ON CONDITIONAL EXPECTATIONS

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The paper contains a new proof concerning the sequence of conditional expectations of an integrable function f on countable Cartesian product of measurable spaces with the given probability μ.

Theorem. *Let (X, S) be a countable Cartesian product of measurable spaces (X_i, S_i). On (X, S) let be given a probability μ. Let f be a μ-integrable function and f_n its conditional expectation given the first n coordinates. Then $\lim_{n\to\infty} f_n = f$ almost everywhere.*

This theorem is demonstrated by Lévy [3] for characteristic functions; a demostration is in Halmos [2] too. For all functions this theorem is a consequence of more general theorem in Sparre Andresen and Jessen [4]; see also Doob [1].

For demonstration we are starting with the following

Lemma. *f_n is convergent to f in $L^{(1)}$.*

Proof. Let ε be a positive number and g a simple function with $\int_X |f - g| \,\mathrm{d}\mu < \varepsilon$. We can suppose that g is a linear combination of characteristic functions of cylindrical sets.

Let g_n be conditional expectation given the first n coordinates. Then $g_n = g[\mu]$ for n sufficiently large.

Let $A_n = \underset{x}{\mathrm{E}}[f_n(x) - g_n(x) > 0]$. A_n is a cylindrical set and the definition of conditional expectation gives

$$\int_{A_n} (f_n - g_n) \,\mathrm{d}\mu = \int_{A_n} (f - g) \,\mathrm{d}\mu \,.$$

Similary

$$\int_{X-A_n} (f_n - g_n) \,\mathrm{d}\mu = \int_{X-A_n} (f - g) \,\mathrm{d}\mu \,.$$

Now

$$\int_X |f_n - g_n| \,\mathrm{d}\mu \leqq \int_{A_n} |f - g| \,\mathrm{d}\mu + \int_{X-A_n} |f - g| \,\mathrm{d}\mu \leqq 2\varepsilon$$

and hence

$$\int_X |f_n - g| \, d\mu \leqq 2\varepsilon$$

for n sufficiently large. We have therefore

$$\int_X |f_n - f| \, d\mu \leqq 3\varepsilon \text{ q. e. d.}$$

Proof of the theorem. 1. We are choosing an integer m and $\varepsilon > 0$. For $n > m$ let $B'_n = \underset{x}{\mathrm{E}}[f_n(x) - f_m(x) > \varepsilon]$, $C'_n = B'_n - \bigcup_{m<i<n} B'_i$, $B''_n = \underset{x}{\mathrm{E}}[f_n(x) - - f_m(x) < -\varepsilon]$, $C''_n = B''_n - \bigcup_{m<i<n} B''_i$.

C'_n, C''_n are n-cylindrical disjoint sets and hence

$$\varepsilon \, \mu(C'_n) \leqq \int_{C'_n} (f_n - f_m) \, d\mu = \int_{C'_n} (f - f_m) \, d\mu \leqq \int_{C'_n} |f - f_m| \, d\mu$$

and similarly

$$\varepsilon \, \mu(C''_n) \leqq \int_{C''_n} |f - f_m| \, d\mu \, .$$

From this

$$\varepsilon \, \mu(\bigcup C'_n) \leqq \int_X |f - f_m| \, d\mu, \quad \varepsilon \, \mu(\bigcup C''_n) \leqq \int_X |f - f_m| \, d\mu \, .$$

Finally, we have

$$\varepsilon \, \mu(B_m) \leqq 2 \int |f - f_m| \, d\mu$$

where B_m is the set of x for that $\sup_{n>m} |f_n(x) - f_m(x)| > \varepsilon$.

2. Following the lemma we can now choose m in the manner to have

$$\int_X |f - f_m| \, d\mu < \tfrac{1}{2}\varepsilon^2$$

and hence

$$\mu(B_m) < \varepsilon \, .$$

3. Let the subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ have the properties
 1. $f_{n_i} \to f$ almost uniformly,
 2. $n_i = m$ for $\varepsilon = i^{-2}$.

Let $\delta > 0$ and j so large that $\sum_{i>j} i^{-2} < \frac{1}{2}\delta$. Let G be a measurable set with $\mu(G) < \frac{1}{2}\delta$ and $f_{n_i} \to f$ on $X - G$ uniformly. Then $\mu(\bigcup_{n>j} B_{ni} \cup G) < \delta$ and $f_n \to f$ outside this set uniformly.

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Резюме

УСЛОВНЫЕ ОЖИДАНИЯ

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Статья содержит новое непосредственное доказательство теоремы, касающейся последовательности условных ожидаемых значений интегрируемой функции f, определенной на счетном декартовом произведении измеримых пространств с данной мерой вероятности μ.