicate a non-negative, non-increasing and continuous function of the positive argument $t$. We shall say that the system

$$\vartheta_{i1}x_1 + \vartheta_{i2}x_2 + \ldots + \vartheta_{im}x_m + x_{m+i} \quad (1 \leqq i \leqq n)$$

admits of the approximation $\varphi(t)$, if to every number $A > 0$ there exist integers $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}$ so that the relations

$$X = \max_{1 \leqq j \leqq m} (|x_j|) > A,$$

$$|\vartheta_{i1}x_1 + \ldots + \vartheta_{im}x_m + x_{m+i}| < \varphi(X) \quad (1 \leqq i \leqq n)$$

are fulfilled simultaneously.

We then see immediately that the system $\vartheta_{i1}x_1 + \ldots + \vartheta_{im}x_m + x_{m+i}$ $(1 \leqq i \leqq n)$ admits of the approximation $t^{-\frac{m}{n}}$.

Khintchine [4] investigated only the following special systems (corresponding to the cases $m = 1$ or $n = 1$):

1. systems formed by a single linear form of $m + 1$ variables: $\vartheta_1 x_1 + \vartheta_2 x_2 + \ldots + \vartheta_m x_m + x_{m+1}$, or
2. systems formed by $m$ linear forms of $m + 1$ variables: $\vartheta_j y_1 + y_{j+1}$ $(1 \leqq j \leqq m)$.

For $t \geqq 1$ he introduced the non-negative functions:

$$\psi_1(t) = \psi_1(t, \vartheta_1, \ldots, \vartheta_m) = \min_{\substack{0 < \max|x_j| \leqq t \\ 1 \leqq j \leqq m}} (|\vartheta_1 x_1 + \ldots + \vartheta_m x_m + x_{m+1}|)$$

and

$$\psi_2(t) = \psi_2(t, \vartheta_1, \ldots, \vartheta_m) = \min_{0 < y_1 \leqq t} (\max_{1 \leqq j \leqq m} |\vartheta_j y_1 + y_{j+1}|).$$

In order to investigate the connection between the functions $\psi_1(t)$ and $\psi_2(t)$, when $\vartheta_1, \vartheta_2, \ldots, \vartheta_m$ are given, he introduced the following numbers: $\beta_1 = \beta_1(\vartheta_1, \ldots, \vartheta_m)$ and $\beta_2 = \beta_2(\vartheta_1, \ldots, \vartheta_m)$ respectively is the upper bound of those $\lambda$ for which $\liminf_{t \to \infty} \psi_1(t)\, t^{m+\lambda} < +\infty$ and $\liminf_{t \to \infty} \psi_2(t) t^{\frac{1+\lambda}{m}} < +\infty$ respectively. In other words: $\beta_1$ is the upper bound of those $\lambda$ for which the form $\vartheta_1 x_1 + \vartheta_2 x_2 + \ldots + \vartheta_m x_m + x_{m+1}$ admits of the approximation $Kt^{-(m+\lambda)}$ and similarly $\beta_2$ is the upper bound of those

$\lambda$ for which the system $\vartheta_j y_1 + y_{j+1}$ $(1 \leqq j \leqq m)$ admits of the approximation $K' t^{-\frac{1+\lambda}{m}}$, $K$ and $K'$ being some finite positive numbers.

From the previous theorem it follows immediately that $\beta_1 \geqq 0$, $\beta_2 \geqq 0$.

In what concerns the relations between $\beta_1$ and $\beta_2$ KHINTCHINE [4] proved the so called principle of transfer:

*If $\vartheta_1, \ldots, \vartheta_m$ $(m \geqq 1)$ are given, we have*

$$\beta_1 \geqq \beta_2 \geqq \frac{\beta_1}{(m-1)\beta_1 + m^2}.$$

JARNÍK [1], [2] showed that the latter inequalities are sharp in the following sense: If $m > 1$ and $\mu > 0$, then there exist systems $\{\vartheta_1, \ldots, \vartheta_m\}$ and $\{\vartheta'_1, \ldots, \vartheta'_m\}$ of such kind that

$$\beta_1(\vartheta_1, \ldots, \vartheta_m) = \beta_1(\vartheta'_1, \ldots, \vartheta'_m) = \mu,$$

$$\beta_2(\vartheta_1, \ldots, \vartheta_m) = \mu; \; \beta_2(\vartheta'_1, \ldots, \vartheta'_m) = \frac{\mu}{(m-1)\mu + m^2}.$$

JARNÍK [3] investigated further the same systems, examining, however, the maximum order of the functions $\psi_1(t)$ and $\psi_2(t)$ instead of their minimum order. For the purpose of this investigation he again introduced the numbers $\alpha = \alpha(\vartheta_1, \ldots, \vartheta_m)$ and $\beta = \beta(\vartheta_1, \ldots, \vartheta_m)$ in the following way: $\alpha$ and $\beta$ respectively is the upper bound of those $\lambda$ for which the respective inequalities $\limsup_{t \to \infty} \psi_1(t)\, t^{m+\lambda} < +\infty$ and $\limsup_{t \to \infty} \psi_2(t)\, t^{\frac{1+\lambda}{m}} < +\infty$ hold. He dealt only with non-trivial "proper" sytems, i. e. such ones for which $\psi_1(t) > 0$ and $\psi_2(t) > 0$ for every finite $t > 0$.

Now we sum up the most important results of his paper [3] into following theorems:

1. *Let $m = 2$; let $\{\vartheta_1, \vartheta_2\}$ be a proper system. Then*

$$\alpha = \frac{2\beta}{1-\beta}$$

*(one to one correspondence of $\alpha$ and $\beta$).*

2. *Let $\{\vartheta_1, \ldots, \vartheta_m\}$ be a proper system, $m \geqq 2$, $\alpha = \alpha(\vartheta_1, \ldots, \vartheta_m)$, $\beta = \beta(\vartheta_1, \ldots, \vartheta_m)$. Then the following relations take place:*

*a)* $m - 1 \geqq \beta \geqq 0, \; \alpha \geqq \beta \geqq \dfrac{\alpha}{(m-1)\alpha + m^2}.$

*b) If* $\alpha > 2m(m-2)$, $\beta$ *fulfils the inequality*

$$\beta \geqq \frac{\alpha - 2(m-2)}{(m-1)\,\alpha - m^2 + 5m - 4}.$$

*c) If* $\beta > m - 2$, *then similarly* $\alpha \geqq \dfrac{2\beta - (m-2)}{m - 1 - \beta}$.

It is obvious that the assertion b) as well as the assertion c) is a sharpening of the assertion a).

These results of Jarník are sharp, too, at least in the case $\alpha = +\infty$:

3. *For* $m > 2$ *there exist two proper systems* $\{\vartheta_1, \ldots, \vartheta_m\}$ *and* $\{\vartheta'_1, \ldots, \ldots, \vartheta'_m\}$ *so that*

$$\alpha(\vartheta_1, \ldots, \vartheta_m) = \alpha(\vartheta'_1, \ldots, \vartheta'_m) = +\infty,$$

$$\beta(\vartheta_1, \ldots, \vartheta_m) = m - 1,\ \beta(\vartheta'_1, \ldots, \vartheta'_m) = \frac{1}{m-1}.$$

The present paper forms a generalization of Jarník's results to systems containing any number of linear homogeneous forms in any number of variables. We shall get results quite analogical to those of Jarník [3], and shall finally show that there are no such integers $m > 1$ and $n > 1$ for which a single-valued or even a mutually single-valued dependence between $\alpha$ and $\beta$ would hold, as it is in the cases $m = n = 1$ and $m = 2$, $n = 1$.

## § 2. Auxiliary Theorems.

Throughout this treatise the letters $x, y, u, v, z$ with any indices will always denote integers and $\vartheta$ with any indices real numbers.

Let $m$ and $n$ be two positive integers and let $\vartheta_{ij}$ be a set of $mn$ real numbers ($1 \leqq i \leqq n$; $1 \leqq j \leqq m$). From these numbers let us construct two systems of linear forms (we shall keep on using this notation)

$$S_i(\mathbf{x}) = \vartheta_{i1}x_1 + \ldots + \vartheta_{im}x_m + x_{m+i} \quad (1 \leqq i \leqq n) \tag{1}$$

and

$$T_j(\mathbf{y}) = \vartheta_{1j}y_1 + \ldots + \vartheta_{nj}y_n + y_{n+j} \quad (1 \leqq j \leqq m). \tag{2}$$

It will be our task to investigate the relations between the approximate solution of the system $S_i(\mathbf{x}) = 0$ ($1 \leqq i \leqq n$) by means of the integers $x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}$ and of the system $T_j(\mathbf{y}) = 0$ ($1 \leqq j \leqq m$) by integers $y_1, \ldots, y_n, y_{n+1}, \ldots, y_{m+n}$. The respective trivial solutions $x_1 = x_2 = \ldots = x_{m+n} = 0$ and $y_1 = y_2 = \ldots = y_{m+n} = 0$ will be not considered here.

**Definition I.** *We shall call* (1) *a* ***regular*** *system if for each point* $[x_1, \ldots, x_m]$ *different from the origin the inequalities* $|S_i(\mathbf{x})| > 0$ ($1 \leqq i \leqq n$)

*are fulfilled. We shall call* (1) *a* **proper** *system, if the equalities* $S_i(\mathbf{x}) = 0$ $(1 \leqq i \leqq n)$ *imply that* $x_1 = x_2 = \ldots = x_{m+n} = 0$. *The other systems, i. e. those which are neither regular nor proper, shall be called* **improper** *systems.*

Similarly, we define the regular, proper and improper systems corresponding to the system (2).

We see that a proper system has the property that to every point $[x_1, x_2, \ldots, x_{m+n}]$ there is at least one form of the above system which is different from zero in that point.

For the respective systems (1) and (2) and real $t \geqq 1$ we define further the functions

$$\psi_1(t) = \min_{\substack{0 < \max|x_j| \leqq t \\ 1 \leqq j \leqq m}} \left( \max_{1 \leqq i \leqq n} |\vartheta_{ij} x_1 + \ldots + \vartheta_{im} x_m + x_{m+i}| \right), \tag{3}$$

$$\psi_2(t) = \min_{\substack{0 < \max|y_i| \leqq t \\ 1 \leqq i \leqq n}} \left( \max_{1 \leqq j \leqq m} |\vartheta_{1j} y_1 + \ldots + \vartheta_{nj} y_n + y_{n+j}| \right). \tag{4}$$

It is obvious that both functions $\psi_1(t)$ and $\psi_2(t)$ are non-negative and non-increasing functions of the variable $t$. If we denote by $[t]$ the integer defined by $[t] \leqq t < [t] + 1$, then $\psi_1(t) = \psi_1([t])$ and $\psi_2(t) = \psi_2([t])$.

We shall study (if the numbers $\vartheta_{ij}$ are given) the relation between the asymptotic behaviour of $\psi_1(t)$ and that of $\psi_2(t)$ for $t \to +\infty$.

In our further discussions we will often use MINKOWSKI's well-known theorem, which I quote without proof [6].

**Theorem 1:** *Let* $\xi_i(x_1, \ldots, x_r)$ $(1 \leqq i \leqq r)$ *be a system of r linear homogeneous forms in r variables* $x_1, x_2, \ldots, x_r$ *with any real coefficients and with the determinant* $\Delta \neq 0$. *Further let* $t_1, t_2, \ldots, t_r$ *be positive numbers such that* $t_1 t_2 \ldots t_r = |\Delta|$. *Then there is at least one lattice point different from the origin and such that*

$$\left.\begin{aligned} |\xi_1(x_1, \ldots, x_r)| &\leqq t_1, \\ |\xi_i(x_1, \ldots, x_r)| &< t_i \quad (2 \leqq i \leqq r). \end{aligned}\right\} \tag{5}$$

Remark: The condition that $[x_1, \ldots, x_r]$ is different from the origin may be also written $0 < \max\limits_{1 \leqq i \leqq r}(|x_i|)$ or $\sum\limits_{i=1}^{r} x_i^2 > 0$.

**Theorem 2:** *Let m and n be positive integers,* $\vartheta_{ij}$ $(1 \leqq i \leqq n;\ 1 \leqq j \leqq m)$ *and* $t \geqq 1$ *real numbers. Then the inequalities*

$$\left.\begin{aligned} |\vartheta_{i1} x_1 + \ldots + \vartheta_{im} x_m + x_{m+i}| &\leqq t^{-\frac{m}{n}} \quad (1 \leqq i \leqq n), \\ 0 < \max_{1 \leqq j \leqq m} (|x_j|) &\leqq t \end{aligned}\right\} \tag{6}$$

*have at least one solution by integers* $x_1, \ldots, x_{m+n}$.

Proof of Theorem 2. In Theorem 1 I put $r = m + n$; $\xi_i(x_1, \ldots, \ldots, x_{m+n}) = S_i(\mathbf{x})$ $(1 \leqq i \leqq n)$; $\xi_{n+j}(x_1, \ldots, x_{m+n}) = x_j$ $(1 \leqq j \leqq m)$; $t_1 = t_2 = \ldots = t_n = t^{-\frac{m}{n}}$; $t_{n+1} = t_{n+2} = \ldots = t_{m+n} = t$. The determinant of this system is then $\Delta = \pm 1$. Thus Theorem 2 is proved.

An immediate consequence is

**Theorem 3:** $\limsup_{t \to \infty} \psi_1(t)\, t^{\frac{m}{n}} \leqq 1, \limsup_{t \to \infty} \psi_2(t)\, t^{\frac{n}{m}} \leqq 1.$

I shall finally need another auxiliary theorem, found by MAHLER and sharpened by KHINTCHINE [5] the proof of which may be reproduced here. KHINTCHINE's sharpening is founded on making a weaker assumption (7): $|f_h(x)| \leqq t_h$ $(1 \leqq h \leqq n)$, $|f_1(x)| > 0$ instead of MAHLER's former assumption $|f_1(x)| = t_1$ and $|f_h(x)| \leqq t_h$ $(2 \leqq h \leqq n)$. This enables me to choose two independent parameters $A$ and $B$ instead of one in the proof of Theorem 5 so that I get the result very easily.

**Theorem 4.** *Let* $n \geqq 2$ *be an integer; besides let us suppose that* $f_h(\mathbf{x}) = \sum_{k=1}^{n} a_{hk} x_k$ $(1 \leqq h \leqq n)$ and $g_h(\mathbf{y}) = \sum_{k=1}^{n} b_{hk} y_k$ $(1 \leqq h \leqq n)$ *are two systems of linear forms and that the determinant of the second system is* $\Delta = |b_{hk}|_{h, k=1, \ldots, n} \neq 0$. *Further suppose that the bilinear form* $\sum_{h=1}^{n} f_h(\mathbf{x})\, g_h(\mathbf{y}) = \sum_{i,k=1}^{n} e_{ik} x_i y_k$ *has integral coefficients and that* $t_1, t_2, \ldots, t_{n-1}, t_n$ *denote positive numbers.*

*If there are such integers* $x_1, x_2, \ldots, x_n$ *that*

$$\left.\begin{aligned} |f_h(\mathbf{x})| \leqq t_h \quad (1 \leqq h \leqq n), \\ \sum_{i=1}^{n} x_i^2 > 0 \end{aligned}\right\} \qquad (7)$$

*and if at least for one* $h$ $(1 \leqq h \leqq n)$ *the inequality* $|f_h(\mathbf{x})| > 0$ *holds then there are integers* $y_1, y_2, \ldots, y_n$ *so that*

$$\left.\begin{aligned} |g_h(\mathbf{y})| \leqq \frac{c\lambda}{t_h} \quad (1 \leqq h \leqq n), \quad \sum_{i=1}^{n} y_i^2 > 0, \\ \text{where} \qquad \lambda_u^{-1} = |\Delta|\, t_1 t_2 \ldots t_n, \quad c = (2n)^{\frac{1}{n-1}}. \end{aligned}\right\} \qquad (8)$$

Proof: Without loss of generality, I may assume that $|f_1(\mathbf{x})| > 0$. I will construct the $n + 1$ forms $g_1(\mathbf{y}) - \frac{nc\lambda}{f_1(\mathbf{x})} u, g_2(\mathbf{y}), \ldots, g_n(\mathbf{y}), u$ which depend on $n + 1$ variables $y_1, y_2, \ldots, y_n, u$. The determinant of this system is equal to the determinant of the system $g_h(\mathbf{y})$ $(1 \leqq h \leqq n)$, i. e. to $\Delta \neq 0$.

If I choose the numbers $s_h$ so that $s_1 s_2 \dots s_n \, . \, s_{n+1} = |\Delta|$, $s_h > 0$ $(1 \leqq h \leqq n + 1)$, there is according to the Theorem 1 at least one lattice point $[y_1, \dots, y_n, u]$ of such kind that

$$\left.\begin{aligned} \left| g_1(\mathbf{y}) - \frac{nc\lambda}{f_1(\mathbf{x})} u \right| &< s_1, \\ |g_h(\mathbf{y})| &< s_h \quad (2 \leqq h \leqq n), \\ |u| &\leqq s_{n+1}, \\ \sum_{i=1}^{n} y_i^2 + u^2 &> 0. \end{aligned}\right\} \tag{9}$$

Next I put $z = \sum_{h=1}^{n} f_h(\mathbf{x})\, g_h(\mathbf{y})$ so that according to (7) and (9) I get $|f_1(\mathbf{x})\, g_1(\mathbf{y}) - nc\lambda u| < s_1 t_1$, $|f_h(\mathbf{x})\, g_h(\mathbf{y})| < s_h t_h$ $(2 \leqq h \leqq n)$, thus

$$|z - nc\lambda u| = \left| \sum_{h=1}^{n} f_h(\mathbf{x})\, g_h(\mathbf{y}) - nc\lambda u \right| < \sum_{h=1}^{n} s_h t_h, \tag{10}$$

hence

$$|z| < nc\lambda |u| + \sum_{h=1}^{n} s_h t_h. \tag{11}$$

Now I put $s_h = \dfrac{c\lambda}{t_h}$ $(1 \leqq h \leqq n)$, $s_{n+1} = \dfrac{1}{2nc\lambda}$ and in this case the relation $s_1 s_2 \dots s_n \, . \, s_{n+1} = |\Delta|$ takes the form

$$(c\lambda)^{n-1} = 2n|\Delta|\, t_1 t_2 \dots t_n \tag{12}$$

and the relation (11), the form $|z| < nc\lambda(1 + |u|)$.

For $|u| \geqq 1$ the last inequality gives further

$$|z| < 2nc\lambda |u| \leqq 2nc\lambda s_{n+1} = 1$$

and thus $z = 0$. But inequality (10) gives then

$$nc\lambda |u| < \sum_{h=1}^{n} s_h t_h = nc\lambda,$$

therefore $|u| < 1$, which is a contradiction with the above assumption $|u| \geqq 1$.

Hence necessarily $u = 0$ and the inequalities (9) take the form of the inequalities (8) which we had to prove. Equation (12) is satisfied by the choice of $\lambda^{n-1} = |\Delta|\, t_1 t_2 \dots t_n$, $c^{n-1} = 2n$, whereby Theorem 4 is entirely proved.

## § 3. The relation between the Maximum Orders of the functions $\psi_1(t)$ and $\psi_2(t)$.

We shall now use the notation introduced in § 2 by the formulae (1), (2), (3), and (4).

**Definition 2:** *We shall denote by $\alpha$ and $\beta$ the upper bound of the numbers $\omega$ and $\omega'$ for which the relations*

$$\limsup_{t \to \infty} \psi_1(t)\, t^{\frac{m+\omega}{n}} < +\infty$$

*and*

$$\limsup_{t \to \infty} \psi_2(t)\, t^{\frac{n+\omega'}{m}} < +\infty$$

*hold* [6].

Theorem 3 immediately implies that $\alpha \geqq 0$ and $\beta \geqq 0$. Throughout this paper we shall eliminate the following cases:

a) $m = n = 1$, where $\psi_1(t) = \psi_2(t)$ and $\alpha = \beta$.

b) The given system (1) is improper, i. e. for $t \geqq t_0$ there is $\psi_1(t) = 0$. In this case we put $\alpha = +\infty$.

c) Finally I will not make detailed investigations of the properties of the functions $\psi_1(t)$ and $\psi_2(t)$, if one of the numbers $m$ or $n$ is equal to unity, since this case has already been solved by Jarník [3].

**Theorem 5.** *Let $m$ and $n$ be positive integers satisfying the relation $m + n \geqq 3$; further let the system (1) be regular or proper and $0 < K < +\infty$. Let $\varphi_1(t)$ be a function which is positive, increasing and continuous in the interval $\langle t_0, +\infty)$, where $t_0 > 0$, and let $\lim_{t \to \infty} \varphi_1(t) = +\infty$. Finally, let us suppose, that*

$$\limsup_{t \to \infty} \varphi_1(t)\, \psi_1(t) < K. \tag{13}$$

*Then the following assertions hold:*

*I. a) If $m = 1$, then*

$$\limsup_{s \to \infty} s^{n-1} \varphi_1\left(\frac{s^n}{2(n+1)}\right) \psi_2(s) \leqq 2(n+1)\, K. \tag{14}$$

*b) If $m > 1$, then*

$$\limsup_{s \to \infty} \left(\frac{s^n}{\varphi_2\left(K^{\frac{m-1}{m+n-1}} s\right)}\right)^{\frac{1}{m-1}} \cdot \psi_2(s) \leqq (2(m+n))^{\frac{1}{m-1}}, \tag{15}$$

*where $\varphi_2(s)$ denotes the inverse function to the function $(t^m \varphi_1^{m-1}(t))^{\frac{1}{m+n-1}}$.*

*II. If* $m \geqq 2$, *if besides* (13) *for* $t \geqq t_0$ *the inequality*

$$\varphi_1(t) \geqq 2^{m+n-2} K t^{\frac{2(m+n-2)^2+m-2}{n}} \tag{16}$$

*is satisfied and if the function* $\varphi_1(t) \,.\, t^{-\frac{2m+n-4}{n}}$ *is increasing, it even holds that*

$$\limsup_{s\to\infty} \left( \frac{s^n}{\varphi_3(K^{\frac{m-1}{m+n-2}} s)} \right)^{\frac{1}{m-1}} \,.\; \psi_2(s) \leqq 3(m+n), \tag{17}$$

*where* $\varphi_3(s)$ *is the inverse function to the function*

$$\left( \varphi_1(t)\, t^{-\frac{(m-2)(2m+n-3)}{(m-1)n}} \right)^{\frac{m-1}{m+n-2}} .$$

Proof of Theorem 5. Theorem 3 implies that $\lim\limits_{t\to\infty} \psi_1(t) = 0$ and since $\psi_1(t) = \psi_1([t])$, it is possible to find a sequence of integers $t_1 < t_2 < t_3 < \ldots$ diverging to infinity in such a way that $\psi_1(t_1) > \psi_1(t_2) > \psi_1(t_3) > \ldots$ and that $\psi_1(t) = \psi_1(t_k)$ for $t \in \langle t_k, t_{k+1})$.

To every $k$ there certainly exists at least one lattice point $\mathbf{x}^{(k)} \equiv \equiv [x_1^{(k)}, \ldots, x_m^{(k)}, x_{m+1}^{(k)}, \ldots, x_{m+n}^{(k)}]$ such that

$$\max_{1\leqq j\leqq m} |x_j^{(k)}| = t_k \text{ and } \psi_1(t_k) = \max_{1\leqq i\leqq n} (|\vartheta_{i1} x_1^{(k)} + \ldots + \vartheta_{im} x_m^{(k)} + x_{m+i}^{(k)}|). \tag{18}$$

The greatest common divisor of the numbers $x_1^{(k)}, x_2^{(k)}, \ldots, x_{m+n}^{(k)}$ is then necessarily equal to unity, for if it were equal to the number $d > 1$, it would hold $\psi_1(t_k) = \max\limits_{1\leqq i\leqq n} (|\vartheta_{i1} x_1^{(k)} + \ldots + \vartheta_{im} x_m^{(k)} + x_{m+i}^{(k)}|) = d \,.$ $. \max\limits_{1\leqq i\leqq n} \left( \left| \vartheta_{i1} \frac{x_1^{(k)}}{d} + \ldots + \vartheta_{im} \frac{x_m^{(k)}}{d} + \frac{x_{m+i}^{(k)}}{d} \right| \right) \geqq d \,.\, \min\limits_{\substack{0 < \max |x_j| \leqq \frac{t_k}{d} \\ 1\leqq j\leqq m}} (\max\limits_{1\leqq i\leqq n} |\vartheta_{i1} x_1 +$ $+ \ldots + \vartheta_{im} x_m + x_{m+i}|)$, so that $\psi_1(t_k) \geqq d \,.\, \psi_1\left(\frac{t_k}{d}\right) > \psi_1\left(\frac{t_k}{d}\right)$ which is a contradiction with the fact that $\psi_1(t)$ is a non-increasing function.

For sufficiently large $t$'s and $k$'s respectively it will be $t_k \geqq t_0$ and $\psi_1(t)\,\varphi_1(t) < K$. For any $\varepsilon > 0$ we get with regard to the definition of the sequence $\{t_k\}$ the following relation: $\psi_1(t_k)\,\varphi_1(t_{k+1} - \varepsilon) = \psi_1(t_{k+1} - \varepsilon)\,\varphi_1(t_{k+1} - \varepsilon) < K$, whence we get (for $\varepsilon \to 0$)

$$\psi_1(t_k)\,\varphi_1(t_{k+1}) \leqq K. \tag{19}$$

As the system (1) is regular or proper, $\psi_1(t)$ is positive for every finite $t > 0$. The relation (18) and (19) give $0 < \max\limits_{1\leqq j\leqq m} |x_j^{(k)}| = t_k$, $0 <$

$< \max\limits_{1 \leq i \leq n} |S_i(\mathbf{x}^{(k)})| \leq \frac{K}{\varphi_1(t_{k+1})}$, so that if we choose two parameters $A \geq 1$ and $B \geq 1$, it will the more appear that

$$\left.\begin{aligned} 0 < \max_{1 \leq j \leq m} |x_j^{(k)}| \leq Bt_k \\ 0 < \max_{1 \leq i \leq n} |S_i(\mathbf{x}^{(k)})| \leq \frac{AK}{\varphi_1(t_{k+1})} \end{aligned}\right\} \tag{20}$$

Let us now apply Theorem 4 so that we therein substitute the number $m + n$ for number $n$ and put:

$f_i(\mathbf{x}) = S_i(\mathbf{x}^{(k)})$ $(1 \leq i \leq n)$, $f_{n+j}(\mathbf{x}) = x_j^{(k)}$ $(1 \leq j \leq m)$, $g_i(\mathbf{y}) = y_i$ $(1 \leq \leq i \leq n)$, $g_{n+j}(\mathbf{y}) = - T_j(\mathbf{y})$ $(1 \leq j \leq m)$, $t_1 = t_2 = \ldots = t_n = \frac{AK}{\varphi_1(t_{k+1})}$, $t_{n+1} = t_{n+2} = \ldots = t_{m+n} = Bt_k$.

We have $|\Delta| = 1$, $c^{m+n-1} = 2(m + n)$ and $\lambda^{m+n-1} = B^m t_k^m \left(\frac{AK}{\varphi_1(t_{k+1})}\right)^n$.

According to Theorem 4 there exists such a lattice point $\mathbf{y} \equiv [y_1, y_2, \ldots, \ldots, y_{m+n}]$ that

$$\left.\begin{aligned} \sum_{i=1}^{m+n} y_i^2 > 0, \quad |y_i| \leq \left(\frac{2(m + n) B^m t_k^m \varphi_1^{m-1}(t_{k+1})}{A^{m-1} K^{m-1}}\right)^{\frac{1}{m+n-1}} \quad (1 \leq i \leq n), \\ |T_j(\mathbf{y})| \leq \left(\frac{2(m + n) A^n K^n}{B^{n-1} t_k^{n-1} \varphi_1^n(t_{k+1})}\right)^{\frac{1}{m+n-1}} \quad (1 \leq j \leq m). \end{aligned}\right\} \tag{21}$$

I. a) Let $m = 1$. If $s$ is large enough, it is possible to find just one $k$ to it so that

$$(2(n + 1)\, t_k)^{\frac{1}{n}} \leq s < (2(n + 1)\, t_{k+1})^{\frac{1}{n}}.$$

In (21) I put $A = 1$, $B = \frac{s^n}{2(n + 1)\, t_k} \geq 1$ and I get

$$|y_i| \leq s \ (1 \leq i \leq n), \ \sum_{i=1}^{n} y_i^2 + y_{n+1}^2 > 0, \ |T_1(y)| \leq \frac{2(n + 1) K}{s^{n-1} \varphi_1(t_{k+1})}.$$

If $s$ is large enough, we have $|T_1(y)| < 1$ and if $\sum\limits_{i=1}^{n} y_i^2$ were equal to zero, $y_{n+1}$ would be necessarily equal to zero, too, and so it would hold also $\sum\limits_{i=1}^{n} y_i^2 + y_{n+1}^2 = 0$ which is not possible. Therefore $\sum\limits_{i=1}^{n} y_i^2 > 0$. As

$\varphi_1(t)$ is increasing, the inequality $\varphi_1\left(\frac{s^n}{2(n+1)}\right) < \varphi_1(t_{k+1})$ takes place so that there exists a lattice point $\mathbf{y} \equiv [y_1, y_2, \ldots, y_{n+1}]$ for which the inequalities

$$0 < \max_{1 \leqq i \leqq n} |y_i| \leqq s, \ \psi_2(s) \leqq |T_1(\mathbf{y})| < \frac{2(n+1)\, K}{s^{n-1}\varphi_1\left(\frac{s^n}{2(n+1)}\right)}$$

hold, what just implies the relation (14).

b) Let $m \geqq 2$. For each sufficiently large $t$ it is possible to find a positive integer $k$ so that $t_k \leqq t < t_{k+1}$. By putting

$$A = (2(m+n))^{\frac{1}{m-1}} \cdot \frac{\varphi_1(t_{k+1})}{\varphi_1(t)} > 1, \ B = \frac{t}{t_k} \geqq 1$$

in formula (21) we get

$$|y_i| \leqq \left(\frac{t^m \varphi_1^{m-1}(t)}{K^{m-1}}\right)^{\frac{1}{m+n-1}} \quad (1 \leqq i \leqq n), \ \sum_{i=1}^{m+n} y_i^2 > 0.$$

$$|T_j(\mathbf{y})| \leqq (2(m+n))^{\frac{1}{m-1}} \cdot \left(\frac{K^n}{t^{n-1}\varphi_1^n(t)}\right)^{\frac{1}{m+n-1}} \quad (1 \leqq j \leqq m).$$

For sufficiently large $t$'s the inequality $\max_{1 \leqq j \leqq m} |T_j(\mathbf{y})| < 1$ will be again satisfied so that $\sum_{i=1}^{n} y_i^2$ cannot be equal to zero for in this case it would also hold $\sum_{i=1}^{m+n} y_i^2 = 0$. Thus $\sum_{i=1}^{n} y_i^2 > 0$. As $t^m\varphi_1^{m-1}(t)$ is a continuous, increasing function, it is possible to find to each sufficiently large $s$ just one $t$ so that $s = \left(\frac{t^m \varphi^{m-1}(t)}{K^{m-1}}\right)^{\frac{1}{m+n-1}}$; accordingly there is a lattice point $\mathbf{y} \equiv \equiv [y_1, y_2, \ldots, y_{m+n}]$ so that $0 < \max_{1 \leqq i \leqq n} |y_i| \leqq s$ and $\psi_2(s) \leqq \max_{1 \leqq j \leqq m} |T_j(\mathbf{y})| \leqq$

$$\leqq (2(m+n))^{\frac{1}{m-1}} \cdot \left(\frac{K^n}{t^{n-1}\varphi_1^n(t)}\right)^{\frac{1}{m+n-1}} = (2(m+n))^{\frac{1}{m-1}} \cdot \left(\frac{t}{s^n}\right)^{\frac{1}{m-1}} =$$

$$= (2(m+n))^{\frac{1}{m-1}} \cdot \left(\frac{\varphi_2(K^{\frac{m-1}{m+n-1}} s)}{s^n}\right)^{\frac{1}{m-1}},$$ what implies (15).

II. Let $m \geqq 2$. Let us take two successive terms of the sequence $\{t_k\}: t_k$ and $t_{k+1}$; we write down the respective lattice points $\mathbf{x}^{(k)} \equiv [x_1^{(k)}, x_2^{(k)}, \ldots, x_{m+n}^{(k)}]$ and $\mathbf{x}^{(k+1)} \equiv [x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_{m+n}^{(k+1)}]$.

Now we show that there exist the values $j$ and $l$ $(j, l = 1, 2, \ldots, m)$ in such a way that $x_j^{(k)} x_l^{(k+1)} - x_j^{(k+1)} x_l^{(k)} \neq 0$. Were it not so, we should have

$$|x_{m+i}^{(k)} x_l^{(k+1)} - x_{m+i}^{(k+1)} x_l^{(k)}| = \left| \sum_{j=1}^{m} \vartheta_{ij} (x_j^{(k)} x_l^{(k+1)} - x_j^{(k+1)} x_l^{(k)}) + x_{m+i}^{(k)} x_l^{(k+1)} - \right.$$

$$\left. - x_{m+i}^{(k+1)} x_l^{(k)} \right| = |x_l^{(k+1)} S_i(\mathbf{x}^{(k)}) - x_l^{(k)} S_i(\mathbf{x}^{(k+1)})| \leqq t_{k+1} \psi_1(t_k) + t_k \psi_1(t_{k+1}) <$$

$$< \frac{2Kt_{k+1}}{\varphi_1(t_{k+1})} \leqq 2^{-(m+n-3)} t_{k+1}^{1 - \frac{2(m+n-2)^2 + m - 2}{n}} < 1,$$

therefore also $x_{m+i}^{(k)} x_l^{(k+1)} - x_{m+i}^{(k+1)} x_l^{(k)} = 0$ for $i = 1, 2, \ldots, n$.

As $\max\limits_{1 \leqq j \leqq m} |x_j^{(k)}| = t_k < t_{k+1} = \max\limits_{1 \leqq j \leqq m} |x_j^{(k+1)}|$, we have for a certain $l$ $(1 \leqq l \leqq m)$ the relation $|x_l^{(k)}| < |x_l^{(k+1)}|$.

If $x_l^{(k)} = 0$, we have necessarily $x_1^{(k)} = x_2^{(k)} = \ldots = x_{m+n}^{(k)} = 0$, a case we have excluded.

If $x_l^{(k)} \neq 0$, let us write $\left| \frac{x_l^{(k)}}{x_l^{(k+1)}} \right| = \frac{p}{q} < 1$, $(p, q) = 1$. Hence we have $p | x_j^{(k)}$ for $j = 1, 2, \ldots, m, m+1, \ldots, m+n$ and $q | x_j^{(k+1)}$ for $j = 1, 2, \ldots, m+n$, $pq > 1$, so that the greatest common divisors satisfy the relations:

$$(x_1^{(k)}, x_2^{(k)}, \ldots, x_{m+n}^{(k)}) \geqq p, \quad (x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_{m+n}^{(k+1)}) \geqq q,$$

which are impossible according to (18).

Without loss of generality it can be, therefore, assumed that

$$|x_1^{(k)} x_2^{(k+1)} - x_1^{(k+1)} x_2^{(k)}| = a \geqq 1, \tag{22}$$

$a$ being an integer.

For $i = 1, 2, \ldots, n$ I put further

$$\vartheta_{i1} x_1^{(k)} + \ldots + \vartheta_{im} x_m^{(k)} + x_{m+i}^{(k)} = \frac{\varepsilon_i^{(k)} K}{\varphi_1(t_{k+1})}, \quad |\varepsilon_i^{(k)}| \leqq 1,$$

$$\vartheta_{i1} x_1^{(k+1)} + \ldots + \vartheta_{im} x_m^{(k+1)} + x_{m+i}^{(k+1)} = \frac{\varepsilon_i^{(k+1)} K}{\varphi_1(t_{k+1})}, \quad |\varepsilon_i^{(k+1)}| < 1,$$

choose any integers $z_1, z_2, \ldots, z_n, z_{n+3}, \ldots, z_{m+n}$ and by means of them I construct the integers $y_1 = az_1$, $y_2 = az_2, \ldots, y_n = az_n$, $y_{n+3} = az_{n+3}, \ldots, y_{m+n} = az_{m+n}$. I can then find such integers $y_{n+1}$ and $y_{n+2}$, depending on the numbers $y_1, y_2, \ldots, y_n, y_{n+3}, \ldots, y_{m+n}$, that the relations

$$\sum_{i=1}^{n} y_i x_{m+i}^{(k)} = \sum_{j=1}^{m} y_{n+j} x_j^{(k)}, \quad \sum_{i=1}^{n} y_i x_{m+i}^{(k+1)} = \sum_{j=1}^{m} y_{n+j} x_j^{(k+1)}$$

are fulfilled simultaneously.

According to this we have $\sum_{i=1}^{n} y_i (\sum_{j=1}^{m} \vartheta_{ij} x_j^{(k)} + x_{m+i}^{(k)}) = \sum_{j=1}^{m} x_j^{(k)} \sum_{i=1}^{n} \vartheta_{ij} y_i +$ $+ \sum_{i=1}^{n} y_i x_{m+i}^{(k)} = \sum_{j=1}^{m} x_j^{(k)} \sum_{i=1}^{n} \vartheta_{ij} y_i + \sum_{j=1}^{m} y_{n+j} x_j^{(k)} = \sum_{j=1}^{m} x_j^{(k)} T_j(\mathbf{y}),$

or

$$\left.\begin{aligned} \sum_{j=1}^{m} x_j^{(k)} T_j(\mathbf{y}) &= \frac{K}{\varphi_1(t_{k+1})} \sum_{i=1}^{n} \varepsilon_i^{(k)} y_i \\ \text{and similarly} \qquad \sum_{j=1}^{m} x_j^{(k+1)} T_i(\mathbf{y}) &= \frac{K}{\varphi_1(t_{k+1})} \sum_{i=1}^{n} \varepsilon_i^{(k+1)} y_i. \end{aligned}\right\} \tag{23}$$

1. If $m = 2$, I choose $z_1 = 1, z_2 = z_3 = \ldots = z_n = 0$, thus according to (22) $0 < \max_{1 \leqq i \leqq n} |y_i| = |y_1| = a \leqq 2t_k t_{k+1}$ and according to (23)

$$x_1^{(k)} T_1(\mathbf{y}) + x_2^{(k)} T_2(\mathbf{y}) = \frac{K \varepsilon_1^{(k)} a}{\varphi_1(t_{k+1})},$$

$$x_1^{(k+1)} T_1(\mathbf{y}) + x_2^{(k+1)} T_2(\mathbf{y}) = \frac{K \varepsilon_1^{(k+1)} a}{\varphi_1(t_{k+1})};$$

therefore there exists a lattice point $\mathbf{y} \equiv [y_1, \ldots, y_n, y_{n+1}, y_{n+2}]$ satisfying the relations

$$\left.\begin{aligned} 0 < \max_{1 \leqq i \leqq n} |y_i| &\leqq 2t_k t_{k+1}, \\ |T_j(\mathbf{y})| &< \frac{2Kt_{k+1}}{\varphi_1(t_{k+1})} \quad (j = 1, 2). \end{aligned}\right\} \tag{24}$$

2. Suppose that $m > 2$, $\zeta \geqq 1$. According to Theorem 2 there are integers $z_1, z_2, \ldots, z_n, z_{n+3}, \ldots, z_{m+n}$ so that $0 < \max_{1 \leqq i \leqq n} |z_i| \leqq \zeta$ and $|\vartheta_{1j} z_1 + \ldots + \vartheta_{nj} z_n + z_{n+j}| \leqq \zeta^{-\frac{n}{m-2}}$ $(3 \leqq j \leqq m)$, whence

$$\left.\begin{aligned} 0 < \max_{1 \leqq i \leqq n} |y_i| &\leqq a\zeta, \\ |T_j(\mathbf{y})| &\leqq a\zeta^{-\frac{n}{m-2}} \quad (3 \leqq j \leqq m). \end{aligned}\right\} \tag{25}$$

According to (23) we have

$$x_1^{(k)} T_1(\mathbf{y}) + x_2^{(k)} T_2(\mathbf{y}) = -\sum_{j=3}^{m} x_j^{(k)} T_j(\mathbf{y}) + \frac{K}{\varphi_1(t_{k+1})} \sum_{i=1}^{n} \varepsilon_i^{(k)} y_i,$$

$$x_1^{(k+1)}T_1(\mathbf{y}) + x_2^{(k+1)}T_2(\mathbf{y}) = -\sum_{j=3}^{m} x_j^{(k+1)}T_j(\mathbf{y}) + \frac{K}{\varphi_1(t_{k+1})}\sum_{i=1}^{n}\varepsilon_i^{(k+1)}y_i,$$

thus we get in account of (25), (22) and of the inequalities $|x_j^{(k)}| \leqq t_k$, $|x_j^{(k+1)}| \leqq t_{k+1}$ the relation:

$$|T_j(\mathbf{y})| < \frac{2(m-2)\,t_k t_{k+1}}{\zeta^{\frac{n}{m-2}}} + \frac{2nKt_{k+1}\zeta}{\varphi_1(t_{k+1})} \quad (j = 1, 2). \tag{26}$$

Since $a \leqq 2t_k t_{k+1}$, I further get by means of (25) and (26): $0 < \max\limits_{1 \leqq i \leqq n}|y_i| \leqq 2t_k t_{k+1}\zeta$,

$$|T_j(\mathbf{y})| < \frac{2(m-2)\,t_k t_{k+1}}{\zeta^{\frac{n}{m-2}}} + \frac{2nKt_{k+1}\zeta}{\varphi_1(k_{k+1})} \quad (1 \leqq j \leqq m).$$

The best estimation is given by the choice $\zeta = \left(\frac{t_k\varphi_1(t_{k+1})}{K}\right)^{\frac{m-2}{m+n-2}}$.

Thus for $m > 2$ there exists a lattice point $\mathbf{y} \equiv [y_1, \ldots, y_n, y_{n+1}, \ldots, y_{m+n}]$ for which the relations

$$\left.\begin{aligned} 0 < \max_{1 \leqq i \leqq n} |y_i| \leqq 2t_k t_{k+1}\left(\frac{t_k\varphi_1(t_{k+1})}{K}\right)^{\frac{m-2}{m+n-2}}, \\ \max_{1 \leqq j \leqq m} |T_j(\mathbf{y})| < 2(m+n-2)\,t_{k+1}\left(\frac{t_k^{m-2}K^n}{\varphi_1^n(t_{k+1})}\right)^{\frac{1}{m+n-2}} \end{aligned}\right\} \tag{27}$$

hold good.

With respect to (24) the inequalities (27) hold good for all integers $m \geqq 2$.

To each sufficiently large $s$ it is then possible to find a positive integer $k$ so that

$$2t_k t_{k+1}\left(\frac{t_k\varphi_1(t_{k+1})}{K}\right)^{\frac{m-2}{m+n-2}} \leqq s < 2t_{k+1}t_{k+2}\left(\frac{t_{k+1}\varphi_1(t_{k+2})}{K}\right)^{\frac{m-2}{m+n-2}}. \tag{28}$$

I put now $s_o = \left(\frac{\varphi_1(t_{k+1})}{K}\right)^{\frac{m-1}{m+n-2}} \cdot t_{k+1}^{-\frac{(m-2)(2m+n-3)}{n(m+n-2)}}$.

Next it appears that $\dfrac{s_o}{2t_k t_{k+1}\left(\frac{t_k\varphi_1(t_{k+1})}{K}\right)^{\frac{m-2}{m+n-2}}} >$

$$> \frac{1}{2}\left(\frac{\varphi_1(t_{k+1})}{K}\right)^{\frac{1}{m+n-2}} \cdot t_{k+1}^{-2-\frac{(m-2)(2m+2n-3)}{n(m+n-2)}} \geqq$$

$$\geqq t_{k+1}^{\frac{2(m+n-2)^2+m-2}{n(m+n-2)}-2-\frac{(m-2)(2m+2n-3)}{n(m+n-2)}} = 1.$$

Let $s \leqq s_o$. According to (27) and (28) we have

$$\psi_2(s) < 2(m+n-2)\, t_k^{\frac{m-2}{m+n-2}} \cdot t_{k+1}^{1-\frac{(m-2)(2m+n-3)}{(m-1)(m+n-2)}} \cdot s_o^{-\frac{n}{m-1}} <$$

$$< 2(m+n-2)\left(\frac{t_{k+1}}{s_o^n}\right)^{\frac{1}{m-1}} = 2(m+n-2)\left(\frac{\varphi_3(K^{\frac{m-1}{m+n-2}}s_o)}{s_o^n}\right)^{\frac{1}{m-1}} \leqq$$

$$\leqq 2(m+n-2)\left(\frac{\varphi_3(K^{\frac{m-1}{m+n-2}}s)}{s^n}\right)^{\frac{1}{m-1}},$$

for

$$\frac{\varphi_3(s)}{s^n} = \frac{t^{1+\frac{(m-2)(2m+n-3)}{m+n-2}}}{(\varphi_1(t))^{\frac{n(m-1)}{m+n-2}}} = \left(\frac{t^{\frac{2m+n-4}{n}}}{\varphi_1(t)}\right)^{\frac{n(m-1)}{m+n-2}}$$

is, according to our supposition, a decreasing function.

For $s > s_o$ we use (21), where we put $k+1$ instead of $k$, and besides

$$A = 2\,\frac{\varphi_1(t_{k+2})}{K}\,(2(m+n))^{\frac{1}{m-1}} \cdot \left(\frac{t_{k+1}^m}{s^{m+n-1}}\right)^{\frac{1}{m-1}}, \quad B = 2^{\frac{m-1}{m}} > 1.$$

According to (28) we have obviously $\frac{1}{s} > \frac{1}{2t_{k+1}t_{k+2}} \cdot \left(\frac{K}{t_{k+1}\varphi_1(t_{k+2})}\right)^{\frac{m-2}{m+n-2}}$, accordingly

$$A > 2^{1+\frac{1}{m-1}-\frac{m+n-1}{m-1}} \cdot \left(\frac{\varphi_1(t_{k+2})}{K}\right)^{\frac{n}{(m-1)(m+n-2)}} \cdot t_{k+2}^{-\frac{m+n-1}{m-1}} \cdot$$

$$\cdot\, t_{k+1}^{-\frac{n-1}{m-1}-\frac{(m-2)(m+n-1)}{(m-1)(m+n-2)}} \geqq t_{k+2}^{\frac{(m+n-2)^2-n}{(m-1)(m+n-2)}} \cdot t_{k+1}^{-\frac{(m+n-2)^2-n}{(m-1)(m+n-2)}} > 1,$$

so that after putting it in (21) we get

$$0 < \max_{1 \leq i \leq n} |y_i| \leq s, \ \psi_2(s) \leq \max_{1 \leq j \leq m} |T_j(\mathbf{y})| \leq 2^{\frac{1}{m}} (2(m+n))^{\frac{1}{m-1}} \left(\frac{t_{k+1}}{s^n}\right)^{\frac{1}{m-1}}$$

and as the relation $s > s_0$ holds good, we have

$$\psi_2(s) < 2^{\frac{1}{m}} (2(m+n))^{\frac{1}{m-1}} \left(\frac{\varphi_3(K^{\frac{m-1}{m+n-2}} s)}{s^n}\right)^{\frac{1}{m-1}}.$$

But further $\max(2(m+n-2), \ 2^{\frac{1}{m}} (2(m+n))^{\frac{1}{m-1}}) \leq \max(2(m+ + n), \ 2\sqrt{2}(m+n)) < 3(m+n)$ so that inequality (17) is also proved, and hereby, therefore, the whole theorem 5.

**Theorem 6.** (*The generalized principle of transfer*). *Let $m$ and $n$ be positive integers; we choose $\alpha$ and $\beta$ according to Definition 2. Then it holds that*:

1. $$\beta \geq \frac{n\alpha}{m(m+n-1)+(m-1)\alpha},$$

$$\alpha \geq \frac{m\beta}{n(m+n-1)+(n-1)\beta}.$$

2. *If $m \geq 2$ and if $\alpha > 2(m+n-1)(m+n-3)$, then we have even*

$$\beta \geq \frac{n\alpha - 2n(m+n-3)}{(m-1)\alpha + m - (m-2)(m+n-3)}.$$

Remark 1: If e. g. we have $\alpha = +\infty$, the mentioned inequalities are to be understood so that

$$\beta \geq \lim_{\alpha \to \infty} \frac{n\alpha}{m(m+n-1)+(m-1)\alpha} = \frac{n}{m-1}$$

and

$$\beta \geq \lim_{\alpha \to \infty} \frac{n\alpha - 2n(m+n-3)}{(m-1)\alpha + m - (m-2)(m+n-3)} = \frac{n}{m-1}.$$

This occurs if system (1) is either improper or if (1) is regular indeed, but can be approximated with a great precision. Such systems will be constructed in the next paragraph.

Remark 2: The reader will easily convince himself that in the case $\alpha > 2(m+n-1)(m+n-3)$ the second statement of Theorem 6 is sharper than the first one.

Proof of Theorem 6: For $m = n = 1$ we have obviously $\alpha = \beta$. Let thus $m + n \geqq 3$. For regular or proper systems the first as well as the second statement follow from Theorem 5, if I put $\varphi_1(t) = t^{\frac{m+\alpha'}{n}}$ ($\alpha' < \alpha$, $\alpha' \to \alpha$).

It remains to show that both the statements remain true for improper systems, too. In this case $\psi_1(t)$ is equal to zero for sufficiently large $t$'s and therefore we put $\alpha = +\infty$. According to Remark 1 it is necessary to show that $\beta \geqq \frac{n}{m-1}$ for $m > 1$, and $\beta = +\infty$ for $m = 1$.

The case $m = 1$ is trivial, as we have, for sufficiently large $s$'s,

$$\psi_2(s) = \min_{\substack{0 < \max|y_i| \leqq s \\ 1 \leqq i \leqq n}} (|\vartheta_{11} y_1 + \ldots + \vartheta_{n1} y_n + y_{n+1}|) \leqq$$

$$\leqq \min_{0 < |y_1| \leqq s} |\vartheta_{11} y_1 + y_{n+1}| \leqq \psi_1(s) = 0, \text{ thus } \beta = +\infty.$$

If $m > 1$, there is such a lattice point $\mathbf{x} \equiv [x_1, \ldots, x_m, x_{m+1}, \ldots, \ldots, x_{m+n}]$ that $0 < \max_{1 \leqq j \leqq m} |x_j| = X$ and $\vartheta_{ij} x_1 + \ldots + \vartheta_{im} x_m + x_{m+i} = 0$ $(1 \leqq i \leqq n)$. Without loss of generality let $|x_1| = X$. According to Theorem 2 there is such a lattice point $[z_1, z_2, \ldots, z_n, z_{n+2}, \ldots, z_{m+n}]$ that

$$\left.\begin{aligned} 0 < \max_{1 \leqq i \leq n} |z_i| \leqq t, \\ |\vartheta_{1j} z_1 + \ldots + \vartheta_{nj} z_n + z_{n+j}| \leqq t^{-\frac{n}{m-1}} \; (2 \leqq j \leqq m). \end{aligned}\right\} \tag{29}$$

I put $\sum_{i=1}^{n} x_{m+i} y_i = \sum_{j=1}^{m} x_j y_{n+j}$, where $y_1 = x_1 z_1, \ldots, y_n = x_1 z_n$, $y_{n+2} = = x_1 z_{n+2}, \ldots, y_{m+n} = x_1 z_{m+n}$ so that I get $y_{n+1} = \sum_{i=1}^{n} x_{m+i} z_i - \sum_{j=2}^{m} x_j z_{n+j}$. We have next $\sum_{j=1}^{m} x_j T_j(\mathbf{y}) = \sum_{j=1}^{m} x_j (\sum_{i=1}^{n} \vartheta_{ij} y_i + y_{n+j}) = \sum_{i=1}^{n} y_i \sum_{j=1}^{m} \vartheta_{ij} x_j + + \sum_{i=1}^{n} x_{m+i} y_i = \sum_{i=1}^{n} y_i S_i(\mathbf{x}) = 0$, so that $x_1 T_1(\mathbf{y}) = -\sum_{j=2}^{m} x_j T_j(\mathbf{y})$.

But according to (29) the relations

$$0 < \max_{1 \leqq i \leqq n} |y_i| \leqq tX,$$

$$|\vartheta_{1j} y_1 + \ldots + \vartheta_{nj} y_n + y_{n+j}| \leqq X t^{-\frac{n}{m-1}} \; (2 \leqq j \leqq m)$$

hold and we have

$$X|T_1(\mathbf{y})| = |\sum_{j=2}^{m} x_j T_j(\mathbf{y})| \leqq (m-1)\, X^2 t^{-\frac{n}{m-1}},$$

$$|T_1(\mathbf{y})| \leqq (m-1)\, X t^{-\frac{n}{m-1}},$$

so that putting $tX = s$ we finally get

$$\psi_2(s) \leqq (m-1)\, X^{\frac{m+n-1}{m-1}} s^{-\frac{n}{m-1}}.$$

But $X$ is some finite fixed integer, thus $\limsup_{s\to\infty} s^{\frac{n}{m-1}} \psi_2(s) \leqq (m-1)$ . $. X^{\frac{m+n-1}{m-1}}$ and hence $\frac{n+\beta}{m} \geqq \frac{n}{m-1}$. or $\beta \geqq \frac{n}{m-1}$, Q. E. D.

The indices $\alpha$ and $\beta$ depend, of course, on the given system $\Theta = \begin{pmatrix} \vartheta_{11}, & \vartheta_{12}, & \ldots, & \vartheta_{1m} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \vartheta_{n1}, & \vartheta_{n2}, & \ldots, & \vartheta_{nm} \end{pmatrix}$ and therefore, if we want to distinguish the indices of various systems, we write $\alpha = \alpha(\Theta)$, $\beta = \beta(\Theta)$.

Now the question arises to what degree Theorem 6 is definitive, i. e. whether it is not possible to replace the inequalities of this theorem by sharper ones, eventually whether it does not suffice to know for the given system $\Theta$ but one of the indices $\alpha$ or $\beta$, as it is the case for $m = n = 1$ or $m = 2$, $n = 1$ [3].

Some answers to these questions are given in the following paragraph.

## § 4. The Existence and the Properties of some special Systems Θ.

**Theorem 7.** *Let $m \geqq 2$, $n \geqq 2$. Let $\varphi(t)$ denote a function of $t$, increasing as rapidly as we wish and having the following properties:*

1. *$\varphi(t)$ is defined for $t \geqq t_o > 0$,*
2. *it is continuous and increasing in the interval $\langle t_o, \infty)$,*
3. *in $\langle t_o, \infty)$ the inequality $\varphi(t) \geqq t$ holds and*
4. *$t^{-1}\varphi(t)$ is not decreasing in $\langle t_o, \infty)$.*

*Then there are systems $\Theta$ which are regular in their rows and columns in such a way that we have $\psi_1(t) = \psi_1(t, \Theta) = o\left(\frac{1}{\varphi(t)}\right)$, $\psi_2(t) = \psi_2(t, \Theta) = o\left(\frac{1}{\varphi(t)}\right)$.*

I will prove Theorem 7 by constructing a system of such kind. I put

$$a_{ij} = ij \quad (1 \leqq i \leqq n;\ 1 \leqq j \leqq m), \tag{30}$$

and therefore $0 < a_{ij} \leqq mn$ and I choose the prime number $p_1 \geqq \max(mn + 1, t_o)$. Besides I choose the sequence of prime numbers $p_2 < p_3 < p_4 < \ldots$ in such a manner that the inequality

$$p_{k+1} \geqq p_1 + \varphi(p_1^{k+1} \cdot p_1 p_2 \cdots p_k), \tag{31}$$

may be fulfilled for each positive integer $k$.

I now construct the series $\sum_{k=1}^{\infty} \frac{a_{ij}^k}{p_1 \cdots p_k}$, which are absolutely convergent, and define the so called LIOUVILLE's numbers [7]

$$\vartheta_{ij} = \sum_{k=1}^{\infty} \frac{a_{ij}^k}{p_1 \cdots p_k} \quad (1 \leqq i \leqq n;\ 1 \leqq j \leqq m). \tag{32}$$

Finally I put

$$q_h = p_1 p_2 \cdots p_h, \quad \sum_{k=1}^{h} \frac{a_{ij}^k}{p_1 \cdots p_k} = \frac{r_{ij}^{(h)}}{q_h} \quad (1 \leqq i \leqq n;\ 1 \leqq j \leqq m) \tag{33}$$

Then I get $\quad 0 < \vartheta_{ij} - \frac{r_{ij}^{(h)}}{q_h} = \sum_{k=h+1}^{\infty} \frac{a_{ij}^k}{q_k} < \sum_{k=h+1}^{\infty} \frac{p_1^k}{q_k} =$

$$= \frac{p_1^{h+1}}{q_h p_{h+1}} \left(1 + \frac{p_1}{p_{h+2}} + \frac{p_1^2}{p_{h+2} p_{h+3}} + \ldots\right).$$

As $\varphi(t) \geqq t$, we have $p_{h+1} \geqq p_1 + \varphi(p_1^{h+1} \cdot p_1 p_2 \cdots p_h) > p_h$ and thus

$$\left|\vartheta_{ij} - \frac{r_{ij}^{(h)}}{q_h}\right| < \frac{p_1^{h+1}}{q_h p_{h+1}} \left(1 + \frac{p_1}{p_{h+1}} + \frac{p_1^2}{p_{h+1}^2} + \ldots\right) =$$

$$= \frac{p_1^{h+1}}{q_h(p_{h+1} - p_1)} \leqq \frac{p_1^{h+1}}{q_h \cdot \varphi(p_1^{h+1} \cdot p_1 p_2 \cdots p_h)} = \frac{p_1^{h+1}}{q_h \cdot \varphi(p_1^{h+1} q_h)}.$$

As $\frac{\varphi(t)}{t}$ is non-decreasing, we have

$$\frac{\varphi(p_1^{h+1} q_h)}{p_1^{h+1} q_h} \geqq \frac{\varphi(q_h)}{q_h},$$

hence

$$\frac{p_1^{h+1}}{\varphi(p_1^{h+1} q_h)} \leqq \frac{1}{\varphi(q_h)}$$

and thus

$$\left|\vartheta_{ij} - \frac{r_{ij}^{(h)}}{q_h}\right| < \frac{1}{q_h\varphi(q_h)} \quad (1 \leqq i \leqq n;\ 1 \leqq j \leqq m). \tag{34}$$

Accordingly, there is a sequence $q_1 < q_2 < \ldots$ so that $\lim\limits_{k\to\infty} q_k = +\infty$ and that (34) holds good.

The system $\Theta = \begin{pmatrix} \vartheta_{11}, \ldots, \vartheta_{1m} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \vartheta_{n1}, \ldots, \vartheta_{nm} \end{pmatrix}$ constructed in this way possesses already all properties stated by Theorem 7.

If for some of the forms (1) a lattice point $[x_1, \ldots, x_m, x_{m+i}]$ would exist in such a way that the relation $\vartheta_{i1}x_j + \ldots + \vartheta_{im}x_m + x_{m+i} = 0$ would be fulfilled, then according to (34) the expression $\frac{r_{i1}^{(h)}x_1 + \ldots + r_{im}^{(h)}x_m + q_hx_{m+i}}{q_h}$ ough to be also equal to the fraction $\frac{\varepsilon_{i1}x_1 + \varepsilon_{i2}x_2 + \ldots + \varepsilon_{im}x_m}{q_h\varphi(q_h)}$, where $|\varepsilon_{ij}| < 1$. As $X = \max\limits_{1\leqq j\leqq m} |x_j|$ is a finite integer, we have for all $h$'s beginning from a certain one the relation

$$|r_{i1}^{(h)}x_1 + \ldots + r_{im}^{(h)}x_m + q_hx_{m+i}| < \frac{mX}{\varphi(q_h)} < 1,$$

thus

$$r_{i1}^{(h)}x_j + \ldots + r_{im}^{(h)}x_m + q_hx_{m+i} = 0. \tag{35}$$

According to definition (33), however, we have:

$$\frac{r_{ij}^{(h+1)}}{q_{h+1}} = \frac{r_{ij}^{(h)}}{q_h} + \frac{a_{ij}^{h+1}}{q_{h+1}} \quad (1 \leqq i \leqq n;\ 1 \leqq j \leqq m),$$

thus

$$r_{ij}^{(h+1)} = \frac{q_{h+1}}{q_h} r_{ij}^{(h)} + a_{ij}^{h+1} \quad (1 \leqq i \leqq n;\ 1 \leqq j \leqq m),$$

so that owing to (35) for $h$ and $h+1$ we have further

$$0 = r_{i1}^{(h+1)}x_1 + \ldots + r_{im}^{(h+1)}x_m + q_{h+1}x_{m+i} = \frac{q_{h+1}}{q_h}(r_{i1}^{(h)}x_1 + \ldots + r_{im}^{(h)}x_m + q_hx_{m+i}) + a_{i1}^{h+1}x_1 + \ldots + a_{im}^{h+1}x_m = a_{i1}^{h+1}x_1 + \ldots + a_{im}^{h+1}x_m;$$

this is satisfied for all positive integers $h$ which are large enough. But then the system of equations

$$
\begin{array}{c}
a_{i1}^{h+1}x_1 + \ldots + a_{im}^{h+1}x_m = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{i1}^{h+m}x_1 + \ldots + a_{im}^{h+m}x_m = 0
\end{array}
$$

the determinant of which is

$$
\Delta = (a_{i1} \ldots a_{im})^{h+1} \begin{vmatrix} 1, & 1, \ldots, & 1 \\ a_{i1}, & a_{i2}, \ldots, & a_{im} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{i1}^{m-1}, & a_{i2}^{m-1}, \ldots, & a_{im}^{m-1} \end{vmatrix} \neq 0
$$

has only the trivial solution $x_1 = x_2 = \ldots = x_m = 0$, consequently the system $\Theta$ is regular in its rows.

For sufficiently large $t$'s such a positive integer $h$ can be found that $p_1^h q_{h-1} \leqq t < p_1^{h+1} q_h$.

I put $x_1 = 2^h q_{h-1}$, $x_2 = -q_{h-1}$, $x_3 = x_4 = \ldots = x_m = 0$,

$$
x_{m+i} = -\sum_{k=1}^{h-1} \frac{(2^h - 2^k)\, i^k}{q_k} q_{h-1} \qquad (1 \leqq i \leqq n).
$$

It is obvious that $0 < \max\limits_{1 \leqq j \leqq m} |x_j| < t$ and thus

$$
\psi_1(t, \Theta) \leqq \max_{1 \leqq i \leqq n} (|\vartheta_{i1} x_1 + \vartheta_{i2} x_2 + x_{m+i}|) = \max_{1 \leqq i \leqq n} \left( \left| \sum_{k=1}^{\infty} \frac{2^h i^k - 2^k i^k}{q_k} q_{h-1} - \right.\right.
$$

$$
\left.\left. - \sum_{k=1}^{h-1} \frac{(2^h - 2^k)\, i^k}{q_k} q_{h-1} \right| \right) = \max_{1 \leqq i \leqq n} \left( \left| \sum_{k=h+1}^{\infty} \frac{(2^h - 2^k)\, i^k}{q_k} q_{h-1} \right| \right) <
$$

$$
< \sum_{k=h+1}^{\infty} \frac{p_1^k}{p_h p_{h+1} \cdots p_k} < \frac{p_1^{h+1}}{p_h p_{h+1}} \left( 1 + \frac{p_1}{p_{h+1}} + \frac{1}{p_{h+1}^2} + \ldots \right) =
$$

$$
= \frac{p_1^{h+1}}{p_h (p_{h+1} - p_1)} \leqq \frac{p_1^{h+1}}{p_h \varphi(p_1^{h+1} \cdot p_1 p_2 \cdots p_h)} < \frac{p_1^{h+1}}{p_h} \cdot \frac{1}{\varphi(t)}.
$$

However, for $h > 2$ we have $p_h \geqq p_1 + \varphi(p_1^h \cdot p_1 p_2 \cdots p_{h-1}) > p_1^h \cdot p_1 p_2 \cdots p_{h-1} \geqq p_1^{h+1} p_{h-1}$, thus $\dfrac{p_1^{h+1}}{p_h} < \dfrac{1}{p_{h-1}}$, hence $\psi_1(t, \Theta) < \dfrac{1}{p_{h-1}\varphi(t)}$; But for $t \to \infty$ we have $h \to \infty$ and so $p_{h-1} \to \infty$, whence $\psi_1(t, \Theta) = o\left(\dfrac{1}{\varphi(t)}\right)$.

In quite the same manner we should prove that $\psi_2(t, \Theta) = o\left(\frac{1}{\varphi(t)}\right)$ and the regularity of the system $\Theta$ in the columns.

From Theorem 7 the following theorem results:

**Theorem 8.** *For $m \geq 2$, $n \geq 2$ there are regular systems $\Theta$ having the indices $\alpha(\Theta) = +\infty$, $\beta(\Theta) = +\infty$.*

Proof: It suffices to put $\varphi(t) = e^t$ in Theorem 7 and according to Definition 2 we get $\alpha(\Theta) = \beta(\Theta) = +\infty$.

**Theorem 9.** *For $m \geq 2$, $n \geq 1$ let $a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+n}$ be integers which are given in such a way that the numbers $a_1, a_2, \ldots, a_m$ are different from zero and that for $i \neq j$ $(i, j = 1, 2, \ldots, m)$ the greatest common divisor of each pair $(a_i, a_j)$ is equal to unity (e. g. $a_1, a_2, \ldots, a_m$ may be equal to $m$ successive prime numbers); further let the relation $|a_1| = \max_{1 \leq j \leq m} |a_j| = t_1$ be true. Moreover let $\varphi(t)$ and $\Phi(t)$ respectively be continuous functions in the interval $\langle t_1, \infty)$, first of them increasing as rapidly as we wish and the second one increasing as slowly as we wish; for $t \to \infty$ both functions tend to infinity. Finally suppose that in the mn-dimensional space of the points $\boldsymbol{\vartheta} \equiv [\vartheta_{11}, \ldots, \vartheta_{1m}; \ldots; \vartheta_{n1}, \ldots, \vartheta_{nm}]$ a mn-dimensional closed cube $K_1$ is given so that there is a point $\boldsymbol{\vartheta}$ within *)$K_1$ for which the relation.*

$$a_1\vartheta_{i1} + a_2\vartheta_{i2} + \ldots + a_m\vartheta_{im} + a_{m+i} = 0 \quad (1 \leq i \leq n) \tag{36}$$

*takes place.*

*Under these assumptions there exist two integers $s_1$ and $t_2$ $(t_1 < s_1 < t_2)$ and $m + n$ integers $a'_1, a'_2, \ldots, a'_m, a'_{m+1}, \ldots, a'_{m+n}$ of which $a'_1, a'_2, \ldots, a'_m$ are different from zero having the following properties: for $i \neq j$ $(i, j = 1, 2, \ldots, m)$ the greatest common divisor of each pair $(a'_i, a'_j)$ is equal to unity; it is $|a'_1| = \max_{1 \leq j \leq m} |a'_j| = t_2$ and there exists a closed cube $K_2 \subset K_1$ with following properties:*

1. *For no system of integers $b_1, \ldots, b_n, b_{n+1}, \ldots, b_{m+n-1}$ and for no point $\boldsymbol{\vartheta} \in K_2$ the inequalities*

$$0 < \max_{1 \leq i \leq n} |b_i| \leq s_1,$$

$$|b_1\vartheta_{1j} + b_2\vartheta_{2j} + \ldots + b_n\vartheta_{nj} + b_{n+j}| \leq \frac{1}{s_1^{\frac{n}{m-1}} \Phi(s_1)} \quad (1 \leq j \leq m - 1) \tag{37}$$

*are fulfilled simultaneously.*

---

*) This means (here and in the following text) in the *interior* of $K_1$.

2. *For no system of integers* $c_1, \ldots, c_m, c_{m+1}, \ldots, c_{m+n}$ *and for no point* $\boldsymbol{\vartheta} \in K_2$ *the inequalities*

$$\left.\begin{aligned} 0 < \max_{1 \leqq j \leqq m} |c_j| < t_1, \\ \min_{1 \leqq i \leqq n} (|c_1\vartheta_{i1} + c_2\vartheta_{i2} + \ldots + c_m\vartheta_{im} + c_{m+i}|) = 0 \end{aligned}\right\} \quad (38)$$

*are satisfied simultaneously.*

3. *For no system of integers* $g_o, g_{ij}$ $(1 \leqq i \leqq n;\ 1 \leqq j \leqq m)$ *and for no point* $\boldsymbol{\vartheta} \in K_2$ *the inequalities*

$$\left.\begin{aligned} 0 < \max_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq m}} |g_{ij}| < t_1, \\ \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij}\vartheta_{ij} + g_o = 0 \end{aligned}\right\} \quad (39)$$

*are fulfilled simultaneously.* (Of course, 2 is a consequence of 3.)

4. *For all points* $\boldsymbol{\vartheta} \in K_2$ *we have*

$$\max_{1 \leqq i \leqq n} (|a_1\vartheta_{i1} + a_2\vartheta_{i2} + \ldots + a_m\vartheta_{im} + a_{m+i}|) < \frac{1}{\varphi(t_2)}. \quad (40)$$

5. *Within* $K_2$ *there is a point* $\boldsymbol{\vartheta}$ *different from the origin for which*

$$a'_1\vartheta_{i1} + \ldots + a'_m\vartheta_{im} + a'_{m+i} = 0 \quad (1 \leqq i \leqq n). \quad (41)$$

P r o o f  o f  T h e o r e m 9: First of all, it is evident that according to the properties of the cube $K_1$ there exist real numbers $\gamma_{ij}$ and $\delta_{ij}$ $(1 \leqq i \leqq n;$ $1 \leqq j \leqq m-1)$ with the following property: If

$$\gamma_{ij} < \vartheta_{ij} < \delta_{ij} \quad (1 \leqq i \leqq n;\ 1 \leqq j \leqq m-1) \quad (42)$$

and if we calculate the values $\vartheta_{im}$ $(1 \leqq i \leqq n)$ from (36), the point $\boldsymbol{\vartheta} \equiv [\vartheta_{11}, \ldots, \vartheta_{1m}; \ldots; \vartheta_{n1}, \ldots, \vartheta_{nm}]$ is situated within $K_1$.

I next choose some integer $s_1 > t_1$ and investigate the inequalities (37) for some given system of the numbers $b_1, \ldots, b_n, b_{n+1}, \ldots, b_{m+n-1}$. Among the numbers $b_1, b_2, \ldots, b_n$ there is certainly at least one that is a maximum in its absolute value and without loss of generality we may suppose that $|b_1| = \max_{1 \leqq i \leqq n} |b_i| = b$. If I make any choice of the numbers $\vartheta_{2j}, \ldots, \vartheta_{nj}$ $(1 \leqq j \leqq m-1)$ according to (42), then every number $\vartheta_{1j}$, provided that the inequalities (37) hold good and $b_{n+j}$ are fixed, is contained in an interval of length

$$O\left(\frac{1}{b s_1^{\frac{n}{m-1}} \Phi(s_1)}\right)$$

so that the volume of the set of such points $[\vartheta_{11}, \ldots, \vartheta_{n1}; \vartheta_{12}, \ldots, \vartheta_{n2}; \ldots; \ldots; \vartheta_{1,m-1}, \ldots, \vartheta_{n,m-1}]$ (if $b_1, \ldots, b_n, b_{n+1}, \ldots, b_{m+n-1}$ are given) is equal to $O\left(\frac{1}{b^{m-1}s_1^n\Phi^{m-1}(s_1)}\right)$.

But for each of the numbers $b_2, \ldots, b_n, b_{n+1}, \ldots, b_{m+n-1}$ we have only $O(b)$ possibilities so that the $n(m-1)$-dimensional measure of the set of the points $[\vartheta_{11}, \ldots, \vartheta_{n1}; \ldots; \vartheta_{1,m-1}, \ldots, \vartheta_{n,m-1}]$ fulfilling the inequalities (37) will be at most equal to:

$$\sum_{b=1}^{s_1} O\left(\frac{1}{b^{m-1}s_1^n\Phi^{m-1}(s_1)} \cdot b^{m+n-2}\right) = O\left(\frac{1}{\Phi^{m-1}(s_1)}\right).$$

It is now sufficient to choose the number $s_1$ large enough in order to assure that the measure of this set is less than that of the set defined by the relations (42). In the set (42) there is, consequently, a point $P^* \equiv [\vartheta^*_{11}, \ldots, \vartheta^*_{n1}; \ldots; \vartheta^*_{1,m-1}, \ldots, \vartheta^*_{n,m-1}]$ which does not fulfil all the inequalities (37) simultaneously. As the eliminated set (37) is closed, there is a certain neighbourhood of the point $P^*$ every point of which has the property that formula (42) is satisfied and that all the inequalities (37) do not simultaneously hold good.

I will besides calculate $\vartheta^*_{im}$ $(1 \leq i \leq n)$ from the relations (36), and then get the point $\boldsymbol{\vartheta}^* \equiv [\vartheta^*_{11}, \ldots, \vartheta^*_{1m}; \ldots; \vartheta^*_{n1}, \ldots, \vartheta^*_{nm}]$ situated within $K_1$ to which I circumscribe a sufficiently small cube $K_1' \subset K_1$. This cube has already the following properties:

a) Within $K_1'$ lies a point that satisfies the conditions (36).

b) To no point $\boldsymbol{\vartheta}$ of $K_1'$ there are such integers $b_1, \ldots, b_n, b_{n+1}, \ldots, \ldots, b_{m+n-1}$ that all the inequalities of (37) would hold good simultaneously.

I now can choose e. g. for the integers $a_1', a_2', \ldots, a_m'$ $m$ successive prime numbers, $|a_1'| = t_2 > |a_2'| > \ldots > |a_m'| > |a_1|$. In doing so, I choose the signs in such a way that the relations

$$a_1a_2' - a_1'a_2 = |a_1a_2'| + |a_1'a_2| > \tfrac{1}{2}t_1t_2 \tag{43}$$

take place, for Chebyshev's inequality for prime numbers implies $|a_2'| < |a_1'| < 2|a_2'|$.

From the construction of the cube $K_1'$, we see immediately: There are real numbers $\gamma_{ij}' < \delta_{ij}'$ $(1 \leq i \leq n;\ 2 \leq j \leq m)$ with the following property: if

$$\gamma_{ij}' < \vartheta_{ij} < \delta_{ij}' \quad (1 \leq i \leq n;\ 2 \leq j \leq m) \tag{44}$$

and if I calculate $\vartheta_{i1}$ $(1 \leq i \leq n)$ from (36), the point $[\vartheta_{11}, \ldots, \vartheta_{1m}; \ldots; \vartheta_{n1}, \ldots, \vartheta_{nm}]$ lies within $K_1'$.

I now choose irrational numbers $\vartheta_{i3}^{+}$ $(1 \leqq i \leqq n)$ so that for $u_3 \neq 0$ the values $\vartheta_{i3}^{+}u_3 + u_{m+i}$ $(1 \leqq i \leqq n)$ are different from zero. Further I choose the numbers $\vartheta_{i4}^{+}$ $(1 \leqq i \leqq n)$ in such a way that for $0 < \max_{j=3,4}|u_j| < t_1$ the expressions $\vartheta_{i3}^{+}u_3 + \vartheta_{i4}^{+}u_4 + u_{m+i}$ are different from zero. This is always possible since, $\vartheta_{i3}^{+}$ being fixed there is at most a finite number, namely $O(t_1^3) = O(1)$, of possibilities in the choice of the number $\vartheta_{i4}$, for which $\vartheta_{i3}^{+}u_3 + \vartheta_{i4}u_4 + u_{m+i} = 0$.

Then I choose $\vartheta_{i5}^{+}$ so that for $0 < \max_{j=3,4,5}|u_j| < t_1$ the values $\vartheta_{i3}^{+}u_3 + \vartheta_{i4}^{+}u_4 + \vartheta_{i5}^{+}u_5 + u_{m+i}$ are different from zero; it is possible to continue this construction by induction. I thus obtain numbers $\vartheta_{ij}^{+}$, $\gamma'_{ij} < \vartheta_{ij}^{+} < \delta'_{ij}$ $(1 \leqq i \leqq n;\ 3 \leqq j \leqq m)$ which have the following property: If $0 < \max_{3 \leqq j \leqq m} |u_j| < t_1$, then

$$\vartheta_{i3}^{+}u_3 + \vartheta_{i4}^{+}u_4 + \ldots + \vartheta_{im}^{+}u_m + u_{m+i} \neq 0 \quad (1 \leqq i \leqq n). \qquad (45)$$

The construction of these numbers falls, of course, off in the case $m = 2$.

Now I put

$$\left.\begin{aligned} \vartheta_{i1}^{+}a_1 + \vartheta_{i2}^{+}a_2 + \vartheta_{i3}^{+}a_3 + \ldots + \vartheta_{im}^{+}a_m + a_{m+i} &= 0, \\ \vartheta_{i1}^{+}a'_1 + \vartheta_{i2}^{+}a'_2 + \vartheta_{i3}^{+}a'_3 + \ldots + \vartheta_{im}^{+}a'_m + a'_{m+i} &= 0 \end{aligned}\right\} \quad (1 \leqq i \leqq n). \qquad (46)$$

By the solution of these equations I get

$$\left.\begin{aligned} \vartheta_{i1}^{+} &= \frac{\vartheta_{i3}^{+}(a_2a'_3 - a_3a'_2) + \ldots + \vartheta_{im}^{+}(a_2a'_m - a_ma'_2) + a_2a'_{m+i} - a_{m+i}a'_2}{a_1a'_2 - a'_1a_2} \\ \vartheta_{i2}^{+} &= \frac{\vartheta_{i3}^{+}(a'_1a_3 - a_1a'_3) + \ldots + \vartheta_{im}^{+}(a'_1a_m - a_1a'_m) + a'_1a_{m+i} - a_1a'_{m+i}}{a_1a'_2 - a'_1a_2} \end{aligned}\right\} \quad (1 \leqq i \leqq n); \qquad (47)$$

at the same time I require that the values $\vartheta_{i2}^{+}$ will satisfy the relations: $\gamma'_{i2} < \vartheta_{i2}^{+} < \delta'_{i2}$ $(1 \leqq i \leqq n)$. I have not yet chosen the numbers $a'_{m+1}, \ldots, a'_{m+n}$. As the denominator of both the fractions (47) is according to (43) at least $\frac{1}{2}t_1t_2$, the value of $\vartheta_{i2}^{+}$ changes by $O\left(\frac{a_1}{t_1t_2}\right) = O\left(\frac{1}{t_2}\right)$, if the number $a'_{m+i}$ changes by unity; simultaneously we get different values of $\vartheta_{i1}^{+}$ as well as different values of $\vartheta_{i2}^{+}$ for different choices of $a'_{m+i}$ (for $a_1 \neq 0$, $a_2 \neq 0$). Thus: in order that the number $\vartheta_{i2}^{+}$ may fall into the given interval $(\gamma'_{i2}, \delta'_{i2})$, I can choose $a'_{m+i}$ in more than (*konst.* $t_2$) manners, i. e. for a sufficiently large $t_2$ I have at least $\sqrt{t_2}$ possibilities for $a'_{m+i}$.

Having chosen $a'_{m+i}$ in this way I calculate $\vartheta_{i2}^{+}$, and then $\vartheta_{i1}^{+}$ from the equations (47). According to (44) the point $\boldsymbol{\vartheta}^{+} \equiv [\vartheta_{11}^{+}, \ldots, \vartheta_{1m}^{+}; \ldots; \ldots; \vartheta_{n1}^{+}, \ldots, \vartheta_{nm}^{+}]$ must lie inside $K_1$.

Let us now suppose („per absurdum") that there is a $i$ $(1 \leqq i \leqq n)$ so that for each of the admissible choices of $a'_{m+i}$'s integers $c_1, c_2, \ldots, \ldots, c_m, c_{m+i}$ exist so that the formulae

$$\left.\begin{aligned} 0 < \max_{1 \leqq j \leqq m} |c_j| < t_1, \\ c_1 \vartheta^+_{i1} + c_2 \vartheta^+_{i2} + \ldots + c_m \vartheta^+_{im} + c_{m+i} = 0 \end{aligned}\right\} \quad (48)$$

hold good. According to (45) we have then necessarily $\max(|c_1|, |c_2|) > 0$.

With $t_1$ fixed, there is a finite number of the above systems $(c_1, \ldots, \ldots, c_m, c_{m+i})$, viz. $O(t_1^{m+1}) = O(1)$. If I choose $t_2$ sufficiently large, it is certainly possible to choose two numbers ${}^{(1)}a'_{m+i} < {}^{(2)}a'_{m+i}$ so that the same system $(c_1, \ldots, c_m, c_{m+i})$ corresponds (in the sense of (48)) to the number ${}^{(1)}a'_{m+i}$ and to ${}^{(2)}a'_{m+i}$.

Thus, according to (46) and (48), I get the following equations:

$$\begin{aligned}
&{}^{(1)}\vartheta^+_{i1} a_1 + {}^{(1)}\vartheta^+_{i2} a_2 + \vartheta^+_{i3} a_3 + \ldots + \vartheta^+_{im} a_m + a_{m+i} = 0 \\
&{}^{(1)}\vartheta^+_{i1} a'_1 + {}^{(1)}\vartheta^+_{i2} a'_2 + \vartheta^+_{i3} a'_3 + \ldots + \vartheta^+_{im} a'_m + {}^{(1)}a'_{m+i} = 0 \\
&{}^{(1)}\vartheta^+_{i1} c_1 + {}^{(1)}\vartheta^+_{i2} c_2 + \vartheta^+_{i3} c_3 + \ldots + \vartheta^+_{im} c_m + c_{m+i} = 0
\end{aligned}$$

and

$$\begin{aligned}
&{}^{(2)}\vartheta^+_{i1} a_1 + {}^{(2)}\vartheta^+_{i2} a_2 + \vartheta^+_{i3} a_3 + \ldots + \vartheta^+_{im} a_m + a_{m+i} = 0 \\
&{}^{(2)}\vartheta^+_{i1} a'_1 + {}^{(2)}\vartheta^+_{i2} a'_2 + \vartheta^+_{i3} a'_3 + \ldots + \vartheta^+_{im} a'_m + {}^{(2)}a'_{m+i} = 0 \\
&{}^{(2)}\vartheta^+_{i1} c_1 + {}^{(2)}\vartheta^+_{i2} c_2 + \vartheta^+_{i3} c_3 + \ldots + \vartheta^+_{im} c_m + c_{m+i} = 0.
\end{aligned}$$

If I put $\Delta_1 = {}^{(1)}\vartheta^+_{i1} - {}^{(2)}\vartheta^+_{i1}$; $\Delta_2 = {}^{(1)}\vartheta^+_{i2} - {}^{(2)}\vartheta^+_{i2}$, then the previous equations imply the following relations:

$$\Delta_1 a_1 + \Delta_2 a_2 = 0, \quad (49)$$

$$\Delta_1 c_1 + \Delta_2 c_2 = 0, \quad (50)$$

$$\Delta_1 a'_1 + \Delta_2 a'_2 \neq 0. \quad (51)$$

According to (51) the values $\Delta_1$ and $\Delta_2$ cannot be both equal to zero and thus according to (49) and (50) we have $a_1 c_2 - a_2 c_1 = 0$. But $a_1 \neq 0$, $a_2 \neq 0$, so that according to (49) both values $\Delta_1$ and $\Delta_2$ are different from zero; as according to (45) we have $\max(|c_1|), |c_2|) > 0$, both $c_1$ and $c_2$ are according to (50) different from zero. Thus $a_1 | c_1$ and hence $|a_1| \leqq |c_1|$, i. e. $|c_1| \geqq t_1$, which is contradictory to (48).

For this reason there are such integers $a'_{m+1}, a'_{m+2}, \ldots, a'_{m+n}$ that (46) is and (48) is not satisfied; hence I have got a point $\boldsymbol{\vartheta}^+ \equiv [\vartheta^+_{11}, \ldots, \ldots, \vartheta^+_{1m}; \ldots; \vartheta^+_{n1}, \ldots, \vartheta^+_{nm}]$ within $K'_1$ for which (36) and (41) hold, but neither (37) nor (38) holds. Round the point $\boldsymbol{\vartheta}^+$ let us now circumscribe a sufficiently small cube $K''_1 \subset K'_1$ which has already the properties 1., 2., 4., and 5. asserted by Theorem 9.

But the $n(m-1)$-dimensional linear variety in the $mn$-dimensional space of the points $\boldsymbol{\vartheta}$ which has the equations

$$\vartheta_{i1} a'_1 + \ldots + \vartheta_{im} a'_m + a'^{+i}_m = 0 \quad (1 \leqq i \leqq n) \quad (41)$$

cannot be contained in any $(mn - 1)$-dimensional hyperplane having the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij}\vartheta_{ij} + g_o = 0, \tag{52}$$

where $0 < \max_{\substack{1 \leq i < n \\ 1 \leq j \leq m}} |g_{ij}| < t_1$.

For were it the case, each point that lies in (41) ought to lie also in (52). In other words: if I choose any $\vartheta_{ij}$ $(1 \leq i \leq n; 2 \leq j \leq m)$ and calculate $\vartheta_{i1}$ $(1 \leq i \leq n)$ from (41), the point $\boldsymbol{\vartheta}$ lies in (41). Substituting these values of $\vartheta_{ij}$ in (52), I get identically for any choice of $\vartheta_{ij}$ $(1 \leq \leq i \leq n;\ 2 \leq j \leq m)$ the relation:

$$\sum_{i=1}^{n} \left( \sum_{j=2}^{m} a'_1 g_{ij}\vartheta_{ij} - g_{i1}(a'_2\vartheta_{i2} + \ldots + a'_m\vartheta_{im} + a'_{m+i}) \right) + a'_1 g_o = 0,$$

since $a'_1 \neq 0$ Hence it naturally follows that $g_{i1}a'_j = g_{ij}a'_1$ $(1 \leq i \leq n; 1 \leq j \leq m)$. As we have $t_1 > \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} |g_{ij}| > 0$, there are such integers $i$ and $j$ $(1 \leq i \leq n;\ 1 \leq j \leq m)$ that $g_{ij} \neq 0$. As the numbers $a'_j$ $(1 \leq \leq j \leq m)$ are altogether different from zero, the values $g_{ij}$ will be different from zero, too, for the above index $i$ and for all $j$'s $(1 \leq j \leq m)$ and besides this the following relation $g_{i1} : g_{i2} : \ldots : g_{im} = a'_1 : a'_2 : \ldots : a'_m$ will take place. Since the numbers $a'_1, \ldots, a'_m$ are by pairs relatively prime, then necessarily $a'_1 | g_{i1}$, thus $|a'_1| = t_2 \leq |g_{i1}|$, which is a contradiction to $t_1 > \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} |g_{ij}|$.

Consequently in the cube $K''_1$ there are points $\boldsymbol{\vartheta}$ that, though lying in (41), do not lie in (52). But there is only a finite number of hyperplanes (52) passing through the cube $K''_1$, viz. at most $O(t_1^{mn+1}) = O(1)$; for this reason I may choose a point $\boldsymbol{\vartheta} \in K''_1$, which lies in (41) and does not lie in any of the hyperplanes (52). If I circumscribe a sufficiently small cube $K_2 \subset K''_1$ round this point $\boldsymbol{\vartheta}$, this cube has already all the properties stated by Theorem 9, Q. E. D.

Now, in the same manner as we have constructed $s_2, t_2, K_2$ from $t_1$, $K_1$, we can by means of induction construct an increasing sequence of integers $0 < t_1 < s_1 < t_2 < s_2 < \ldots$ fulfilling the relation: $\lim_{r\to\infty} t_r = \lim_{r\to\infty} s_r = +\infty$ and a decreasing sequence $K_1 \supset K_2 \supset K_3 \supset \ldots$ of $mn$-dimensional closed cubes. We have then $\lim_{r\to\infty} K_r = K = \prod_{r=1}^{\infty} K_r \neq \emptyset$, so that there is a point $\boldsymbol{\vartheta} \equiv [\vartheta_{11}, \ldots, \vartheta_{1m}; \ldots; \vartheta_{n1}, \ldots, \vartheta_{nm}]$ which lies in all cubes $K_r$ $(r = 1, 2, 3, \ldots)$. Now I construct the system

$$\Theta = \begin{pmatrix} \vartheta_{11}, & \vartheta_{12}, & \ldots, & \vartheta_{1m} \\ \vartheta_{21}, & \vartheta_{22}, & \ldots, & \vartheta_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \vartheta_{n1}, & \vartheta_{n2}, & \ldots, & \vartheta_{nm} \end{pmatrix}$$

and investigate the functions $\psi_1(t, \Theta)$ and $\psi_2(t, \Theta)$.

*A. The system $\Theta$ is regular in its rows and also in its columns*; and even more is true:

*B. There is no system which of integers $g_o, g_{ij}$ $(1 \leqq i \leqq n; 1 \leqq j \leqq m)$ for which*

$$\max_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq m}} |g_{ij}| > 0,$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij}\vartheta_{ij} + g_o = 0.$$

For, if such a system would exist, there would be an $r$ such that $\max_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq m}} |g_{ij}| < t_r$ and according to the property 3. of Theorem 9 the point $\vartheta$ could not lie in $K_{r+1}$.

*C. For $t \geqq t_1$ it is $\psi_1(t, \Theta) < \frac{1}{\varphi(t)}$.* This results at once from the 4th property of Theorem 9, according to which for points of $K_{r+1}$ the inequality $\psi_1(t_r, \Theta) < \frac{1}{\varphi(t_{r+1})}$ is satisfied. Thus for $t_r \leqq t < t_{r+1}$ I have

$$\psi_1(t, \Theta) \leqq \psi_1(t_r, \Theta) < \frac{1}{\varphi(t_{r+1})} < \frac{1}{\varphi(t)}.$$

*D. Finally the 1st property of Theorem 9 implies that there is such a sequence of positive integers $s_1 < s_2 < s_3 < \ldots$ tending to infinity that*

$$\psi_2(s_r, \Theta) > \frac{1}{s_r^{\overline{\frac{n}{m-1}}} \Phi(s_r)};$$

hence we have $\limsup_{s \to \infty} s^{\overline{\frac{n}{m-1}}} \Phi(s)\, \psi_2(s, \Theta) \geqq 1$.

Thereby, accordingly, the following

**Theorem 10** is proved: *Let $m \geqq 2$, $n \geqq 1$. Then there are such systems $\Theta$ regular in their rows and columns that the following relations*

$$\limsup_{t \to \infty} \varphi(t)\, \psi_1(t, \Theta) \leqq 1,$$

$$\limsup_{s \to \infty} s^{\frac{n}{m-1}} \Phi(s)\, \psi_2(s, \Theta) \geqq 1$$

*take place, where* $\varphi(t)$ *and* $\Phi(t)$ *denote any preassigned continuous and increasing functions of* $t$*, defined fot* $t \geqq t_1 > 0$ *with the properties:* $\lim_{t \to \infty} \varphi(t) = +\infty$, $\lim_{t \to \infty} \Phi(t) = +\infty$.

**Theorem 11.** *Let* $m \geqq 2$, $n \geqq 1$; *then there are systems* $\Theta$ *that are regular in both their rows and columns and for the indices of which the following formulae are satisfied:*

$$\alpha(\Theta) = +\infty, \ \beta(\Theta) = \frac{n}{m-1}.$$

The respective proof results from Theorem 10, if we put e. g. $\varphi(t) = e^t, \Phi(t) = \log t$.

As it is evident from Theorems 8 and 11, the results of Theorem 6 are sharp in the following sense: For $m \geqq 2$, $n \geqq 2$, there exist two systems $\Theta$ and $\Theta'$, both of them regular in their rows and columns so that

$$\alpha(\Theta) = \alpha(\Theta') = +\infty; \ \beta(\Theta) = +\infty; \ \beta(\Theta') = \frac{n}{m-1}.$$

In other words there is, provided that $m + n \geqq 4$, no one to one correspondence (in the case $m$, $n \geqq 2$ not even a single-valued correspondence) between the indices $\alpha(\Theta)$ and $\beta(\Theta)$, what we intended to prove.

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