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NOTE ON K. MENGER'S PROBABILISTIC GEOMETRY

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The purpose of this note is to establish a number of completely elementary results in Menger's probabilistic geometry from the point of view of the theory of random processes.

Following roughly the definition of K. MENGER in [1] the probabilistic geometry is a theory of random distance functions in an abstract space $X \neq \emptyset$. It will be considered here as a probability measure in a properly chosen σ -algebra of random events in the space F of all real functions defined in the Cartesian power $X^2 = X \times X$. This σ -algebra \mathfrak{F} is defined to be the smallest σ -algebra of subsets of F containing the class

$$\{ \{f : f(x, y) < r\} : x, y \in X, r \in R \},$$

where R denotes the space of all real numbers. If μ is a probability measure in \mathfrak{F} then, according to the definition in [2], (F, \mathfrak{F}, μ) is a random function.

For $A \subset X$ let us denote by $T(A)$ the set of all functions from F which are distance functions in A . *The random function (F, \mathfrak{F}, μ) is said to be a distance function or a metric with probability one, if $\bar{\mu}(T(X)) = 1$* , where $\bar{\mu}$ denotes the outer measure induced by μ . This definition seems to be the natural one.

Clearly, the class \mathfrak{D} of all denumerable subsets of X satisfies the conditions (2) and (3) in [2] and the transform T satisfies the conditions (6), (7) and (8) in [2]. The property (5) of T in [2] follows from the obvious fact that if $A \subset X$ and ρ is a metric in A then there exists a metric δ in X such that δ and ρ coincide on A . Using theorem 1 in [2] we obtain

Theorem 1. *A necessary and sufficient condition for a random function (F, \mathfrak{F}, μ) to be a metric with probability one is that*

- (1) $\mu\{f : f(x, x) = 0\} = 1$ for $x \in X$,
- (2) $\mu\{f : f(x, y) > 0\} = 1$ for $x, y \in X, x \neq y$,
- (3) $\mu\{f : f(x, y) = f(y, x)\} = 1$ for $x, y \in X$,
- (4) $\mu\{f : f(x, y) + f(y, z) \geq f(x, z)\} = 1$ for $x, y, z \in X$.

If the power of X does not exceed 2^{\aleph_0} , if $A \subset X$ is denumerable and ρ is a metric in A then there exists a metric δ in X such that the space X is separable with respect to δ and δ coincides with ρ on A ; hence, using in addition the fact that the separability is a hereditary property, we can state

Theorem 2. *If the power of X does not exceed 2^{\aleph_0} and the random function (F, \mathfrak{F}, μ) is a metric with probability one then it is a separable metric with probability one.*

Using theorem 1 we can easily verify that a necessary condition for a random function (F, \mathfrak{F}, μ) to be a metric with probability one is that

- (5) $\mu\{f : f(x, x) < r\} = 1$ for $x \in X, r > 0$,
 (6) $\mu\{f : f(x, y) < r\} = 0$ for $x, y \in X, r \leq 0$,
 (7) $\mu\{f : f(x, y) < r\} = \mu\{f : f(y, x) < r\}$ for $x, y \in X, r \in R$,
 (8)
$$\begin{cases} \mu\{f : f(x, y) + f(y, z) < r\} \leq \mu\{f : f(x, z) < r\} \\ \text{for } x, y, z \in X, r \in R. \end{cases}$$

But unfortunately it is not sufficient as will be shown by the following example:

Let $X = \{a, b, c\}$ be a set consisting of three points. Then there is a probability measure μ_0 in \mathfrak{F} such that

$$\begin{aligned} \mu_0\{f : f(a, a) = 0\} &= \mu_0\{f : f(b, b) = 0\} = \mu_0\{f : f(c, c) = 0\} = 1, \\ \mu_0\{f : f(a, b) < t\} &= \mu_0\{f : f(a, c) < t\} = \mu_0\{f : f(b, c) < t\} = \\ &= 1 - e^{-t} \text{ or } 0 \text{ according as } t \geq 0 \text{ or } t < 0, \\ \mu_0(\{f : f(a, b) < t_1\} \cap \{f : f(a, c) < t_2\} \cap \{f : f(b, c) < t_3\}) &= \\ &= \mu_0\{f : f(a, b) < t_1\} \mu_0\{f : f(a, c) < t_2\} \mu_0\{f : f(b, c) < t_3\}, \end{aligned}$$

hence (5), (6), (7) and (8) are satisfied, but for example

$$\begin{aligned} \mu_0\{f : f(a, b) = f(b, a)\} &= 0, \\ \mu_0\{f : f(a, b) + f(b, c) \geq f(a, c)\} &= \frac{3}{4}, \end{aligned}$$

i. e. (3) and (4) do not hold and $\mu_0(T(X)) = 0$. Essentially the same example can be constructed if X is of arbitrary power. We can state that the random function (F, \mathfrak{F}, μ_0) is a metric with probability zero or that it is almost never a metric.

We see from [1] that (5), (6), (7), (8) correspond exactly to the definition of a random metric given by Menger, hence, Menger's conditions do not suffice for the characterization of a random metric in the sense of our definition.

BIBLIOGRAPHY

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 [2] *A. Špaček*: Regularity properties of random transforms. Czechoslovak Mathematical Journal, vol. 80 (1955), pp. 143—151.

Резюме

ЗАМЕТКА К ВЕРОЯТНОСТНОЙ ГЕОМЕТРИИ
К. МЕНГЕРА

АНТОН ШПАЧЕК (Antonín Špaček), Прага.
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Если мы определим приведенным в статье натуральным способом случайную метрику, то мы сможем выразить необходимые и достаточные условия для того, чтобы случайная функция двух переменных, определенная в абстрактном пространстве, была почти наверно метрикой. Если мощность этого пространства не превосходит 2^{\aleph_0} и если приведенное случайное преобразование является почти наверно метрикой, то оно почти наверно будет сепарабельной метрикой. Указано, что аксиомы Менгера недостаточны для определения случайной метрики.