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A NOTE ON A LINEAR PROGRAMMING PROBLEM

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This paper contains some theorems on the behaviour of the solutions of a transportation problem in linear programming.

This paper was motivated by a transportation problem in linear programming. We give here DANTZIG's economic formulation of this problem (see [1] — References): "A homogenous product is to be shipped in the amounts $a_1, \ldots, \ldots, a_m$, respectively, from each of m shipping origins and received in amounts $b_1, \ldots, b_n$, respectively, by each of n shipping destinations. The cost of shipping a unit amount from the ith origin to jth destination is c_{ij} and is known for all combinations (i, j). The problem is to determine the amounts x_{ij} to be shipped over all routes (i, j) so as to minimize the total cost of transportation."

This problem can be mathematically expressed as follows: Let a system of $m + n$ positive numbers $a_1, \ldots, a_m; b_1, \ldots, b_n$ be given such that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ $(m \geqq 1, n \geqq 1)$. According to NOŽIČKA [2] such a system will in following be called an *M-system* and denoted by $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$. Two M-systems $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$, $M(a_1', \ldots, a_\mu'; b_1', \ldots, b_\nu')$, will be held as *equivalent* if $m = \mu$, $n = \nu$ and if the finite sequences $a_1', \ldots, a_m'$ and $b_1', \ldots, b_n'$ are permutations of $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively.

The M-system $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ will be said to be *degenerate* if there are indices $i_1, i_2, \ldots, i_p$ $(1 \leqq p < m)$, $j_1, j_2, \ldots, j_q$ $(1 \leqq q < n)$ such that $\sum_{k=1}^{p} a_{i_k} = \sum_{k=1}^{q} b_{j_k}$.

We shall call a *solution of the M-system* $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ every non-negative $m \times n$ matrix $X = (x_{ij})$ satisfying the conditions

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m\ , \quad \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n\ . \tag{1}$$

It can be proved that the set of all solutions of the given M-system considered as a set of points in an mn-dimensional Euclidean space is non-empty, convex, closed and bounded (see [2]). If $\boldsymbol{M}_1 \equiv M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ and $\boldsymbol{M}_2 \equiv$ $\equiv M\ (a_{i_1}, \ldots, a_{i_m}, b_{j_1}, \ldots, b_{j_n})$ are two equivalent M-systems and if $X = (x_{ij})$ is a solution of $\boldsymbol{M}_1$, then $X' = (x_{i_r j_s})$, $r = 1, 2, \ldots, m$, $s = 1, 2, \ldots, n$ is obviously a solution of $\boldsymbol{M}_2$.

Let now mn real numbers c_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ be given. The problem to be solved is to find the minimal (or maximal) values of the linear functional $f(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ over the set of all nonnegative matrices satisfying (1).

Dantzig [1], Nožička [2] and others have given methods for solving the described problem. We shall now give a short summary of Nožička's method. The proofs will be ommited.

We can without any loss of generality suppose that the considered M-system is non-degenerate. If $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ is degenerate the modified M-system $M(a_1 - \varepsilon, \ldots, a_m - \varepsilon; b_1, \ldots, b_{n-1}, b_n - m\varepsilon)$ which is non-degenerate for $\varepsilon\ (> 0)$ sufficiently small can be studied (see [1], [2]).

In solving the problem described above we construct firstly a *basic solution* (not necessarily minimizing $f(X)$) as follows: At first, indices i_1, j_1 $(1 \leqq i_1 \leqq m$, $1 \leqq j_1 \leqq n)$ will be chosen and we put $x_{i_1 j_1} = \min(a_{i_1}, b_{j_1})$. Because of (1) the "smaller" M-system $\boldsymbol{M}_1 \equiv (a_1, \ldots, a_{i_1-1}, a_{i_1+1}, \ldots, a_m; b_1, \ldots, b_{j_1-1}, b_{j_1} - a_{i_1}, b_{j_1+1}, \ldots,$ $\ldots, b_n)$ or $\boldsymbol{M}_1' \equiv M(a_1, \ldots, a_{i_1-1}, a_{i_1} - b_{j_1}, a_{i_1+1}, \ldots, a_m; b_1, \ldots, b_{j_1-1}, b_{j_1+1}, \ldots,$ $\ldots, b_n)$ can be studied according to whether $a_{i_1} < b_{j_1}$ or $a_{i_1} > b_{j_1}$ (the case $a_{i_1} = b_{j_1}$ is eliminated by the non-degeneracy assumption). For $\boldsymbol{M}_1$ or $\boldsymbol{M}_1'$ the whole construction will be repeated until an M-system with $m = 1$ or $n = 1$ is obtained. This will then be solved trivially.

A basic solution (i. e. a solution obtainable by the above-described construction) contains at least $(m - 1)(n - 1)$ zero components, under the non-degeneracy assumption exactly $(m - 1)(n - 1)$ zero components.

From a basic solution $X = (x_{ij})$ a new basic solution will be got for which $f(X)$ acquires a smaller value (if such a solution exists) as follows:

Let us consider a zero component of the given basic solution X, e. g. x_{gh} and let us distinguish two cases.

(I) There exist indices k, l $(1 \leqq k \leqq m,\ 1 \leqq l \leqq n)$ such that $x_{gl} > 0, x_{kl} > 0$, $x_{hk} > 0$. If the corresponding coefficients c_{ij} fulfil $c_{gl} + c_{kh} > c_{kl} + c_{gh}$ a new basic solution X' will be defined by the relations $x'_{ij} = x_{ij}$ for $g \neq i \neq k$, $h \neq j \neq l$, $x'_{gl} = x_{gl} - \alpha$, $x'_{kh} = x_{kh} - \alpha$, $x'_{kl} = x_{kl} + \alpha$, $x'_{gh} = x_{gh} + \alpha$ where $\alpha = \min(x_{gl}, x_{kh})$. We have obviously $f(X') < f(X)$.

(II) If (I) does not hold for some zero component x_{gh} we construct the basic solution X'' in the following way: We put $x''_{gh} = t > 0$ and leave all other zero components unchanged. These conditions determine X'' uniquely. If $f(X'')$ as a function of t is smaller than $f(X)$ for $t > 0$ we choose t as large as possible.

If no solution X' such that $f(X') < f(X)$ can be constructed by (I) and if for every zero component of the case (II) the corresponding X'' is such that $f(X'') \geqq f(X)$, then $f(X) = \min$.

Every step of the above-described procedure has its geometrical interpretation given in [2].

We shall now prove some theorems on the behaviour of the solutions of the considered M-system minimalizing (maximalizing) the linear functional $f(X)$ if the coefficients c_{ij} satisfy certain conditions.

Theorem 1. *Let p, q $(1 \leqq p < m,\ 1 \leqq q < n)$ be positive integers such that the coefficients of the linear functional $f(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ fulfil the condition*

$$c_{\mu\nu} + c_{\varrho\sigma} \geqq c_{\mu\sigma} + c_{\varrho\nu} \tag{2}$$

for all μ, ν, ϱ, σ such that $1 \leqq \mu \leqq p$, $q + 1 \leqq \nu \leqq n$, $p + 1 \leqq \varrho \leqq m$, $1 \leqq \sigma \leqq q$ and let us denote

$$\left|\sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j\right| = \tau\ . \tag{3}$$

Then there exists a solution $X = (x_{ij})$ of the given M-system such that either

$$x_{ij} = 0\ , \quad i = 1, 2, \ldots, p\ , \quad j = q + 1, \ldots, n \tag{4a}$$

and

$$\sum_{i=p+1}^{m} \sum_{j=1}^{q} x_{ij} = \tau \tag{4b}$$

or

$$x_{ij} = 0\ , \quad i = p + 1, \ldots, m,\ j = 1, 2, \ldots, q \tag{5a}$$

and

$$\sum_{i=1}^{p} \sum_{j=q+1}^{n} x_{ij} = \tau \tag{5b}$$

and that $f(X) = \min$.

Theorem 2. *If under the assumptions of Theorem 1 the condition*

$$c_{\mu\nu} + c_{\varrho\sigma} > c_{\mu\sigma} + c_{\varrho\nu} \tag{6}$$

holds instead of (2) for all μ, ν, ϱ, σ as in (2), then every solution of the M-system considered in Theorem 1 such that $f(X) = \min$ has the properties (4a,b) or (5a,b).

Proof of Theorems 1 and 2. First we prove that (4a) and (5a) imply (4b) and (5b) respectively. For, let (4a) hold. Then

$$\sum_{i=1}^{p}\sum_{j=1}^{q} x_{ij} = \sum_{i=1}^{p} a_i\,, \quad \sum_{i=1}^{m}\sum_{j=1}^{q} x_{ij} = \sum_{j=1}^{q} b_j = \sum_{i=1}^{p} a_i + \tau \qquad \text{(by (3))}$$

and by subtracting the first equation from the second we get (4b). The implication (5a) $\Rightarrow$ (5b) will be proved analogically. It therefore suffices to prove (4a) or (5a).

In the following we shall distinguish two cases:

a) Let $\sum_{i=1}^{p} a_i + \tau = \sum_{j=1}^{q} b_j$. It will be proved that in this case (4a) holds. For, let $X' = (x'_{ij})$ be a solution of the given M-system such that $x'_{rs} > 0$ for some $r \leqq p$, $s \geqq q + 1$ and such that $f(X') = \min$. From $\sum_{i=1}^{p} a_i \leqq \sum_{j=1}^{q} b_j$ it follows that there exists an $x'_{tu} > 0$ with $t \geqq p + 1$, $u \leqq q$. This is clear from the following consideration:

$$\sum_{i=1}^{p}\sum_{j=1}^{q} x'_{ij} + \sum_{i=p+1}^{m}\sum_{j=1}^{q} x'_{ij} = \sum_{j=1}^{q} b_i\,, \quad \sum_{i=1}^{p}\sum_{j=1}^{q} x'_{ij} < \sum_{i=1}^{p} a_i$$

and if $\sum_{i=p+1}^{m}\sum_{j=1}^{q} x'_{ij} = 0$ it follows that $\sum_{j=1}^{q} b_j < \sum_{i=1}^{p} a_i$, which is a contradiction.

It suffices to consider the case $x'_{rs} \leqq x'_{tu}$ (in the contrary case the proof is completely analogical). Let us now define the $m \times n$ matrix $X'' = (x''_{ij})$ as follows:

$$x''_{ij} = x'_{ij} \quad \text{for} \quad r \neq i \neq t\,, \quad s \neq j \neq u\,, \quad x''_{rs} = 0\,,$$

$$x''_{tu} = x'_{tu} - x'_{rs}\,, \quad x''_{rs} = x'_{ru} + x'_{rs}\,, \quad x''_{ts} = x'_{ts} + x'_{rs}\,.$$

The matrix X'' is obviously non-negative and satisfies (1). By direct computation we find

$$f(X') - f(X'') = x'_{rs}(c_{rs} + c_{tu} - c_{ru} - c_{ts})\,. \qquad (7)$$

Due to relation (6) the last expression is positive, i. e. $f(X'') < f(X')$, which contradicts the assumption $f(X') = \min$ and therefore (4a) must hold for every solution X such that $f(X) = \min$. Theorem 2 is thus proved.

According to (2) the conditions $f(X'') \leqq f(X')$ and $f(X') = \min$ imply $f(X'') = f(X')$. If for X'' (4a) holds then Theorem 1 is proved. If not the whole consideration can be repeated and we must obviously after a finite number of steps come to a solution satisfying (4a).

b) From $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j + \tau$ we get (5a). The proof is analogical and therefore will be ommited.

Remark 1. The conditions (2) and (6) are somewhat difficult to verify. The reader may easily prove that the validity of the conditions (2) also follows from the validity of conditions

$$\max_{1 \leqq j \leqq q} c_{ij} \leqq \min_{q+1 \leqq j \leqq n} c_{ij}\,, \quad i = 1, 2, \ldots, p\,, \tag{8}$$

$$\max_{q+1 \leqq j \leqq n} c_{ij} \leqq \min_{1 \leqq j \leqq q} c_{ij}\,, \quad i = p + 1, \ldots, m\,. \tag{9}$$

If in (8) or (9) only the inequality sign $<$ holds then (8) and (9) imply (6).

Analogically to the Theorems 1 and 2 the following Theorems can also be proved:

Theorem 3. *Let the M-system* $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ *have the property* (3) *and suppose that the coefficients of the linear functional* $f(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ *fulfil*

$$c_{\mu\sigma} + c_{\varrho\nu} \geqq c_{\mu\nu} + c_{\varrho\sigma} \tag{10}$$

for all $\mu, \nu, \varrho, \sigma$ *as in* (2). *Then there exists a solution X of the given M-system such that* (4a, b) *or* (5a,b) *hold and such that* $f(X) = \max$.

Theorem 4. *If under the assumptions of Theorem* 3 *the condition* $c_{\mu\sigma} + c_{\varrho\nu} > c_{\mu\nu} + c_{\varrho\sigma}$ *holds instead of* (10) *for all* $\mu, \nu, \varrho, \sigma$ *as in* (2), *then every solution X of the M-system considered in Theorem* 3 *such that* $f(X) = \max$ *has the properties* (4a, b) *or* (5a,b).

Remark 2. Theorems 1 and 2 apply to degenerate M-systems as follows: Suppose that for the degenerate M-system $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ there exist positive integers p, q $(1 \leqq p < m,\ 1 \leqq q < n)$ such that $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j$ (i. e. relation (3) with $\tau = 0$). In this case it follows from Theorem 1 that there exists a solution $X = (x_{ij})$ of the considered M-system such that $f(X) = \min$ and

$$\begin{aligned} x_{ij} = 0 \quad \text{for} \quad & i = 1, \ldots, p,\ j = q + 1, \ldots, n \\ \text{and for} \quad & i = p + 1, \ldots, m,\ j = 1, \ldots, q\,. \end{aligned} \tag{11}$$

If in the above-described case relation (6) holds, then every solution of the considered M-system has the property (11). In these cases we can find a solution X of the degenerate M-system $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ such that $f(X) = \min$ by solving this problem separately for the M-systems $M(a_1, \ldots, a_p; b_1, \ldots, b_q)$ and $M(a_{p+1}, \ldots, a_m; b_{q+1}, \ldots, b_n)$. We then obtain the value of the total minimum by adding the minima for both "partial" M-systems and we find the form of a total solution by (11). Under the condition (6) we can find in this way all solutions, under the condition (2) only some of them (but at least one).

Analogical rules can be formulated for finding solutions for which the linear functional $f(X)$ acquires its maximum.

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Резюме

ЗАМЕТКА К ОДНОЙ ПРОБЛЕМЕ ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ

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Систему $m + n$ положительных чисел $a_1, \ldots, a_m; b_1, \ldots, b_n$ $(m \geqq 1,\ n \geqq 1)$ таких, что $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ назовем M-системой и обозначим через $M(a_1, \ldots, \ldots, a_m; b_1, \ldots, b_n)$. Скажем, что M-система $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ *разложима*, если существуют такие группы индексов $i_1, \ldots, i_p$ $(1 < p < m)$, $j_1, \ldots, j_q$ $(1 \leq q < n)$, что $\sum_{k=1}^{p} a_{i_k} = \sum_{s=1}^{q} b_{j_s}$.

Решением M-системы $M(a_1, \ldots, a_m; b_1, \ldots, b_n)$ назовем любую неотрицательную матрицу $X = (x_{ij})$ типа m, n такую, что

$$\sum_{j=1}^{n} x_{ij} = a_i\ , \quad i = 1, 2, \ldots, m\ , \quad \sum_{i=1}^{m} x_{ij} = b_j\ , \quad j = 1, 2, \ldots, n\ .$$

Основной задачей линейного программирования является нахождение такого решения X заданной M-системы, для которого линейный функционал $f(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ $(c_{ij},\ i = 1, \ldots, m;\ j = 1, \ldots, n$ — вещественные числа) достигает на множестве всех решений рассматриваемой M-системы своего минимума или максимума.

В работе дано сжатое изложение алгорифма Ножички [2] (см. список литературы) для решения данной задачи и доказаны теоремы, описывающие при некоторых условиях относительно матрицы (c_{ij}) поведение таких решений заданной M-системы. В случае разложимой M-системы они иногда позволяют решить описанную задачу для „частичных" M-систем $M(a_1, \ldots, a_p;\ b_1, \ldots, b_q)$ и $M(a_{p+1}, \ldots, a_m;\ b_{q+1}, \ldots, b_n)$ независимо друг от друга.