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AN INEQUALITY FOR UNCORRELATED RANDOM VARIABLES

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This paper contains the proof of an inequality established by the authors. The inequality can be easily applied to the proof of the strong law of large numbers for pairwise orthogonal random variables.

Introduction

A. N. KOLMOGOROFF [1] proved that if $\xi_1, \xi_2, \dots, \xi_k, \dots$ are mutually independent random variables with mean value $M(\xi_k) = 0$ and finite variance $D^2(\xi_k) = D_k^2$ ($k = 1, 2, \dots$) and further if

$$\sum_{k=1}^{\infty} \frac{D_k^2}{k^2} < +\infty, \tag{1}$$

then the strong law of large numbers is valid for the sequence $\{\xi_k\}$, i. e. we have

$$P\left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \xi_k}{n} = 0\right) = 1 \tag{2}$$

(here and in what follows $P(A)$ denotes the probability of the event A). In proving this theorem, Kolmogoroff used the following inequality, due to him:

$$P\left(\max_{1 \leq k \leq m+1} \left|\sum_{j=1}^k \xi_j\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{m+1} D_j^2 \quad (m = 0, 1, \dots) \tag{3}$$

for any $\varepsilon > 0$, which is valid under the conditions on the variables ξ_k stated above.

J. HÁJEK [2] has recently discovered a similar inequality which, when used instead of (3), simplifies considerably the proof of the above-mentioned and other similar theorems. The inequality of Hájek states that if c_k is a non-increasing sequence of positive numbers ($k = 1, 2, \dots$), we have under the same conditions as above

$$P\left(\max_{n \leq k \leq n+m} c_k \left|\sum_{j=1}^k \xi_j\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \left(c_n^2 \sum_{j=1}^n D_j^2 + \sum_{j=n+1}^{n+m} D_j^2 c_j^2\right) \tag{4}$$

for $n, m = 1, 2, \dots$ and any $\varepsilon > 0$. The inequality (4) is clearly a generalization of (3); as a matter of fact, for $n = 1, c_k = 1 (1 \leq k \leq m + 1)$ we obtain from (4) as a special case the inequality (3).

It is well known that the strong law of large numbers is valid also if, instead of the mutual independence of the random variables considered, only pairwise uncorrelatedness is supposed, provided that instead of (1) the stronger condition

$$\sum_{k=1}^{\infty} \frac{D_k^2 \log^2 k}{k^2} < +\infty \quad (5)$$

is fulfilled (see e. g. [3]). As a matter of fact, this result follows, from the well-known theorem of H. RADEMACHER and D. MENCHOFF [4]. The proof of this theorem is based on the following inequality (see [3] p. 156. Lemma 4.1.): If the random variables $\xi_1, \xi_2, \dots, \xi_n$ have zero mean values ($M(\xi_k) = 0$) and finite dispersions $D^2(\xi_k) = D_k^2 (k = 1, 2, \dots, n)$, and if in addition they are pairwise uncorrelated, i. e. $M(\xi_j \xi_k) = 0$ for $j \neq k (j, k = 1, 2, \dots, n)$, we have*)

$$M \left(\max_{1 \leq k \leq n} \left(\sum_{j=1}^k \xi_j \right)^2 \right) \leq \left(\frac{\log 4n}{\log 2} \right)^2 \cdot \sum_{j=1}^n D_j^2, \quad (6)$$

from which it follows that, for any $\varepsilon > 0$,

$$P \left(\max_{1 \leq k \leq n} \left| \sum_{j=1}^k \xi_j \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2} \left(\frac{\log 4n}{\log 2} \right)^2 \sum_{j=1}^n D_j^2. \quad (7)$$

In the present paper we shall prove the inequality (9) which is in the same relation to inequality (7) as the inequality (4) of J. Hájek to the inequality (3) of Kolmogoroff.

This inequality simplifies the proof of the strong law of large numbers for uncorrelated random variables.

Let $\xi_1, \xi_2, \dots, \xi_k, \dots$ denote a sequence of uncorrelated random variables with mean values 0, i. e. we suppose $M(\xi_k) = 0 (k = 1, 2, \dots)$ and $M(\xi_j \xi_k) = 0$ for $j \neq k (j, k = 1, 2, \dots)$. Let us suppose that the variances $D_k^2 = D^2(\xi_k)$ exist ($k = 1, 2, \dots$).

Let further c_k denote a non-increasing sequence of positive numbers, satisfying the inequality $1 < c \leq \frac{c_k}{c_{2k}} \leq C (k = 1, 2, \dots)$. We shall prove that under these conditions we have the inequality

$$M \left(\sup_{n \leq k} c_k^2 \left| \sum_{j=1}^k \xi_j \right|^2 \right) \leq K \left(c_n^2 \sum_{j=1}^n D_j^2 + \sum_{j=n+1}^{\infty} D_j^2 c_j^2 \log^2 j \right) \quad (8)$$

*) To prove (6), instead of the pairwise uncorrelatedness and the vanishing of the mean values of the random variables ξ_k it suffices to suppose only the orthogonality of those variables, i. e. to suppose that $M(\xi_j \xi_k) = 0$ for $j \neq k$, without supposing that $M(\xi_k) = 0$. In this case however D_k^2 must be replaced in (6) resp. (7) by $M(\xi_k^2)$.

for $n = 1, 2, \dots$, where the constant K depends only on the constants c and C .

From (8) it follows easily, that

$$P\left(\sup_{n \leq k} c_k \left| \sum_{j=1}^k \xi_j \right| \geq \varepsilon\right) \leq \frac{K}{\varepsilon^2} \left(c_n^2 \sum_{j=1}^n D_j^2 + \sum_{j=n+1}^{\infty} D_j^2 c_j^2 \log^2 j \right) \quad (9)$$

for any $\varepsilon > 0$ and $n = 1, 2, \dots$

To prove (8), let us fix the integer n and put

$$\zeta_n = \sum_{j=1}^n \xi_j \quad \text{and} \quad \delta_{a,b} = \begin{cases} 0 & \text{if } a \leq n = b \text{ and if } b \leq a, \\ \sum_{j=a+1}^b \xi_j & \text{if } n \leq a < b, \\ \sum_{j=n+1}^b \xi_j & \text{if } a \leq n < b. \end{cases} \quad (10)$$

If k is a positive integer, $n \leq k$ and the integer s is defined by the inequalities $2^s \leq k < 2^{s+1}$, we have

$$\zeta_k = \zeta_n + \delta_{n,2^s} + \delta_{2^s,k}. \quad (11)$$

It follows, by Cauchy's inequality and by $c_{m+1} \leq c_m$, that

$$c_{k \wedge k}^2 \leq 3(c_n^2 \zeta_n^2 + c_{2^s}^2 \delta_{n,2^s}^2 + c_{2^s}^2 (\max_{2^s \leq j < 2^{s+1}} \delta_{2^s,j}^2)) \quad \text{for } 2^s \leq k < 2^{s+1} \quad (12)$$

and therefore

$$c_{k \wedge k}^2 \leq 3(c_n^2 \zeta_n^2 + \sum_{s=r}^{\infty} c_{2^s}^2 \delta_{n,2^s}^2 + \sum_{s=r}^{\infty} c_{2^s}^2 (\max_{2^s \leq j < 2^{s+1}} \delta_{2^s,j}^2)) \quad (13)$$

for $k = n, n+1, \dots$, where the integer r is defined by the inequalities $2^r \leq n < 2^{r+1}$; thus

$$M(\sup_{n \leq k} c_{k \wedge k}^2) \leq 3(c_n^2 M(\zeta_n^2) + \sum_{s=r}^{\infty} c_{2^s}^2 M(\delta_{n,2^s}^2) + \sum_{s=r}^{\infty} c_{2^s}^2 M(\max_{2^s \leq j < 2^{s+1}} \delta_{2^s,j}^2)). \quad (14)$$

Due to the uncorrelatedness of the variables ξ_j we have

$$M(\zeta_n^2) = \sum_{j=1}^n D_j^2 \quad (15)$$

and

$$\sum_{s=r}^{\infty} c_{2^s}^2 M(\delta_{n,2^s}^2) = \sum_{j=n+1}^{\infty} D_j^2 (\sum_{j \leq 2^s} c_{2^s}^2). \quad (16)$$

Since by supposition $c \leq \frac{c_{2^s}}{c_{2^{s+1}}}$ we have $\sum_{j \leq 2^s} c_{2^s}^2 \leq \frac{c^2}{c^2 - 1} c_j^2$ and thus from (16)

$$\sum_{s=r}^{\infty} c_{2^s}^2 M(\delta_{n,2^s}^2) \leq \frac{c^2}{c^2 - 1} \sum_{j=n+1}^{\infty} c_j^2 D_j^2. \quad (17)$$

As regards the third term on the right of (13) we use inequality (6) and obtain

$$M(\max_{2^s \leq j < 2^{s+1}} \delta_{2^s,j}^2) \leq K_1 \sum_{j=2^{s+1}}^{2^{s+1}} D_j^2 \log^2 j, \quad \text{for } s \geq r+1,$$

and

$$M(\max_{2^r \leq j < 2^{r+1}} \delta_{2^r, j}^2) \leq K_1 \sum_{j=n+1}^{2s+1} D_j^2 \log^2 j, \quad (18)$$

where K_1 is a constant, not depending on s .

Thus it follows that

$$\sum_{s=r}^{\infty} c_{2^s}^2 M(\max_{2^s \leq j < 2^{s+1}} \delta_{2^s, j}^2) \leq K_1 (c_{2^r}^2 \sum_{j=n+1}^{2^{r+1}} D_j^2 \log^2 j + \sum_{s=r+1}^{\infty} c_{2^s}^2 \sum_{j=2^{s+1}}^{2^{s+1}} D_j^2 \log^2 j) \quad (19)$$

and as $c_{2^s} \leq C c_{2^{s+1}} \leq C c_j$ for $2^s + 1 \leq j < 2^{s+1}$, we obtain

$$\sum_{s=r}^{\infty} c_{2^s}^2 M(\max_{2^s \leq j < 2^{s+1}} \delta_{2^s, j}^2) \leq C^2 K_1 (\sum_{j=n+1}^{\infty} D_j^2 c_j^2 \log^2 j). \quad (20)$$

It follows from (14), (15), (17) and (20) that

$$M(\sup_{n \leq k} c_{k^2}^2) \leq K (c_n^2 \sum_{j=1}^n D_j^2 + \sum_{j=n+1}^{\infty} D_j^2 c_j^2 \log^2 j) \quad (n = 1, 2, \dots),$$

where K is a constant, depending only on c and C . Thus (8) is proved and as we pointed out, (9) follows.

By applying the inequality (9) it follows immediately that if the variables ξ_k have zero mean values, finite variances $D_k^2 = D^2(\xi_k)$ and are uncorrelated and if in addition the series (5) converges, then the strong law of large numbers is valid, i. e. (2) holds. As a matter of fact, by choosing $c_k = 1/k$ in (9) we obtain for any $\varepsilon > 0$

$$P\left(\sup_{n \leq k} \frac{|\sum_{j=1}^k \xi_j|}{k} \geq \varepsilon\right) \leq \frac{K}{\varepsilon^2} \left(\frac{\sum_{j=1}^n D_j^2}{n^2} + \sum_{j=n+1}^{\infty} \frac{D_j^2 \log^2 j}{j^2}\right). \quad (22)$$

Since by (5) the right hand side of (22) tends to zero for $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} P\left(\sup_{n \leq k} \frac{|\sum_{j=1}^k \xi_j|}{k} \geq \varepsilon\right) = 0, \quad (23)$$

for any $\varepsilon > 0$, which is clearly equivalent to (2).

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Резюме

НЕРАВЕНСТВО ДЛЯ ОРТОГОНАЛЬНЫХ СЛУЧАЙНЫХ ПЕРЕМЕННЫХ

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В работе доказывается неравенство (8), которое справедливо при условии, что случайные переменные $\xi_1, \xi_2, \xi_3, \dots$ попарно ортогональны, что их математические ожидания равны нулю и их дисперсии конечны, и что невозрастающая последовательность положительных чисел c_1, c_2, c_3, \dots удовлетворяет при данных постоянных c и C неравенству $1 < c \leq \frac{c_k}{c_{2k}} \leq C$: Постоянная K , встречающаяся в неравенстве (8), зависит только от c и C . Доказательство основывается на неравенстве (6) (см. [3]).

Для $c_k = \frac{1}{k}$ и при условиях, при которых справедливо неравенство (8), можно при помощи неравенства (9), которое является непосредственным следствием неравенства (8), легко доказать справедливость сильного закона больших чисел.