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ON TRANSITIVE BOOLEAN RELATIONS

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The author investigates under what conditions a relation R, defined on a Boolean algebra, is a quasi-ordering, an equivalence relation, a partial ordering, a lattice ordering or a partial ordering determining a Boolean algebra.

Let B be a Boolean algebra, a, b, c and d be elements of B, and $\mathbf{R}$ be the relation over B defined by the condition that if x and y are elements of B, then $x\mathbf{R}y$ if and only if

$$axy + bxy' + cx'y + dx'y' = 0.$$

$\mathbf{R}$ is *transitive* if and only if either

$$a(b + c)\, d \neq 0$$

or

$$a + d \leqq b + c.$$

For if x, y and z are elements of B, the assertion that

$$x\mathbf{R}y \text{ and } y\mathbf{R}z$$

is equivalent to the equation

$$\begin{aligned} axyz + (a + b)\, xyz' + (b + c)\, xy'z + (b + d)\, xy'z' + \\ + (a + c)\, x'yz + (b + c)\, x'yz' + (c + d)\, x'y'z + dx'y'z' = 0. \end{aligned} \tag{1}$$

Hence if $a(b + c)\, d \neq 0$, there exist no elements x, y and z of B such that $x\mathbf{R}y$ and $y\mathbf{R}z$, and the assertion that $\mathbf{R}$ is transitive is vacuous. Let $a(b + c)\, d = 0$.

Then (1) implies $x\mathbf{R}z$ or

$$\begin{aligned} axyz + bxyz' + axy'z + bxy'z' + cx'yz + dx'yz' + cx'y'z + \\ + dx'y'z' = 0 \end{aligned}$$

if and only if $a + d \leqq b + c$ (see III, Satz 17, p. 182); it is a corollary that in this case

$$ad = 0.$$

In (I), the case that $a(b + c)\, d \neq 0$ is not considered. The author of the present note wishes to state necessary and sufficient conditions for $\mathbf{R}$ being a quasi ordering (see, e. g., (II), p. 444), an equivalence relation, a quasi-lattice ordering, a partial ordering, and a lattice ordering.

It is almost obvious that $\mathbf{R}$ is reflexive if and only if $a = d = 0$ and symmetric if and only if $abcd \neq 0$ or $b = c$.

Hence $\mathbf{R}$ is transitive if it is reflexive, and the following Theorems 1 and 2 hold:

Theorem 1. $\mathbf{R}$ *is a quasi-ordering if and only if* $a = d = 0$.

Theorem 2. $\mathbf{R}$ *is an equivalence relation if and only if* $a = d = 0$ *and* $b = c$.

Let $\mathbf{r}$ be the relation over B defined by the condition that if x and y are elements of B, then $x\mathbf{r}y$ if and only if $x\mathbf{R}y$ and $y\mathbf{R}x$. Hence if x and y are elements of B, then $x\mathbf{r}y$ if and only if

$$axy + (b + c)\, xy' + (b + c)\, x'y + dx'y' = 0.$$

It follows that $\mathbf{R}$ is antisymmetric (i. e., $x \in B$, $y \in B$ and $x\mathbf{r}y$ imply $x = y$) if and only if $a(b + c)\, d \neq 0$ or $b + c = 1$.

If $\mathbf{R}$ is a quasi ordering, then $\mathbf{r}$ is an equivalence relation, and the principal ideal $B(b + c)$ of B generated by $b + c$ is a complete set of representative elements of the $\mathbf{r}$-categories.

Theorem 3. *Let* $\mathbf{R}$ *be a quasi-ordering. Then the following assertions* (1) *to* (7) *are equivalent*:

(1) $bc = 0$.

(2) *There exists an element* x_0 *of* B *such that* $x_0\mathbf{R}x$ *for* $x \in B$.

(3) *There exists an element* x_1 *of* B *such that* $x\mathbf{R}x_1$ *for* $x \in B$.

(4) *Whenever* $u \in B$ *and* $v \in B$, *there exists an element* x *of* B *such that* $x\mathbf{R}u$ *and* $x\mathbf{R}v$.

(5) *Whenever* $u \in B$ *and* $v \in B$, *there exists an element* x *of* B *such that* $u\mathbf{R}x$ *and* $v\mathbf{R}x$.

(6) $\mathbf{R}$ *determines a lattice algebra over the set of the* $\mathbf{r}$*-categories.*

(7) $\mathbf{R}$ *determines a Boolean algebra over the set of the* $\mathbf{r}$*-categories.*

Proof. If (2) holds, then $bx_0 = 0$, $cx_0' = 0$, $c \leqq x_0 \leqq b'$, and (1) holds. Interchanging b and c, we find that (3) implies (1).

If (4) holds, let x be an element of B such that $x\mathbf{R}0$ and $x\mathbf{R}1$. Hence $bx = 0$, $cx' = 0$, and (1) holds. Interchanging b and c, we find that (5) implies (1).

It follows that each of the assertions (2) to (7) implies (1). As (7) implies (2) to (6), it suffices to prove that (1) implies (7).

Suppose that $bc = 0$. Let u and v be elements of B. Then it is easily verified that

$$x\mathbf{R}u \text{ and } x\mathbf{R}v$$

if and only if

$$x\mathbf{R}[buv + c(u + v)],$$

and that

$$u\mathbf{R}x \text{ and } v\mathbf{R}x$$

if and only if

$$[b(u + v) + cuv]\ \mathbf{R}x.$$

Hence (6) holds, and the equations

$$u \cup v = b(u + v) + cuv$$

and

$$u \cap v = buv + c(u + v)$$

for $u \in B(b + c)$, $v \in B(b + c)$ define a lattice algebra over $B(b + c)$. If we assign to each element x of B the element $bx + cx'$ of $B(b + c)$, we obtain a homomorphism of the original lattice algebra over B (operations: $x + y$ and xy) on the lattice algebra over $B(b + c)$ just mentioned; this can easily be verified. Hence (7) holds, completing our proof.

Now the following Theorems 4, 5 and 6 are obvious:

Theorem 4. $\mathbf{R}$ *is a partial ordering if and only if* $a = d = 0$ *and* $b + c = 1$.

Theorem 5. $\mathbf{R}$ *is a lattice ordering if and only if* $a = d = 0$ *and* $b = c'$.

Theorem 6. $\mathbf{R}$ *is a partial ordering determining a Boolean algebra if and only if* $a = d = 0$ *and* $b = c'$.

If $\mathbf{R}$ is a lattice ordering, the homomorphism considered in the proof of Theorem 3 is an isomorphism. This isomorphism was considered in **(IV)**.

REFERENCES.

(I) B. A. BERNSTEIN: A Generalization of the Syllogism, Bulletin of the American Mathematical Society **30** (1924), 125—127.

(II) G. BIRKHOFF: Rings of Sets, Duke Mathematical Journal **3** (1937), 443—454.

(III) L. LÖWENHEIM: Über die Auflösung von Gleichungen im logischen Gebietekalkul, Mathematische Annalen **68** (1910), 169—207.

(IV) N. WIENER: Certain Formal Invariances in Boolean Algebras, Transactions of the American Mathematical Society **18** (1917), 65—72.