

Jan Mařík

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THE BAIRE AND BOREL MEASURE

JAN MAŘÍK, Praha.

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This paper contains the main results of [5]. It is shown that in many important cases the Baire measure can be extended to a Borel measure.

1. Let \mathfrak{A} be a non-empty family of sets. We say that \mathfrak{A} is a *field* if the sum and the difference of each pair of elements $A, B \in \mathfrak{A}$ also belongs to \mathfrak{A} . If \mathfrak{A} is a field and if $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$ (resp. $\bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$) whenever $A_n \in \mathfrak{A}$ ($n = 1, 2, \dots$), then \mathfrak{A} is called a σ -*field* (resp. δ -*field*). If \mathfrak{A} is a σ -field and if $\mathbf{U}\mathfrak{A} \in \mathfrak{A}$, we say that \mathfrak{A} is a σ -*algebra*.

If A_1, A_2, \dots are sets and if $A_1 \subset A_2 \subset \dots, \bigcup_{n=1}^{\infty} A_n = A$, we write $A_n \nearrow A$.

A non-negative σ -additive function μ (on a field \mathfrak{A}) such that $\mu(\emptyset) = 0$ is termed a *measure* (on \mathfrak{A}). If μ is a measure on a σ -algebra \mathfrak{A} , we put for each $M \subset \mathbf{U}\mathfrak{A}$

$$\begin{aligned} \bar{\mu}(M) &= \inf \mu(A), \quad \text{where } A \in \mathfrak{A}, \quad A \supset M, \\ \underline{\mu}(M) &= \sup \mu(A), \quad \text{where } A \in \mathfrak{A}, \quad A \subset M. \end{aligned}$$

Let P be a topological space.¹⁾ Let \mathfrak{G} (resp. \mathfrak{F}) be the family of all open (resp. closed) subsets of P . Let \mathfrak{G}^* (resp. \mathfrak{F}^*) be the family of all sets $E[x; f(x) > 0]$ (resp. $E[x; f(x) = 0]$), where f is a continuous function on P . Let \mathfrak{B} (resp. \mathfrak{B}^*) be the smallest σ -algebra containing \mathfrak{G} (resp. \mathfrak{G}^*). The elements of \mathfrak{B} (resp. \mathfrak{B}^*) are called *Borel* (resp. *Baire*) sets; a measure on the system \mathfrak{B} (resp. \mathfrak{B}^*) is termed a *Borel* (resp. *Baire*) measure.

Given a Baire measure μ , let \mathfrak{Y} be the family of all sets $A \subset P$ for which there exist $G_n \in \mathfrak{G}^*$ such that

$$A \subset \bigcup_{n=1}^{\infty} G_n, \quad \mu(G_n) < \infty \quad (n = 1, 2, \dots).$$

¹⁾ We suppose that the topology is defined by means of a system \mathfrak{G} (whose elements are subsets of P) with the following properties: 1) $\emptyset, P \in \mathfrak{G}$; 2) $G_1, G_2 \in \mathfrak{G} \Rightarrow G_1 \cap G_2 \in \mathfrak{G}$; 3) $\mathfrak{G}_0 \subset \mathfrak{G} \Rightarrow \mathbf{U}\mathfrak{G}_0 \in \mathfrak{G}$.

We say that μ has the property V_p if each set $A \in \mathfrak{B}^*$ such that $\mu(A) < \infty$ belongs to \mathfrak{F} .

2. We state now an elementary lemma which is important for further considerations:

Let P be an arbitrary set; let $\mathfrak{M}, \mathfrak{N}$ be systems of subsets of P and let $\emptyset \in \mathfrak{M} \cap \mathfrak{N}$. Let α (resp. β) be a finite non-negative function on \mathfrak{M} (resp. on \mathfrak{N}). Suppose that the following conditions are fulfilled:

- 1) $M \in \mathfrak{M}, N \in \mathfrak{N} \Rightarrow M - N \in \mathfrak{M}, N - M \in \mathfrak{N}$;
- 2) $M_1, M_2 \in \mathfrak{M}, M_1 \cap M_2 = \emptyset \Rightarrow M_1 \cup M_2 \in \mathfrak{M}, \alpha(M_1) + \alpha(M_2) = \alpha(M_1 \cup M_2)$;
- 3) $M \in \mathfrak{M}, N \in \mathfrak{N}, M \subset N \Rightarrow \beta(N - M) = \beta(N) - \alpha(M)$;
- 4) $N \in \mathfrak{N} \Rightarrow \beta(N) \leq \sup \alpha(M)$, where $M \subset N, M \in \mathfrak{M}$;
- 5) $N_n \in \mathfrak{N} (n = 1, 2, \dots), \sum_{n=1}^{\infty} \beta(N_n) < \infty \Rightarrow \bigcup_{n=1}^{\infty} N_n \in \mathfrak{N}, \beta(\bigcup_{n=1}^{\infty} N_n) \leq \sum_{n=1}^{\infty} \beta(N_n)$.

For each $A \subset P$ put

$$\begin{aligned} \underline{\gamma}(A) &= \sup \alpha(M), \quad \text{where } M \in \mathfrak{M}, M \subset A, \\ \bar{\gamma}(A) &= \inf \beta(N), \quad \text{where } N \in \mathfrak{N}, N \supset A. \end{aligned} \quad .^2)$$

Let \mathfrak{X} be the system of all sets $T \subset P$ for which $\underline{\gamma}(T) = \bar{\gamma}(T) < \infty$; let \mathfrak{A} be the system of all $A \subset P$ such that $A \cap T \in \mathfrak{X}$ for each $T \in \mathfrak{X}$.

Then \mathfrak{X} is a δ -field, \mathfrak{A} is a σ -algebra and $\bar{\gamma}$ is a measure on \mathfrak{A} . Furthermore, $\mathfrak{N} \subset \mathfrak{X} \subset \mathfrak{A}$ and $\underline{\gamma}(N) = \bar{\gamma}(N) = \beta(N)$ for each $N \in \mathfrak{N}$.

(The proof is not difficult.)

3. Now let the measure μ have the property V_p . Let \mathfrak{M} (resp. \mathfrak{N}) be the family of all $A \in \mathfrak{F}^*$ (resp. \mathfrak{G}^*) such that $\mu(A) < \infty$. If we put $\alpha(M) = \mu(M)$, $\beta(N) = \mu(N)$ for $M \in \mathfrak{M}, N \in \mathfrak{N}$, then all the conditions of the preceding lemma are satisfied and we easily obtain the following assertion:

If the measure μ has the property V_p , then

$$\bar{\mu}(A) = \inf \mu(G), \quad \text{where } G \in \mathfrak{G}^*, G \supset A, \quad (1)$$

for each $A \subset P$, and

$$\underline{\mu}(A) = \sup \mu(F), \quad \text{where } F \in \mathfrak{F}^*, F \subset A, \mu(F) < \infty, \quad (2)$$

for each $A \in \mathfrak{F}$.

4. We say that the measure μ has the property W_p , if μ is a Baire measure and if there exists a Borel measure ν with the following properties:

- 1) $B \in \mathfrak{B}^* \Rightarrow \nu(B) = \mu(B)$;
- 2) $G \in \mathfrak{G} \cap \mathfrak{F} \Rightarrow \nu(G) = \underline{\mu}(G)$;

²⁾ If there exist no $N \in \mathfrak{N}$ such that $N \supset A$, we have $\bar{\gamma}(A) = \inf \emptyset = \infty$.

- 3) $B \in \mathfrak{B} - \mathfrak{Y} \Rightarrow \nu(B) = \infty$;
 4) $B \in \mathfrak{B} \Rightarrow \nu(B) = \inf \nu(G)$, where $G \in \mathfrak{G}$, $G \supset B$.

Now we can state the following theorem:

Let the measure μ have the property V_p and let the implications

$$G_1, G_2 \in \mathfrak{G} \cap \mathfrak{Y} \Rightarrow \underline{\mu}(G_1) + \underline{\mu}(G_2) \geq \underline{\mu}(G_1 \cup G_2), \quad (3)$$

$$G_n \in \mathfrak{G} \cap \mathfrak{Y} \quad (n = 1, 2, \dots), \quad G_n \nearrow G \Rightarrow \underline{\mu}(G_n) \rightarrow \underline{\mu}(G) \quad (4)$$

be valid. Then the measure μ has the property W_p .

The proof is based on the following ideas: Let \mathfrak{M} (resp. \mathfrak{N}) be the family of all $A \in \mathfrak{F}$ (resp. $\mathfrak{G} \cap \mathfrak{Y}$) such that $\bar{\mu}(A) < \infty$ (resp. $\underline{\mu}(A) < \infty$). If we put $\alpha(M) = \bar{\mu}(M)$, $\beta(N) = \underline{\mu}(N)$ for $M \in \mathfrak{M}$, $N \in \mathfrak{N}$, then it follows from (1), (2), (3), (4), that all the conditions of Section 2 are fulfilled. It is easy to see that the corresponding system \mathfrak{A} contains each open subset of P and that the conditions 1)–4) are satisfied if we write $\nu(B) = \bar{\gamma}(B)$ for each $B \in \mathfrak{B}$.

Remark. If the space P is normal, we have clearly

$$\bar{\mu}(F) \leq \underline{\mu}(G), \quad (5)$$

whenever $F \in \mathfrak{F}$, $G \in \mathfrak{G}$, $F \subset G$. It follows easily from (5) that (3) holds in each normal space.

5. Now we are able to prove the following assertion:

Let the measure μ have the property V_p and let some of the three following conditions be fulfilled:

1) *P is completely regular and for each $F \in \mathfrak{F}^*$, where $\mu(F) < \infty$, there exist compact sets K_n such that $\underline{\mu}(F - \bigcup_{n=1}^{\infty} K_n) = 0$.*

2) *P is normal and for each $F \in \mathfrak{F}^*$, where $\mu(F) < \infty$, there exist pseudo-compact³⁾ sets A_n such that $\underline{\mu}(F - \bigcup_{n=1}^{\infty} A_n) = 0$.*

3) *P is normal and countably paracompact.⁴⁾*

Then μ has the property W_p .

We have to prove that the conditions (3) and (4) are satisfied. If the space P has one of the properties 1) or 2), then the proof requires only elementary considerations. Now let P be normal and countably paracompact; let μ have the property V_p . If $G_n \in \mathfrak{G} \cap \mathfrak{Y}$, $G_n \nearrow G$, we choose a set $F \in \mathfrak{F}$, $F \subset G$. Making use of the theorem which asserts that a normal space P is countably paracompact if and only if for each sequence U_1, U_2, \dots , where $U_n \in \mathfrak{G}$, $U_n \nearrow P$,

³⁾ The space A is pseudocompact if each continuous function on A is bounded.

⁴⁾ P is countably paracompact if for each countable open covering of P there exists a locally finite refinement.

there exist sets $D_n \in \mathfrak{F}$ such that $D_n \subset U_n$ ($n = 1, 2, \dots$) and $D_n \not\supset P$ (see [1] or [3]), we see that there exist $F_n \in \mathfrak{F}$ such that $F_n \subset G_n$, $F_n \not\supset F$. By (5) we get $\underline{\mu}(G_n) \geq \underline{\mu}(F_n) \rightarrow \underline{\mu}(F)$, whence $\lim \underline{\mu}(G_n) \geq \underline{\mu}(F)$, $\lim \underline{\mu}(G_n) \geq \sup \underline{\mu}(F) = \underline{\mu}(G)$, which proves (4). Because the relation (3) holds in each normal space, we see that the proof is complete.

Remark. Combining this result with [4], p. 479, we obtain various theorems concerning the representation of a non-negative functional by means of an integral $\int_P f d\nu$, where ν is a Borel measure.

6. If J is a non-negative linear functional which is defined on the family of all continuous functions on a topological space P , then there exists (see [4], p. 479) a unique Baire measure μ such that

$$J(f) = \int_P f d\mu$$

for each continuous f . (The measure μ is obviously finite and has therefore the property V_p .) For each $G \in \mathfrak{G}$ put

$$\delta(G) = \sup J(f),$$

where f is continuous, $f(x) \leq 1$ on G , $f(x) \leq 0$ on $P - G$. It is easy to see that

$$\delta(G) \leq \underline{\mu}(G) \quad (G \in \mathfrak{G}); \tag{6}$$

if P is normal, then

$$\delta(G) = \underline{\mu}(G) \quad (G \in \mathfrak{G}). \tag{7}$$

If the space P fulfils the condition 1) (Section 5), then (7) holds again. If P is a completely regular Q -space, then there exists a compact set K such that $\underline{\mu}(P - K) = 0$ (see [2]) and (7) is fulfilled. We see at the same time that the measure μ has in this case the property W_p .

Now let P be an arbitrary topological space. If there exists a Borel measure ν such that

$$J(f) = \int_P f d\nu \quad (f \text{ continuous on } P), \tag{8}$$

then obviously $\nu(B) = \mu(B)$ for each $B \in \mathfrak{B}^*$ and, consequently,

$$\underline{\mu}(B) \leq \nu(B) \leq \bar{\mu}(B) \tag{9}$$

for each $B \in \mathfrak{B}$. If, moreover,

$$\nu(G) = \delta(G) \quad (G \in \mathfrak{G}), \tag{10}$$

then it follows from (6) and (9) that (7) holds again.

In [2], p. 170, HEWITT raised the following question: Let J be a non-negative linear functional which is defined on the system of all continuous functions on a normal space P . Does there exist a Borel measure ν such that the relations

(8) and (10) are true? It follows from Section 5 that the answer is affirmative if P is countably paracompact; but we do not yet know a normal space which has not this property.

If the space P is not normal, it may happen that $\delta(G) < \mu(G)$ for some open set G ; then there exists no Borel measure ν such that the relations (8) and (10) hold good. Such example (where P is completely regular) is constructed in [2], pp. 169—170 (Remark 1); but the corresponding Baire measure has the property W_p again.

Now let Ω be the smallest non-countable ordinal number; let T be the space of all the ordinal numbers $\xi \leq \Omega$ and put $P = T \times T - \{[\Omega, \Omega]\}$. It is easy to see that we can put

$$J(f) = \lim_{[\xi, \eta] \rightarrow [\Omega, \Omega]} f(\xi, \eta)$$

for each continuous f . Then the corresponding Baire measure has not the property W_p , but it is possible to extend it to a Borel measure.

REFERENCES

- [1] *C. H. Dowker*: On countably paracompact spaces, *Canadian J. Math.* 3 (1951), 219—224.
- [2] *E. Hewitt*: Linear functionals on spaces of continuous functions, *Fund. Math.* 37 (1950), 161—189.
- [3] *M. Katětov*: On real-valued functions in topological spaces, *Fund. Math.* 38 (1951), 85—91.
- [4] *Ян Маржик* (Jan Mařík): Представление функционала в виде интервала, *Чех. Мат. Ж.* 5 (80) (1955), 467—487.
- [5] *Jan Mařík*: Baireova a Borelova míra, *Časopis pro pěstování matematiky* 81 (1956), 431—450.

Резюме

МЕРЫ БЭРА И БОРЕЛЯ

ЯН МАРЖИК (Jan Mařík), Прага.

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Пусть P — топологическое пространство. Пусть \mathfrak{F} (соотв. \mathfrak{G}) — система всех замкнутых (соотв. открытых) подмножеств пространства P ; пусть \mathfrak{F}^* (соотв. \mathfrak{G}^*) — система всех множеств вида $E[x; f(x) = 0]$ (соотв. $E[x; f(x) > 0]$), где f — непрерывная функция на пространстве P . Далее пусть

\mathfrak{B} (соотв. \mathfrak{B}^*) — наименьшая σ -алгебра, содержащая систему \mathfrak{F} (соотв. \mathfrak{F}^*). Неотрицательную σ -аддитивную функцию на системе \mathfrak{B} (соотв. \mathfrak{B}^*) назовем мерой Бореля (соотв. Бэра).

Пусть μ — мера Бэра: пусть \mathfrak{P} — система всех множеств $A \subset P$, для которых существуют $G_n \in \mathfrak{G}^*$ так, что $\mu(G_n) < \infty$ ($n = 1, 2, \dots$), $A \subset \bigcup_{n=1}^{\infty} G_n$. Предположим, что мера μ обладает следующим свойством: Если $B \in \mathfrak{B}^*$, $\mu(B) < \infty$, то $B \in \mathfrak{P}$. Далее положим для каждого $A \subset P$

$$\begin{aligned} \underline{\mu}(A) &= \sup \mu(B), \quad \text{где } B \subset A, \quad B \in \mathfrak{B}^*, \\ \bar{\mu}(A) &= \inf \mu(B), \quad \text{где } B \supset A, \quad B \in \mathfrak{B}^*. \end{aligned}$$

В работе намечены главные идеи доказательства следующих двух теорем:

Теорема 1. Для любого $A \subset P$ имеет место

$$\bar{\mu}(A) = \inf \mu(G), \quad \text{где } G \in \mathfrak{G}^*, \quad G \supset A;$$

если $\underline{\mu}(A) < \infty$, будет также

$$\underline{\mu}(A) = \sup \mu(F), \quad \text{где } F \subset A, \quad F \in \mathfrak{F}^*.$$

Теорема 2. Пусть выполняется какое-либо из следующих трех условий:

а) Пространство P вполне регулярно и для любого $F \in \mathfrak{F}^*$, где $\mu(F) < \infty$, существуют компактные множества K_n так, что $\underline{\mu}(F - \bigcup_{n=1}^{\infty} K_n) = 0$.

б) Пространство P нормально и для любого $F \in \mathfrak{F}^*$, где $\mu(F) < \infty$, существуют псевдокомпактные множества A_n так, что $\underline{\mu}(F - \bigcup_{n=1}^{\infty} A_n) = 0$.

в) Пространство P нормально и счетно-паракомпактно.

Тогда существует мера Бореля ν , обладающая следующими свойствами:

а) $B \in \mathfrak{B}^* \Rightarrow \nu(B) = \mu(B)$;

б) $G \in \mathfrak{G} \cap \mathfrak{P} \Rightarrow \nu(G) = \underline{\mu}(G)$;

γ) $B \in \mathfrak{B} - \mathfrak{P} \Rightarrow \nu(B) = \infty$;

δ) $B \in \mathfrak{B} \Rightarrow \nu(B) = \inf \nu(G)$, где $G \in \mathfrak{G}$, $G \supset B$.

Подробное доказательство этих теорем приведено в [5].