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EXPERIENCE IN STATISTICAL DECISION PROBLEMS

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In this paper we consider a statistical decision problem the solution of which is a random decision process constructed sequentially using past experience. The sequence of losses is shown to be $(C, 1)$ convergent to the Bayes loss with probability one.

1. Basic notations

Let $X = (X, \mathcal{E}, \mu)$ be a probability space. For avoiding misunderstandings, the random variables on $X$ (i.e. absolutely integrable functions on $X$) or random vectors on $X$ (i.e. finite sequences of random variables on $X$) will be denoted by Greek letters, the numbers or vectors (finite sequences of numbers) will be denoted by Latin letters.

An $m$-dimensional random vector $\tau$ generates the probability space

$$X^{\tau} = (R_m, \mathcal{B}_m, \mu^{-1}) ,$$

where $R_m$ is the $m$-dimensional Euclidean space, $\mathcal{B}_m$ the $\sigma$-algebra of all Borel sets $B \subset R_m$, and

$$\mu^{-1}(B) = \mu(\tau^{-1}(B))$$

for every $B \in \mathcal{B}_m$. If $h$ is a random variable on $X^{\tau}$, the composite function $h\tau$ is a random variable on $X$. If $\varphi$ is a random variable on $X$, then

$$\mathbb{E}\varphi = \int_X \varphi \, d\mu, \quad \mathbb{E}^\tau h = \int_{R_m} h \, d\mu^{-1}.$$ 

The symbol $\mathbb{E}_\tau \varphi$ denotes the conditional expectation of $\varphi$ with respect to $\tau$; thus $\mathbb{E}_\tau \varphi$ is a random variable on $X^{\tau}$. An integration variable will be never denoted by a letter; we denote it, if necessary, by an asterisk.

For the sake of simplicity, we introduce the following conventions: Let

$$\tau_1 = (\tau_{11}, \tau_{12}, \ldots, \tau_{1m_1})$$

and

$$\tau_2 = (\tau_{21}, \tau_{22}, \ldots, \tau_{2m_2})$$

be two random vectors. We denote by $(\tau_1, \tau_2)$ the random vector $(\tau_{11}, \tau_{12}, \ldots, \tau_{1m_1}, \tau_{21}, \ldots, \tau_{2m_2})$. Further we shall say that $H$ is $\tau_1, \tau_2$ integrable if (i) $H$ is
a function on \( \mathbb{R}_n \times \mathbb{R}_m \), (ii) for every fixed \( y \in \mathbb{R}_m \), \( H(y, x) \) is a random variable on \( \mathbb{X}^2 \), (iii) \( H(\tau_1, \tau_2) \) is a random variable on \( \mathbb{X} \).

We shall write \( \tau_1 = \tau_2 \) if \( \tau_1(x) = \tau_2(x) \) with probability one; by the convergence of a sequence of random variables or vectors it is meant the convergence with probability one.

In the following section we state Theorem I, immediate consequence of which is Theorem II (section 3), the main result of this paper.

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The following lemma stated here for convenience follows from Theorem III of [3] and the fact, that dealing with expectations of finite number of random variables we may suppose without loss of generality the basic space \( \mathbb{X} \) to be \( (\mathbb{R}, \mathcal{B}, \mu) \), where \( t \) is a suitably chosen integer.

**Lemma 1.** Let \( \tau_1, \tau_2 \) be independent random vectors, \( H \) a function integrable \( \tau_1, \tau_2 \). Then, denoting

\[
g(y) = \mathbb{E}H(y, \tau_2),
\]

it holds

\[
g = \mathbb{E}_\tau H(\tau_1, \tau_2).
\]

**Theorem 1.** Let \( \{\pi_n\}, \{\theta_n\} \) are two sequence of \( m_1 \) respectively \( m_2 \)-dimensional random vectors. Let \( \theta_n \) be independent of \( \theta_1, \theta_2, \ldots, \theta_{n-1}, \pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n \).

Let \( h \) be a function integrable \( \pi_i, \theta_i \) and

\[
\mathbb{E}h(r, \theta_n) = b \quad \text{for every } r \in \mathbb{R}_{m_1},
\]

\[
\mathbb{E}h^2(r, \theta_n) < M \quad \text{for every } r \in \mathbb{R}_{m_1},
\]

where \( b \) and \( M \) are real numbers. Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} h(\pi_i, \theta_i) = b.
\]

This assertion holds also if we replace the condition

\[
K(r) = \mathbb{E}h(r, \theta_n) = b
\]

(say)

by the weakened condition

\[
K \pi_n \to b.
\]

**Proof.** First we shall assume that \( \mathbb{E}h(y, \pi_n) = b = 0 \). Let \( 0 < n_1 < n_2 \) be two integers. Denote

\[
\tau_1 = [\pi_{n_1}, \pi_{n_2}, \theta_{n_1}], \quad \tau_2 = \theta_{n_2}.
\]

\( ^{1} \) In particular this condition of independence is satisfied if \( \theta_n \) are independent and \( \pi_n \) is function at most of \( \theta_1, \theta_2, \ldots, \theta_{n-1} \).
By hypothesis \( \tau_1 \) and \( \tau_2 \) are independent. Clearly, from integrability \( \tau_i, \theta_i \) of \( h \) it follows the integrability \( \tau_1, \tau_2 \) of \( H \) defined by

\[
H(\tau_1, \tau_2) = h(\tau_{n_1}, \theta_{n_1}) h(\tau_{n_2}, \theta_{n_2}).
\]

Thus lemma 1 applied, we get \( g = 0 \) and

\[
\mathbb{E} h(\tau_{n_1}, \theta_{n_1}) h(\tau_{n_2}, \theta_{n_2}) = \mathbb{E} H(\tau_1, \tau_2) = \mathbb{E}^t \mathbb{E} H(\tau_1, \tau_2) = \mathbb{E}^t g = 0.
\]

Thus, denoting \( \zeta_n = h(\tau_{n_1}, \theta_{n_1}) \), we have proved that \( \zeta_1, \zeta_2, \ldots \) are mutually orthogonal. Clearly \( \mathbb{E} \zeta_n^2 = \sigma_n^2 < M \) and thus

\[
\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} \log^2 n < +\infty.
\]

From [2] Theorem 5.2 it follows \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \zeta_i = 0 \).

We have proved the first part of the theorem for \( b = 0 \). The generalization for \( a \neq 0 \) and the second part follow immediately.

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Let \( k, s, m \), be fixed positive integers. We shall consider a sequence of decision problems. Each problem is of this type:

The statistician observes (sequentially) a random vector \( \varphi = [\varphi^1, \varphi^2, \ldots, \varphi^s] \). The random variable \( \varphi^i \) are distributed independently with the common distribution \( \nu \). The statistician knows that \( \nu \) is an element of the finite set \( \mathcal{N} = \{\nu_1, \nu_2, \ldots, \nu_k\} \) and selects, using observations on \( \varphi \), a decision \( d \) from the set \( \mathcal{D} = \{d_1, \ldots, d_m\} \). If a decision function \( \delta \) is used, then every value of \( \varphi \) determines a decision \( d \in \mathcal{D} \) and the number of coordinates of \( \varphi \) observed. Let us denote by \( l(\delta, i, \varphi) \) the loss plus the cost of experimentation if \( \delta \) is used, \( \varphi \) observed and \( i \) is the "true" distribution.

It will be assumed that also \( i \), the index of the true distribution \( \nu_i \) is a random variable \( \gamma \), unobservable for the statistician and taking the values 1, 2, \ldots, \( k \). Obviously

\[
\nu_i(B) = \mu(\{x; \varphi^1(x) \in B\} | \gamma(x) = i).
\]

Thus we may denote by \( \theta \) the random vector \( [\gamma, \varphi] \) and call it random decision problem. For each \( \delta \) and each value \( \theta(x) = [\gamma(x), \varphi(x)] \), the symbol \( l(\delta, \theta(x)) = l(\delta, \gamma(x), \varphi(x)) \) denotes the loss when \( \delta \) is used and the elementary event \( x \) occurs.

The conditional expected loss, given \( \gamma(x) = i \), will be denoted by \( L_i(\delta) \).

If the statistician estimates by \( r = (r_1, \ldots, r_k) \), \( r_i \geq 0, \sum_{i=1}^{k} r_i = 1 \) the true apriori distribution \( p \) given by

\[
p = (\mu \gamma^{-1}(\{1\}), \ldots, \mu \gamma^{-1}(\{k\})),
\]

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he uses the \( r \)-Bayes decision function \( \delta_r \) which minimises the expression

\[
\sum_{i=1}^{k} r_i L_i(\delta) .
\]

Clearly \( L(\delta_p) = \min_{r} L(\delta_r) = \min_{\delta} L(\delta) \), where \( L(\delta) \) denotes the expected losses, if the decision function \( \delta \) is used, i.e., \( L(\delta) = \mathbb{E} l(\delta, \theta) \).

We note that the existence of Bayes decision functions is guaranteed by the finiteness of \( \mathcal{R}, \mathcal{D} \), and because \( \varphi \) is finite-dimensional.

From the Theorem 3.6 p. 89 of [4] it follows further that

\[
\lim_{r \to p} L(\delta_r) = L(\delta_p) .
\]

This suggests to consider the sequence of decisions problems of the same type and to use an estimate based on past experience to construct better and better decision functions. In Theorem II we show that not only the expected losses converge to the minimum loss \( L(\delta_p) \). In fact, if we denote by \( \lambda_n \) the loss occurring at the \( n \)-th step of the process, then

\[
\frac{1}{n} \sum_{i=1}^{n} \lambda_i \to L(\delta) .
\]

Thus we shall consider a sequence \( \theta_1 = \theta, \theta_2, \theta_3, \ldots \) of independently and identically distributed random vectors (i.e. random decision problems). A sequence of random vectors \( \pi_1, \pi_2, \ldots \) (where for each \( \pi_n = (\pi_n^1, \ldots, \pi_n^k) \), \( \pi_n^j \geq 0, \sum_{j=1}^{k} \pi_n^j = 1 \) hold), will be called a regular sequence estimating \( p \) if \( \pi_n \to p \) and \( \theta_n \) is independent of \( \pi_1, \pi_2, \ldots, \pi_n, \theta_1, \ldots, \theta_{n-1} \) for every \( n = 1, 2, \ldots \).

The statistician, who uses at the \( n \)-th step the decision function \( \delta_{\pi_n} \) pays the loss \( \lambda_n = l(\delta_{\pi_n}, \theta_n) = H(\pi_n, \theta_n) \). But from the constructive definition of \( \delta_r \) it follows easily that \( H \) is measurable \( \pi_n, \theta_n \). From the finiteness of \( \mathcal{D}, \mathcal{R} \) and from the fact that \( \varphi \) are finite dimensional it follows that \( H \) is bounded and therefore \( H \) satisfies the conditions of the Theorem I. Obviously, if we denote

\[
K(r) = \mathbb{E} H(r, \theta_n) ,
\]

we get

\[
K(r) = L(\delta_r) ;
\]

from (*) and from \( \pi_n \to p \) we have finally

\[
K\pi_n \to L(\delta_p) .
\]

Hence from Theorem I it follows

**Theorem II.** If \( \pi_n \) is a regular sequence estimating \( p \), then the sequence of losses, if the sequence \{\( \delta_{\pi_n} \)\} of decision functions is used, is \((C,1)\) convergent with probability one to the minimum loss \( L(\delta_p) \). In symbols

\[
2) \text{ We note that } K\pi_n \text{ is a random variable and is not } \mathbb{E} H(\pi_n, \theta_n).
\]
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \lambda_i = L(\delta_p).
\]

Example: Suppose \( k = 2, m = 1, s = 2 \), the weight function

\[
W(i, j) = \begin{cases} 
0 & \text{if } i = j \\
1 & \text{if } i \neq j 
\end{cases}
\]

and the cost of experimentation to be identically zero. Then if \( h_i \) is a generalized probability density of \( r_i \) with respect to some measure \( v \) (for \( i = 1, 2 \)), then the decision function \( \delta_{(r_1, r_2)} \) accepts \( d_1 \) if

\[ q_n(x) \in \{ y \in R_1; r_1h_1(y) > r_2h_2(y) \} \]

and \( d_2 \) otherwise.

If \( v_1 \) and \( v_2 \) are not identical, then a regular sequence \( \pi_n \) estimating

\[ p = (\mu \gamma_n^{-1}(1); \mu \gamma_n^{-1}(2)) \]

can be defined as follows:

\[ \pi_n(x) = \left[ \frac{1}{n-1} \sum_{i=1}^{n-1} c_i q_n(x) \right] \left[ \frac{1}{n-1} \sum_{i=1}^{n-1} c_{R_i \setminus E} q_n(x) \right] \]

where \( v_1(E) \neq v_2(E) \) and \( c_E \) and \( c_{R_i \setminus E} \) are characteristic functions of the sets \( E \) and \( R_1 \setminus E \) respectively.

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Резюме

ОПЫТ В СТАТИСТИЧЕСКИХ РЕШАЮЩИХ ПРОБЛЕМАХ

ВАЦЛАВ ФАБИАН, АНТОНИН ШПАЧЕК (VÁCLAV FABIAN, ANTONÍN ŠPAČEK), Прага.

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В статье рассматривается статистическая проблема, решением которой является случайная последовательность решающих функций; построение этой последовательности опирается на каждом шагу о предыдущий опыт. Последовательность потерь сходится \((C, 1)\) к потере Бэйesa с вероятностью 1.