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STRUCTURAL RELATION

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The author finds two conditions for a function C to be identifiable from a relation $\eta = C(\xi)$ between two random variables and studies the dependence between random variables if the observed quantities are obtained with random errors.

1. Introduction and summary.

Let ξ, α, β, X, Y be random variables and

$$\begin{aligned} X &= \xi + \alpha , \\ Y &= C(\xi) + \beta , \end{aligned} \tag{1.1}$$

where C is a function measurable in the Borel sense (further B-measurable). (ξ, α, β are called latent variables, X, Y observable variables, (1.1) structural relation, α, β components of error.)

Under the assumption $C(x) = ax + b$, Reiersøl [8][1]) found necessary and sufficient conditions for C to be identifiable by knowledge of $F_{X,Y}$[2]).

In the present paper the identification problem is studied without the assumption that C is a linear function. Certain knowledge of $F_{\xi,\alpha,\beta}$ is required:

In sections 3, 4 we suppose that F_α and F_β are known, that the sets $A = \{t; \varphi_\alpha(t) \neq 0\}$ and $B = \{t; \varphi_\beta(t) \neq 0\}$ are dense in E_1 (by E_k we denote the k-dimensional Euclidian space) and that ξ is independent of α and of β. Thus φ_ξ is determined by φ_X and φ_α, for

$$\varphi_\xi(t) = \frac{\varphi_X(t)}{\varphi_\alpha(t)} \quad \text{if } t \in A ,$$

φ_ξ is continuous and A is dense. Similarly $\varphi_{C(\xi)}$ is determined by φ_Y and φ_β. Hence in the sections **3, 4** we give sufficient conditions for C to be identifiable by the knowledge of F_ξ and $F_{C(\xi)}$.

In section **5** we suppose that F_α is known, $\varphi_\alpha(t) \neq 0$ on a set dense in E_1, ξ, α, β are independent and $E(\beta) = 0$. Then F_ξ is determined by the knowledge

[1]) See references at the end of the paper.

[2]) The distribution function of a vector random variable $(T_1, \ldots, T_n)$ is denoted by $F_{T_1, \ldots, T_n}$. Similarly we use the letters φ and f for denoting the characteristic and frequency functions respectively.

of F_X and F_α, $E_X(Y) = E_X(C(\xi))$. Hence in the section **5** we give a sufficient condition for C to be identifiable by knowledge of F_ξ and $E_X(C(\xi))$.

In section **5** the conditions imposed on $F_{\xi,\alpha,\beta}$ and on C are essentially less restrictive than those in sections **3**, **4**. But when the conditions of sections **3**, **4** are satisfied, we can determine C by F_X and F_Y, which is easier than in sec. 5., where we need the knowledge of $E_X(Y)$.

In section **6** the results of sections **3**, **4**, **5** are applied to prove that (i) if F_X and F_Y are normal distribution functions, C is monotone or C has a continuous derivative, then C is a linear function (ii) if F_X is normal or F_X is a Poisson distribution function, then C is linear when and only when $E_X(Y)$ is linear.

2. Basic definitions and notations.

If T is a transformation of a set A into a set B, V is a transformation of the set B into a set C, we denote by VT the transformation of A into C defined by the relation $VT(x) = V(T(x))$ for all $x \in A$. For $M \subset A$ we denote by $T(M)$ the set $\{y;\ y = T(x),\ x \in M\}$ and by T_M the transformation of M into B defined by $T_M(x) = T(x)$ for $x \in M$. For $N \subset B$ we denote $T^{-1}(N) = \{x;\ x \in A,\ T(x) \in N\}$. If there exists an inverse function to the function T, we denote it T^{-1}.

A probability space (Ω, F, P) is a set Ω, a σ-algebra F of subsets $A \subset \Omega$ and a probability measure P on F. A random variable X is a function measurable (F) and defined and finite on a set $A \in \mathsf{F}$, $P(A) = 1$. Frequently we shall write $P(X \in A)$ instead of $P(X^{-1}(A))$. The d. f. F_X is defined by the relation $F_X(x) = P(X \leq x)$ for all real x.

By $\int_A h \mathrm{d}F_X$ we understand the Lebesgue-Stielties intergral $\int_A h\, \mathrm{d}\nu$, where $\nu(B) = P(X^{-1}(B))$ for every Borel set B. Thus if $|\int_A h\, \mathrm{d}F_X| < +\infty$, then $\int_A |h|\, \mathrm{d}F_X < +\infty$. When $A = E_1$, we write $\int h\, \mathrm{d}F_X = \int_{E_1} h\, \mathrm{d}F_X$. If two functions h and g, defined on E_1, satisfy $h(x) = g(x)$ for all $x \in A$, where $P(X \in A) = 1$, we say that $h = g$ almost everywhere (F_X), or $h(x) = g(x)$ for almost all $x\ (F_X)$, or write shortly $h = g\ (F_X)$. If X and Y are random variables, $|E(Y)| < +\infty$, then we denote $E_X(Y)$ the conditional expectation of Y with respect to X, which is defined by the relation

$$\left[\int_{X^{-1}(A)} Y\, \mathrm{d}P = \int_A E_X(Y)\, \mathrm{d}F_X \quad \text{for every Borel set } A\right] \tag{2.1}$$

almost everywhere (F_X). (See Kolmogorov [6], Halmos [5] and Doob [3].) From (2.1) it follows, that

$$\int E_X(Y)\, \mathrm{d}F_X = \int_\Omega Y\, \mathrm{d}P = E(Y) \tag{2.2}$$

It is known that there exists a class of distribution functions $\{_{X=x}F_Y\}_{x \in E_1}$ such that for every integrable function h

$$\int h \, d_{X=x} F_Y = E_{X=x} h(Y)) \tag{2.3}$$

for almost all x (F_X).

3.

Theorem 3.1. *Let ξ and η be random variables, $\eta = C(\xi)$, where C is a non-decreasing function, defined almost everywhere (F_ξ). Then C is identifiable by the knowledge of F_ξ and F_η, i. e., the value $C(x)$ is uniquely determined by F_ξ and F_η for almost all x (F_ξ).*

Proof: For brevity, let us denote $F_\xi = F$, $F_{C(\xi)} = H$. It is for all x (we can eventually complete the definition of C in such a way, that C remains non decreasing and defined on E_1)

$$\xi \leq x \Rightarrow C(\xi) \leq C(x) \,,$$

$$\xi < x \Leftarrow C(\xi) < C(x)$$

and thus (denoting $F(x - 0) = \lim_{y \to x-} F(y)$ and similarly for H)

$$F(x) \leq HC(x),\ F(x - 0) \geq H(C(x) - 0) \ ;$$

hence

$$H(C(x) - 0) \leq F(x) \leq HC(x) \,, \tag{3.2}$$

$$H(C(x) - 0) \leq F(x - 0) \leq HC(x) \tag{3.3}$$

for all $x \in E_1$.

Let us denote by B_1 resp. B_2 resp. B_3 the set of such $x \in E_1$, that H acquires the value $F(x)$ in no point resp. in one and only one point resp. in more points. Obviously $\bigcup_{i=1}^{3} B_i = E_1$ and B_i are disjoint.

1. Let $x \in B_1$.

If $0 < F(x) < 1$, then there exists one and only one $y \in E_1$ such that

$$H(y - 0) \leq F(x) < H(y) \tag{3.4}$$

and from 3.2 it follows, that $C(x) = y$.

If $F(x) = 0$ $(F(x) = 1)$, then obviously $C(x) = -\infty$ $(C(x) = +\infty)$.

2. Let $x \in B_2$. Then there exists one and only one y such that $H(y) = F(x)$ and $C(x) = y$, according to (3.2) and the following implications:

$$C(x) < y \Rightarrow HC(x) < H(y) = F(x)$$

$$C(x) > y \Rightarrow H(C(x) - 0) > H(y) = F(x)$$

3. Let $x \in B_3$. Define

$$I_x = \{y; H(y) = F(x)\}, \qquad J_x = \{y; F(y) = F(x)\}.$$

I_x and J_x are non empty intervals, closed or closed to the left (for the distribution functions are continuous to the right) and the intervals I_x have non empty interiors. For every $x_1 \in B_3$, $x_2 \in B_3$ it is either $I_{x_1} = I_{x_2}$, $J_{x_1} = J_{x_2}$ or $I_{x_1} \cap I_{x_2} = 0$, $J_{x_1} \cap J_{x_2} = 0$.

Denote by $\mathfrak{M}$ the class of the sets I_x and $\mathfrak{N}$ the class of the sets J_x. $\mathfrak{M}$ is at most countable, because it contains disjoint intervals with non empty interiors. We can easily define a one-to-one correspondence between $\mathfrak{M}$ and $\mathfrak{N}$ and thus $\mathfrak{N}$ is at most countable.

Let now A be the set of such left edges a of intervals $J \in \mathfrak{N}$, that $P(\xi = a) > 0$. Following implications hold:

$$J \in \mathfrak{N}, \quad J = \langle a, b) \Rightarrow P(\xi \in (a, b)) = F(b - 0) - F(a) = 0$$
$$J \in \mathfrak{N}, \quad J = \langle a, b\rangle \Rightarrow P(\xi \in (a, b\rangle) = F(b) - F(a) = 0 ,$$

for F is constant on $J \in \mathfrak{N}$.

Thus, from countability of $\mathfrak{N}$ it follows, that

$$P(\xi \in B_3 - A) = P(\xi \in \bigcup_{J \in \mathfrak{N}} J - A) = 0 .$$

Let $x \in A$. It is then $C(x) = \inf I_x$. Let us denote $\langle a, b\rangle$ the closure of I_x. We shall prove, that $C(x) = a$. But this is an immediate consequence of (3.2), (3.3) and the following implications:

$C(x) < a \Rightarrow HC(x) < F(x)$, which contradicts (3.2)
$C(x) > b \Rightarrow H(C(x) - 0) > F(x)$, which contradicts (3.2)
$C(x) \in (a, b) \Rightarrow 0 = H(b - 0) - H(a) = P(C(\xi) \in (a, b)) \geq P(\xi = x) > 0.$
$C(x) = b \Rightarrow H(C(x) - 0) = F(x)$ and $F(x - 0) < F(x)$ for $P(\xi = x) > 0 \Rightarrow$
$\Rightarrow H(C(x) - 0) > F(x - 0)$, which contradicts (3.3).

Since $P(\xi \in B_1 \cup B_2 \cup A) = 1$, the proof is completed.

In the proof, the assumption that C is non-decreasing, is essential. In the next section we shall prove a theorem analogous to (3.1) without this assumptions. Other conditions, of course, are required.

4.

Theorem 4.1. *Let U be a random variable, h a function with a continuous derivative on $(0, 1)$. Let $F_U(x) = F_{h(U)}(x) = x$ for $x \in \langle 0, 1\rangle$.*

Then it is $h(x) = x$ for all $x \in (0, 1)$, or $h(x) = 1 - x$ for all $x \in (0, 1)$.

Proof: Denote by L the Lebesgue measure. It is for every Borel set $A \subset \langle 0, 1\rangle$

$$L(A) = L(h^{-1}(A)) . \tag{4.2}$$

It is sufficient to prove that h is a monotone function, the assertion of (4.1) being then a consequence of the theorem (3.1).

Let h be non monotone on $(0, 1)$. Then there exists a $x_0 \in (0, 1)$ such that $h'(x_0) = 0$. But h' is continuous and thus there exists a $\delta > 0$ such that

$$x \in I = \langle x_0 - \delta, x_0 + \delta \rangle \Rightarrow |h'(x)| < 1 . \tag{4.3}$$

The function h acquires on I a minimum value y_1 and a maximum value y_2; let x_1 and x_2 be these points. Without loss of generality, we may assume, that $x_1 < x_2$, $h(x_1) = y_1 \leqq y_2 = h(x_2)$.

From the first mean value theorem

$$0 \leqq y_2 - y_1 = h'(\mu)\,(x_2 - x_1), \qquad \mu \in I$$

and thus, according to (4.3)

$$0 \leqq y_2 - y_1 < x_2 - x_1 . \tag{4.4}$$

Further, if $x \in \langle x_1, x_2 \rangle$, then $y_1 \leqq h(x) \leqq y_2$ and thus

$$\langle x_1, x_2 \rangle \subset h^{-1}(\langle y_1, y_2 \rangle) . \tag{4.5}$$

From (4.5) and (4.2) it follows

$$L(\langle x_1, x_2 \rangle) \leqq L(h^{-1}(\langle y_1, y_2 \rangle)) = L(\langle y_1, y_2 \rangle) \tag{4.6}$$

and

$$x_2 - x_1 \leqq y_2 - y_1 \tag{4.7}$$

which is a contradiction to (4.4) .

Thus h is monotone and the theorem is proved.

Theorem 4.8. *Let ξ and η be random variables, (a, b) an interval (finite or infinite), $F_\xi(b) - F_\xi(a) = 1$, C a function with a continuous derivative on (a, b), $\eta = C(\xi)$. Let F_ξ and F_η be continuous on E_1. Let F_ξ have a continuous positive derivative on (a, b), let F_η have a continuous derivative on $C((a, b))$.*

Denote $H = (F_\eta)_{C(a,b)}$, $G = (F_\xi)_{(a,b)}$,

Then H is a increasing function and either

$$C = H^{-1}G$$

or

$$C = H^{-1}(1 - G)$$

Proof: Define new variables

$$U = F_\xi(\xi), \qquad V = F_\eta(\eta) = F_\eta C(\xi) . \tag{4.9}$$

Thus $\xi = G^{-1}(U)$ almost everywhere (P) and

$$V = F_\eta C G^{-1}(U) \tag{4.10}$$

almost everywhere (P) .

Denote $F_\eta C G^{-1} = h$. From (4.9) and (4.10) it follows, that $F_U(x) = F_{h(U)}(x) = x$ for all $x \in \langle 0, 1 \rangle$. Futher G^{-1} has a continuous derivative in $(0, 1)$, C has a continuous derivative on $G^{-1}(0, 1) = (a, b)$ and F_η has a continuous derivative on $C(a, b)$. Thus h has a continuous derivative on $(0, 1)$. Hence it follows from the preceding theorem, that $h(x) = x$ or $h(x) = 1 - x$ for all $x \in (0, 1)$. Denoting E the identical transformation of $(0, 1)$ on $(0, 1)$, we have

$$E = F_\eta C G^{-1} \quad \text{or} \quad 1 - E = F_\eta C G^{-1} ;$$

hence

$$G = F_\eta C \quad \text{or} \quad 1 - G = F_\eta C .$$

F_η is increasing on $C(a, b)$, since G is increasing. Thus

$$C = H^{-1} G \quad \text{or} \quad C = H^{-1}(1 - G), \text{ q. e. d.}$$

5.

Theorem 5.1. *Let ξ, α be independent random variables, $X = \xi + \alpha$, C a B-measurable function. Let $E(\xi)$, $E(\alpha)$, $E(C(\xi))$ be finite. Let $\varphi_\alpha(t) \neq 0$ on a set dense in E_1. Let for some functions h, G,*

$$E(e^{itX} h(X)) = \varphi_\alpha(t) E(e^{it\xi} G(\xi)). \tag{5.2}$$

Then:

(5.2) *holds for*

$$h = E_X(C(\xi)) \text{ and } G = C . \tag{5.3}$$

If (5.2) *holds, then*

$$h = E_X(C(\xi)) \ (F_X) \quad \text{when and only when } G = C \ (F_\xi). \tag{5.4}$$

Consequence 5.5. *Let ξ, α, X, φ_α, C be as in the preceding theorem, let C_1 be a B- measurable function and $E_X(C(\xi)) = E_X(C_1(\xi))$ (F_X).* Then $C = C_1$ (F_ξ).

Proof of Consequence: It is $E(C_1(\xi)) = E(E_X(C_1(\xi))) = E(E_X(C(\xi))) = E(C(\xi))$ and thus $E(C_1(\xi))$ is finite and C_1 satisfies the conditions of theorem 5.1 imposed on C. From $E_X(C(\xi)) = E_X(C_1(\xi))$ and from (5.3) it follows, that (5.2) holds for $h = E_X(C(\xi))$ and $G = C_1$. From (5.4) it follows, that $C = C_1$ (F_ξ), q. e. d.

For the proof of theorem (5.1) we need the following lemma, which is a consequence of the one-to-one correspondence between characteristic and distribution functions.

Lemma 5.6. *Let F be a distribution function, H a measurable function and*

$$\int e^{itx} H(x) \, dF(x) = 0 \quad \textit{for all } t \in E_1$$

Then $H = 0$ (F).

Proof of Theorem 5.1. It is

$$\varphi_{X, C(\xi)}(t, v) = E(e^{i(t\xi + t\alpha + vC(\xi))}) = \varphi_\alpha(t) \int e^{i(ty + vC(y))} \, dF_\xi(y) .$$

But

$$\varphi_{X,C(\xi)}(t, v) = E(\mathrm{e}^{\mathrm{i}tX}\mathrm{e}^{\mathrm{i}vC(\xi)}) = E(E_X(\mathrm{e}^{\mathrm{i}tX}\mathrm{e}^{\mathrm{i}vC(\xi)})) = \int \mathrm{e}^{\mathrm{i}tx}G(x, v)\,\mathrm{d}F_X(x) ,$$

where

$$G(x, v) = E_{X=x}(\mathrm{e}^{\mathrm{i}vC(\xi)}) = \int \mathrm{e}^{\mathrm{i}vC(y)}\,\mathrm{d}_{X=x}F_\xi(y) \quad (F_x) . \tag{5.7}$$

Thus

$$\varphi_\alpha(t)\int \mathrm{e}^{\mathrm{i}(ty+vC(y))}\,\mathrm{d}F_\xi(y) = \int \mathrm{e}^{\mathrm{i}tx}G(x, v)\,\mathrm{d}F_X(x) . \tag{5.8}$$

Let us compute $\dfrac{\partial G(x, v)}{\partial v}$

The function $\dfrac{\partial}{\partial v}\mathrm{e}^{\mathrm{i}vC(y)} = iC(y)\mathrm{e}^{\mathrm{i}vC(y)}$ has the integrable majorante C. Thus

$$\frac{\partial G(x, v)}{\partial v} = \mathrm{i}\int C(y)\mathrm{e}^{\mathrm{i}vC(y)}\,\mathrm{d}_{X=x}F_\xi(y)$$

for almost all x (F_x).

It is obvious that

$$\left|\frac{\partial G(x, v)}{\partial v}\right| \leq \int C(y)\,\mathrm{d}_{X=x}F_\xi(\mathrm{y})$$

for almost all $x(F_X)$ and the last integral, as function of x, is integrable with respect to F_X.

Thus

$$\frac{\partial}{\partial v}\int \mathrm{e}^{\mathrm{i}tx}G(x, v)\,\mathrm{d}F_X(x) = \mathrm{i}\int \mathrm{e}^{\mathrm{i}tx}\left[\int C(y)\mathrm{e}^{\mathrm{i}vC(y)}\,\mathrm{d}_{X=x}F_\xi(y)\right]\mathrm{d}F_X(x)$$

Similarly

$$\frac{\partial}{\partial v}\int \mathrm{e}^{\mathrm{i}(ty+vC(y))}\,\mathrm{d}F_\xi(y) = \mathrm{i}\int C(y)\mathrm{e}^{\mathrm{i}(ty+vC(y))}\,\mathrm{d}F_\xi(y) .$$

Hence and from (5.8), putting $v = 0$, we get

$$\int \mathrm{e}^{\mathrm{i}tx}E_{X=x}(C(\xi))\,\mathrm{d}F_X(x) = \varphi_\alpha(t)\int \mathrm{e}^{\mathrm{i}ty}C(y)\,\mathrm{d}F_\xi(y) \tag{5.9}$$

or

$$E(\mathrm{e}^{\mathrm{i}tX}E_X(C(\xi))) = \varphi_\alpha(t)\,E(C(\xi)\mathrm{e}^{\mathrm{i}t\xi}) . \tag{5.10}$$

Thus (5.3) is proved.

We shall prove (5.4). Let $h = E_X(C(\xi))$ (F_X) and let (5.2) hold. Then

$$E(\mathrm{e}^{\mathrm{i}tX}E_X(C(\xi))) = \varphi_\alpha(t)E(G(\xi)\mathrm{e}^{\mathrm{i}t\xi}) , \tag{5.11}$$

and hence and from (5.10)

$$\varphi_\alpha(t)\,E(\mathrm{e}^{\mathrm{i}t\xi}(C(\xi) - G(\xi))) = 0 \tag{5.12}$$

for all t. Thus

$$E(\mathrm{e}^{\mathrm{i}t\xi}(C(\xi) - G(\xi))) = 0 \tag{5.13}$$

for all t, since this is a continuous function of t and $\varphi_\alpha(t) \neq 0$ on a set dense in E_1. From (5.13) and lemma 5.6 it follows, that $C = G\,(F_\xi)$. The proof of the implication $C = G\,(F_\xi) \Rightarrow E_x(C(\xi)) = h\,(F_x)$ is analogous and simpler. Thus the theorem is proved.

6. Special cases.

Theorem 6.1. *Let in* (1.1) *either C be monotone, or C have continuous derivative on $(-\infty, +\infty)$. Let F_X and F_Y be normal distribution functions. Let ξ be independent of α and β.*

Then C is linear (F_ξ).

Proof: According to Cramér [2] ξ and $C(\xi)$ being normally distributed, the theorem follows from theorems 3.1 and 4.8.

Theorem 6.2. *Let in* (1.1) *ξ, α, β are independent, $|E(\beta)| < +\infty$. Let F_X be a normal distribution function.*

Then C is a linear function (F_ξ) if and only if $E_X(Y)$ is a linear function (F_X).

Proof: The assertion that, if C is linear (F_ξ), then $E_X(Y)$ is linear (F_X), is obvious and known (see e. g. Fix[4]). We shall prove the inverse implication. Now, let $E_{X=x}(Y) - E(\beta) = E_{X=x}(C(\xi)) = ax + b\ (F_X)$,

$$E(\alpha) = m_1 \qquad E(\alpha - m_1)^2 = \sigma_1^2$$
$$E(\xi) = m_2 \qquad E(\xi - m_2)^2 = \sigma_2^2 .$$

Then, using the symbol f for denote frequency functions, it is

$$_xf_\xi = \frac{f_X}{f_\xi}\, _\xi f_X .$$

After easy calculations, we verify that $_xf_\xi$ is a normal frequency function with mean

$$(x - m_1)\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} + m_2 \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

and variance

$$\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Putting

$$C_1(x) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}\, ax + b + \frac{a}{\sigma_2^2}\,(m_1\sigma_2^2 - m_2\sigma_1^2)\,,$$

we get $E_X(C_1(\xi)) = ax + b = E_X(C(\xi))$.

Applying theorem (5.5), we get $C = C_1\,(F_\xi)$ and thus C is linear (F_ξ), q. e. d.

Theorem 6.3. *Let in* (1.1) *ξ, α, β are idependent, $|E(\beta)| < +\infty$. Let*

$$F_X(x) = e^{-\lambda} \sum_{n<x} \frac{\lambda^n}{n!} .$$

Then C is linear (F_ξ) *if and only if* $E_X(Y)$ *is linear* (F_X).

Proof: Denote $F(x;\lambda) = e^{-\lambda} \sum_{n \leq x} \frac{\lambda^n}{n!}$

From a theorem of Rajkov [7] it follows that

$$F_\alpha(x) = F\left(\frac{x-\mu_1}{\sigma_1};\lambda_1\right),\ F_\xi(x) = F\left(\frac{x-\mu_2}{\sigma_2};\lambda_2\right),$$

where $\sigma_1 > 0$, $\lambda_i \geq 0$ $(i = 1,2)$. We can assume $\lambda_i > 0$ $(i = 1,2)$, the proof being trivial in the opposite case. From the relation $\varphi_X = \varphi_\xi \cdot \varphi_\alpha$ we can verify, that $\sigma_1 = \sigma_2 = 1$; without loss of generality we may assume, that $\mu_1 = \mu_2 = 0$. Put $h(x) = ax + b$. It is $E(h(X)e^{itX}) = aE(Xe^{itX}) + bE(e^{itX}) = e^{\lambda(e^{it}-1)}[a\lambda e^{it} + b]$. Put $G(x) = a\frac{\lambda}{\lambda_2}x + b$. It is similarly

$$E(G(\xi)e^{it\xi}) = e^{\lambda_2(e^{it}-1)}[a\lambda e^{it} + b]$$

Further

$$\varphi_\alpha(t) = e^{\lambda_1(e^{it}-1)}$$

and thus

$$\varphi_\alpha(t)\ E(G(\xi)e^{it\xi}) = e^{\lambda(e^{it}-1)}[a\lambda e^{it} + b]\,.$$

Now, (5.2) holds for ours h and G and the theorem follows from (5.4).

Remark: Evelyne Fix [4] has proved this very interesting

Theorem: *Let* ξ, α, β *be independent random variables, and* $E(\xi^2) < +\infty$, *or* $E(\alpha^2) < +\infty$. *Let*

$$X = c\xi + \alpha$$
$$Y = a\xi + \beta,\quad a \neq 0\,.$$

Let there exists a non empty interval (c_1, c_2) *such that for every* $c \in (c_1, c_2)$ $E_X(Y)$ *is linear. Then* F_X *is a normal distribution function.*

From the theorem (6.3) it follows, that it does not suffice to require linearity of $E_X(Y)$ only for one value of c.

BIBLIOGRAPHY

[1] *Harald Cramér*: Mathematical methods of statistics. Princeton 1951.

[2] *Harald Cramér*: Random variables and probability distributions. Cambridge 1937.

[3] *J. L. Doob*: Stochastic Processes. New York 1953.

[4] *Evelyne Fix*: Distributions which lead linear regressions. Preceedings of the Berkeley Symposium 1949.

[5] *P. R. Halmos*: Measure theory. New York 1950.

[6] *A. N. Kolmogorov*: Grundbegriffe der Wahrscheinlichkeitsrechnung. Berlin 1933.

[7] *Rajkov*: On the decomposition of Poisson laws. C. R. Acad. Sci URSS N. s. *14*, 9–11 (1937). (See Zentralblatt *15* (1937).)

[8] *Olaf Reiersøl*: Identifiability of a linear structural relation between variables which are subject to error. Econometrica *18* (1950).

Резюме.

СТРУКТУРНЫЕ СООТНОШЕНИЯ

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Пусть ξ, α, β, X, Y — случайные переменные, удовлетворяющие соотношению (1.1). В работе выводятся достаточные условия, при которых функцию C можно идентифицировать, если известны $F_{X,Y}$ и F_α, или если известны $F_{X,Y}$, F_α, F_β. Общие результаты применяются к случаям, когда X и Y обладают нормальным распределением вероятностей, или когда X имеет нормальное распределение вероятностей, или когда, наконец, X обладает пуассоновым распределением вероятностей. В последних двух случаях необходимым и достаточным условием линейности функции C является линейность регрессии $E_X(Y)$, если предположить, что ξ, α, β независимы и $|E(\beta)| < +\infty$.