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NOTES ON STOCHASTIC APPROXIMATION METHODS

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In section 1 and 2, asymptotic properties of the Robbins-Monro and the Kiefer-Wolfowitz stochastic approximation methods are studied under the assumption, that the solution lies in an a priori known finite interval. In section 3, a stochastic approximation method is considered for solving systems of linear equations with a symmetric matrix of coefficients.

0. Introduction and summary

Stochastic approximation methods deal with the following problems: $M(x)$ is the (unknown) regression function of a family of random variables $\{Y_x\}$; we have to solve the equation $M(x) = \alpha$, or we have to find the value of $x$ for which $M(x)$ achieves its maximum, by means of an iterative process, using observations of $Y$ on various levels of $x$. The former problem has been solved by ROBBINS and MONRO [1], the latter by KIEFER and WOLFOWITZ [2]; both problems — so as the methods of solution — have multidimensional analogs (BLUM, [3]). The theoretical investigations of these methods go in two directions:

1° they are conditions studied, under which the approximations $x_n$ converge to the solution $\Theta$ with probability one;

2° the asymptotic order of second moments $E[(x_n - \Theta)^2]$, or the asymptotic distribution of $x_n$ are studied, and conclusions are drawn about the optimal choice of some eligible constants occurring in the approximation scheme.

In the second direction, the CHUNG’S paper [4], concerning Robbins-Monro procedure, is the most advanced. Chung’s methods were adapted by DERMAN [5] and — independently — by the author [6] to derive asymptotic properties of Kiefer-Wolfowitz procedure.

The present paper contains two contributions to the investigations sub 2°. First it is shown, that the conditions under which the approximation procedure
has satisfactory asymptotic properties, can be considerably weakened, if the approximations \( x_n \) are all restricted to a finite interval known as containing the solution \( \Theta \). This is done in Section 1 for the Robbins-Monro procedure, and in Section 2 for the Kiefer-Wolfowitz method. Secondly, a multidimensional modification of the Robbins-Monro procedure is considered in a special case of linear regression with a symmetrical matrix. The upper bounds for the quantities \( E[||x_n - \Theta||^2] \) are given (Section 3).

In the following, \( K_0, K_1, \ldots, K_9 \) are positive constants numbered in order of appearance. As \( [f(x)]^2 \) will be denoted the function

\[
g(x) = \begin{cases} A, & \text{if } f(x) < A, \\ f(x), & \text{if } A \leq f(x) \leq B, \\ B, & \text{if } f(x) > B. \end{cases}
\]

A lemma, due to Chung (Lemma 1 in [4]), will be used repeatedly:

Let \( \{b_n\}, n \geq 1 \), be a sequence of real numbers such that for \( n \geq n_0 \),

\[
b_{n+1} \leq \left(1 - \frac{c}{n}\right) b_n + \frac{c_1}{n^{p+1}},
\]

where \( c > p > 0, c_1 > 0 \). Then

\[
b_n \leq \frac{c_1}{c - p} \cdot \frac{1}{n^p} + O \left( \frac{1}{n^{p+1}} + \frac{1}{n^e} \right).
\]

1. The Robbins-Monro stochastic approximation method

Let to each value \( x \) from a finite interval \( \langle A, B \rangle \) correspond a distribution function \( H(y \mid x) \), let \( M(x) = \int_A^y dH(y \mid x) \) be a Borel measurable function bounded in \( \langle A, B \rangle \). Suppose that the equation \( M(x) = \lambda \) has a unique root \( x = \Theta \) in \( \langle A, B \rangle \), and that the inequality \( (M(x) - \lambda)(x - \Theta) > 0 \) holds for all \( x \neq \Theta, x \epsilon \langle A, B \rangle \).

Let \( a \) be a positive constant. Take \( x_1 \epsilon \langle A, B \rangle \) arbitrarily and for \( n \geq 1 \) set recursively

\[
x_{n+1} = \left[ x_n + \frac{a}{n} (\lambda - y_n) \right]_A^B, \tag{1}
\]

where \( y_n \) is a random variable whose distribution function, for given \( x_1, \ldots, x_n, y_1, \ldots, y_n \), is \( H(y \mid x_n) \).

We shall add the following assumptions:

Assumption (I_1): There exists a constant \( \sigma^2 \), such that

\[
\int_A^B (y - M(x))^2 \ dH(y \mid x) \leq \sigma^2 \ \text{for all} \ x \epsilon \langle A, B \rangle.
\]
Assumption (II): For every \( \delta > 0 \) we have
\[
\inf_{|x - \Theta| > \delta, x \in \langle A, B \rangle} |M(x) - x| = K_0(\delta) > 0.
\]

Assumption (III): We have \( M'(\Theta) > 0 \).

We derive a simple consequence of the assumptions. For a given \( \eta, 0 < \eta < 1 \), let \( \delta_0(\eta) \) be the supremum of all \( \delta \), such that
\[
|x_n - \Theta| \leq \delta \Rightarrow |M(x_n) - \lambda| \geq \eta M'(\Theta).
\]
The existence of such \( \delta \)'s follows from (II). From (I) it follows that
\[
|M(x_n) - \lambda| \geq K_0(\delta_0(\eta)) \geq \frac{K_0(\delta_0(\eta))}{B - A} |x_n - \Theta| \text{ for } |x_n - \Theta| > \delta_0(\eta),
\]
since \( |x_n - \Theta| \leq B - A \), by the definition of \( x_n \). Now set
\[
\varrho(\eta) = \min \left( \eta M'(\Theta), \frac{K_0(\delta_0(\eta))}{B - A} \right) \text{ and } K_1 = \sup_{0 < \eta < 1} \varrho(\eta).
\]
Evidently,
\[
|M(x_n) - \lambda| \geq K_1|x_n - \Theta| \text{ holds for all } n = 1, 2, \ldots. \tag{2}
\]
Similarly, from (III) and from the boundedness of \( M(x) \) in \( \langle A, B \rangle \) it follows that
\[
|M(x_n) - \lambda| \leq K_2|x_n - \Theta| \text{ for all } n = 1, 2, \ldots. \tag{3}
\]

We shall denote the second moment \( E[(x_n - \Theta)^2] \) as \( b_n \).

**Theorem I.** Suppose that the assumptions (I), (II) and (III) are satisfied, and that \( a > \frac{1}{2K_1} \). Then
\[
\hat{b}_n = O \left( \frac{1}{n} \right).
\]

Remark. The choice of \( a \) depends on the unknown constant \( K_1 \). We can avoid this fact by replacing the factor \( \frac{a}{n} \) in (1) through \( \frac{a'}{n} \log n \), where now \( a' \) is an arbitrary positive constant. Then — under the same assumptions —
\[
b_n = o \left( \frac{1}{n^{1-\varepsilon}} \right) \text{ for every } \varepsilon > 0, \text{ as could be easily shown.}
\]

Proof of the Theorem 1: From (1) it follows
\[
(x_n - \Theta)^2 = \begin{cases} (A - \Theta)^2 \text{ for } x_n - \Theta + \frac{a}{n} (x - y_n) < A - \Theta, \\ (B - \Theta)^2 \text{ for } x_n - \Theta + \frac{a}{n} (x - y_n) > B - \Theta, \\ (x_n - \Theta)^2 + \frac{a^2}{n^2} (y_n - \lambda)^2 - \frac{2a}{n} (x_n - \Theta)(y_n - \lambda) \text{ otherwise.} \end{cases}
\]
If we square the inequalities $x_n - \Theta + \frac{a}{n} (\xi - y_n) < A - \Theta$, or $B - \Theta$, respectively, and note that $A - \Theta$ is negative, $B - \Theta$ positive, we get

$$(x_{n+1} - \Theta)^2 \leq (x_n - \Theta)^2 + \frac{a^2}{n^2} (y_n - \xi)^2 - \frac{2a}{n} (x_n - \Theta)(y_n - \xi)$$

(4)

for all three possibilities.

Writing $y_n - \alpha = y_n - M(x_n) + M(x_n) - \alpha$, taking conditional expectations on both sides of (4), and using (I1), we get

$$E [(x_{n+1} - \Theta)^2| x_n] \leq (x_n - \Theta)^2 + \frac{a^2}{n^2} \{\sigma^2 + (M(x_n) - \xi)^2\} - \frac{2a}{n} (x_n - \Theta)(M(x_n) - \alpha) ;$$

hence by (2) and (3)

$$b_{n+1} \leq b_n + \frac{a^2}{n^2} (\sigma^2 + K^2 b_n) - \frac{2a}{n} K^1 b_n ,$$

i.e.,

$$b_{n+1} \leq \left(1 - \frac{2K^1 a + o(1)}{n}\right) b_n + \frac{\sigma^2 a^2}{n^2} .$$

Hence by Chung's lemma

$$b_n \leq \frac{\sigma^2 a^2}{2K^1 a - 1} \cdot \frac{1}{n} + O\left(\frac{1}{n^2} + \frac{1}{n^2 K^1 a}\right) .$$

In order to prove the asymptotic normality of $x_n$ we shall make further assumptions.

**Assumption (IV1):** For every even integer $p > 2$ there exists a constant $C_p$, such that

$$\int_{-\infty}^{\infty} (y - M(x))^p \, dH(y \mid x) \leq C_p \text{ for all } x \in \langle A, B \rangle .$$

**Assumption (V1):** The function

$$\sigma^2(x) = \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y \mid x)$$

is continuous and nonvanishing at $x = \Theta$.

**Theorem 2.** Suppose that the assumptions (I1), (II1), (III1), (IV1) and (V1) are satisfied, and that $a > \frac{1}{2K^1}$. Then the random variable $n^2 (x_n - \Theta)$ tends in distribution to the normal distribution with mean 0 and variance $\frac{\sigma^2(\Theta) a^2}{M'(\Theta) a - 1}$.

Proof is very similar to that of an analogous theorem in [4], and will be only sketched here, with some differences pointed out.
1° Under the additional assumption (IV), the asymptotic order of the higher absolute moments $\beta_n^p = E[|x_n - \Theta|^p] = O(n^{-2})$ will be deduced by induction with respect to even $r$; for odd $r$ it follows then by Lyapunov’s inequality. As a consequence we get by Chebyshev’s inequality,

$$\int_{|x_n - \Theta|^r > \delta} F_n(x) \, dP = O(n^{-q}) \quad (5)$$

for every Borel measurable function $F$ bounded in $\langle A, B \rangle$ and for every $\delta > 0,$ $q > 0$ (i.e. for $\delta$ arbitrarily small and $q$ arbitrarily large).

2° We observe that

$$P\left( x_n + \frac{a}{n} (x - y_n) > B \right) \leq P\left( x_n - \Theta > \frac{B - \Theta}{2} \right) \leq$$

$$\leq \frac{\beta_n^{2s}}{\left( \frac{B - \Theta}{2} \right)^{2s}} + \frac{E[|x - y_n|^q]}{\left( \frac{B - \Theta}{2} \right)^q n^q} = O(n^{-q}),$$

and, similarly, $P\left( x_n + \frac{a}{n} (x - y_n) < A \right) = O(n^{-q}),$ $q > 0$ arbitrary. Therefore

$$E[(x_{n+1} - \Theta)^r] = E\left[ (x_{n+1} - \Theta)^r \mid x_n + \frac{a}{n} (x - y_n) \epsilon \langle A, B \rangle \right] + O(n^{-q}) =$$

$$= E\left[ (x_n - \Theta + \frac{a}{n} (x - y_n))^r \right] x_n + \frac{a}{n} (x - y_n) \epsilon \langle A, B \rangle + O(n^{-q}) =$$

$$= E\left[ (x_n - \Theta + \frac{a}{n} (x - y_n))^r \right] + O(n^{-q}), \quad \text{for arbitrary } q > 0.$$

Denoting $b_n^{r} = E[(x_n - \Theta)^r],$ we get

$$b_n^{r+1} = b_n^{r} + \sum_{t=1}^{r} (-1)^t \frac{r!}{t!} \frac{a^t}{n^t} E[(x_n - \Theta)^{r-t} (y_n - x)^t] + O(n^{-q}).$$

Evaluating expectations on the right side, we can by (5) reduce the integration to the interval $|x_n - \Theta| \leq \delta,$ where by means of (III) and (V) more precise estimates are available; this enables us to prove (inductively) that

$$\lim_{n \to \infty} n^{2s} b_n^{s} = \left\{ \begin{array}{ll} 0 & \text{for } r = 2s - 1, \\
\left( \frac{\sigma^2(\Theta) a^2}{2M'(\Theta) a - 1} \right)^s & \text{for } r = 2s, \end{array} \right.$$ 

which implies the statement of the theorem.

2. The Kiefer-Wolfowitz stochastic approximation method

Let again $\{H(y \mid x)\}$ be a family of distribution functions and $M(x) = \int y \, dH(y \mid x)$ the corresponding regression function. Suppose that $M(x)$ achieves its maximum for a value $x = \Theta$ from a (known) finite interval $(A, B)$
and that $M(x)$ is increasing or decreasing according to $x < \Theta$ or $x > \Theta$ in a larger interval $\langle A - c', B + c' \rangle$.

Let $a > 0$, $0 < c \leq c'$, $0 < c' < \frac{1}{2}$ be constants; denote $\frac{a}{n} = a_n$, $\frac{c}{n^r} = c_n$. Take $x_1 \in \langle A, B \rangle$ arbitrarily and for $n \geq 1$ set recursively

$$x_{n+1} = \left[ x_n + a_n \frac{y_{2n} - y_{2n-1}}{c_n} \right]_a,$$

where $y_{2n}, y_{2n-1}$ are random variables, which for given $x_1, \ldots, x_n, y_1, \ldots, y_{2n-2}$ have distribution functions $H(y \mid x_n + c_n), H(y \mid x_n - c_n)$ respectively, and are independent.

We shall still add the following assumptions.

Assumption (I$_2$): There exists a constant $\sigma^2$, such that

$$\int_{-\infty}^{\infty} (y - M(x))^2 dH(y \mid x) \leq \sigma^2 \text{ for all } x \in \langle A - c', B + c' \rangle.$$

Assumption (II$_2$): There exist $K_3 > 0$, $K_4 > 0$ such that

$$K_3|x - \Theta| \leq |M'(x)| \leq K_4|x - \Theta| \text{ in some neighbourhood of } \Theta.$$

Assumption (III$_2$): There exists a $K_5 > 0$ and for every $\delta > 0$ a $K_6(\delta) > 0$, such that

$$|M'(x)| \leq K_5 \text{ for all } x \in \langle A - c', B + c' \rangle,$$

$$|M'(x)| \geq K_6(\delta) \text{ for all } |x - \Theta| > \delta, x \in \langle A - c', B + c' \rangle.$$  

Remark. The assumption (II$_2$) is certainly satisfied, if $M''(\Theta) < 0$ exists.

We deduce first some consequences of the assumptions. Denote $M_\varepsilon(x) = \frac{M(x + \varepsilon) - M(x - \varepsilon)}{2\varepsilon}$ for $x \in \langle A, B \rangle$, $0 < \varepsilon < c'$; we have

$$M_\varepsilon(x) = M'(x + \vartheta_1 \varepsilon) + M'(x - \vartheta_2 \varepsilon) \text{ with } 0 < \vartheta_i < 1, \ i = 1, 2. \quad (6)$$

Set $\varkappa(x) = -\frac{M'(x)}{x - \Theta}$ for $x \neq \Theta$, $\varkappa(\Theta) = K_3$; by (II$_2$) it holds $K_3 \leq \varkappa(x) \leq K_4$ in some neighbourhood of $\Theta$, say for $|x - \Theta| \leq \delta$.

Suppose that $\varepsilon < \frac{1}{3}\delta$. We have

$$M_\varepsilon(x) = [\varkappa(x + \vartheta_1 \varepsilon) + \varkappa(x - \vartheta_2 \varepsilon)](x - \Theta) + [\vartheta_2 \varkappa(x - \vartheta_2 \varepsilon) - \vartheta_1 \varkappa(x + \vartheta_1 \varepsilon)] \varepsilon,$$

hence

$$(x - \Theta) M_\varepsilon(x) \leq -2K_3(x - \Theta)^2 + K_4 \varepsilon |x - \Theta| \text{ for } |x - \Theta| \leq \delta - \varepsilon. \quad (7)$$

On the other hand, by (III$_2$) and (6), we have

$$|M_\varepsilon(x)| \leq 2K_6 \left( \frac{\delta}{3} \right) \text{ for } |x - \Theta| > \delta - \varepsilon, \quad (8)$$

$$M_\varepsilon^2(x) \leq 4K_5^2 \text{ for all } x \in \langle A, B \rangle. \quad (9)$$
Returning to the approximation scheme, we see that $|x_n - \Theta| < B - A$ for all $n$, and $c_n < \frac{1}{3} \delta$ for all $n > n_0(\delta)$; hence

$$|M_{c_n}(x_n)| > \frac{2K_6 \frac{\delta}{3}}{B - A} |x_n - \Theta| \text{ for } |x_n - \Theta| > \delta - c_n, \ n > n_0(\delta),$$

or, taking into account that $M(x)$ is increasing or decreasing as $x < \Theta$ or $x > \Theta$,

$$(x_n - \Theta) M_{c_n}(x_n) \leq - \frac{2K_6 \frac{\delta}{3}}{B - A} (x_n - \Theta)^2 \text{ for } |x_n - \Theta| > \delta - c_n, \ n > n_0(\delta).$$

Combining this with (7), we get

$$(x_n - \Theta) M_{c_n}(x_n) \leq - K_7 \cdot (x_n - \Theta)^2 + K_4 c_n |x_n - \Theta| \text{ for } n > n_0(\delta) \quad (10)$$

(without restriction on $x_n$).

**Theorem 3.** Suppose that the assumptions (I), (II) and (III) are satisfied, and that $a > \frac{1}{2K_7}$. Then

$$b_n = \begin{cases} O\left(\frac{1}{n^{1-2\gamma}}\right) & \text{for } \gamma \geq \frac{1}{4}, \\ O\left(\frac{1}{n^{2\gamma}}\right) & \text{for } \gamma < \frac{1}{4}. \end{cases}$$

Remark. These upper bounds for $b_n$ cannot be lowered in general; therefore the choice $\gamma = \frac{1}{4}$, giving $b_n = O\left(\frac{1}{n^{\frac{1}{2}}}\right)$, is the optimal one (under the assumptions made above).

In order to prove the statement in the remark, we use a family $\{H(y \mid x)\}$ with $M(x) = \begin{cases} - (x - \Theta)^2 & \text{for } x \leq \Theta \\ - \frac{1}{2}(x - \Theta)^2 & \text{for } x > \Theta \end{cases}$ and with $\sigma^2(x) = \sigma^2 > 0$. This special case leads — for every choice of $\gamma$ — to $b_n$ of exactly that order which is given as upper bound in Theorem 3. (Cf. [6]!)

**Proof of Theor. 3.** As in Sect. 1, it is easily seen that

$$(x_{n+1} - \Theta)^2 \leq (x_n - \Theta)^2 + a_n^2 \frac{(y_{2n} - y_{2n-1})^2}{c_n^2} + 2a_n(x_n - \Theta) \frac{y_{2n} - y_{2n-1}}{c_n},$$

hence

$$E[(x_{n+1} - \Theta)^2 \mid x_n] \leq (x_n - \Theta)^2 + 2\sigma^2 a_n^2 c_n^{-2} + a_n^2 M_{c_n}^2(x_n) + 2a_n(x_n - \Theta) M_{c_n}(x_n).$$

Taking once more expectations and using (9) and (10) we get

$$b_{n+1} \leq b_n + \frac{2\sigma^2 a_n c_n^{-2}}{n^{2-2\gamma}} + \frac{4K_6^2 a_n^2}{n^2} - \frac{2K_7 \sigma}{n} b_n + \frac{2K_4 a_c}{n^{1+\gamma}} E[|x_n - \Theta|].$$
By means of the inequality \( E[|x_n - \Theta|] \leq \varepsilon_n + \frac{1}{\varepsilon_n} b_n \) we obtain

\[
\begin{aligned}
\left( \text{setting } \varepsilon_n = \frac{2K_7 e}{eK_7 n^\gamma} \text{ with } 0 < \varepsilon < \frac{2}{a} \left( a - \frac{1}{2K_7} \right) \right) \\
\quad b_{n+1} \leq \left( 1 - \frac{(2 - \varepsilon) K_7 a}{n} \right) b_n + \frac{2\sigma^2 e^{-2} + o(1)}{n^{2-\gamma}} + \frac{4K_7^2 e^{-1} a c^2}{n^{1+2\gamma}}.
\end{aligned}
\]

The application of Chung's lemma completes the proof.

The proofs of the following three theorems will be omitted; they are entirely analogous to the proofs of corresponding theorems in [6].

**Assumption (IV\(_2\)):** The bounded third derivative \( M'''(x) \) exists in some neighbourhood of \( \Theta \).

**Theorem 4.** Suppose that the assumptions (I\(_2\)), (II\(_2\)), (III\(_2\)) and (IV\(_2\)) are satisfied, and that \( a > \frac{1}{2K_7} \). Then

\[
b_n = \begin{cases} 
O\left(\frac{1}{n^{1-2\gamma}}\right) & \text{for } \gamma \geq \frac{1}{6}, \\
O\left(\frac{1}{n^{1+\gamma}}\right) & \text{for } \gamma < \frac{1}{6}.
\end{cases}
\]

**Remark.** These bounds for \( b_n \) cannot be lowered without adding further restrictive assumptions; therefore the choice \( \gamma = \frac{1}{6} \), giving \( b_n = O\left(\frac{1}{n^3}\right) \), is the optimal one.

**Assumption (V\(_2\)):** The function \( M(x) \) is analytical and symmetrical about \( \Theta \) in some neighbourhood of \( \Theta \).

**Theorem 5.** Suppose that the assumptions (I\(_2\)), (II\(_2\)), (III\(_2\)) and (V\(_2\)) are satisfied, and that \( a > \frac{1}{2K_7} \). Then

\[
b_n = O\left(\frac{1}{n^{1-2\gamma}}\right) \text{ for all } 0 < \gamma < \frac{1}{2}.
\]

**Assumption (VI\(_2\)):** For every even integer \( p > 2 \) there exists a constant \( C_p \) such that

\[
\int_{-\infty}^\infty (y - M(x))^p \, dH(y \mid x) \leq C_p \text{ for all } x \in \langle A, B \rangle.
\]

**Assumption (VII\(_2\)):** The function

\[
\sigma^2(x) = \int_{-\infty}^\infty (y - M(x))^2 \, dH(y \mid x)
\]

is continuous and nonvanishing at \( x = \Theta \).
Theorem 6. Suppose that the assumptions (I2), (II2), (III2), (VI2) and (VII2) are satisfied, and that \( \alpha > -\frac{1}{2K} \). Then in each of the following three cases

1. \( \gamma \geq \frac{1}{2} \) and the continuous \( M''(x) < 0 \) exists in some neighbourhood of \( \Theta \),
2. \( \gamma > \frac{1}{2} \) and the assumption (IV2) is satisfied,
3. the assumption (V2) is satisfied,

the random variable \( n_{\frac{1}{2}-\gamma}(x_n - \Theta) \) tends in distribution to the normal distribution with mean 0 and variance \( \frac{\sigma^2(\Theta) \alpha^2}{(-2M''(\Theta) \alpha - \frac{1}{2} + \gamma)^2} \).

3. Solving systems of linear equations by a stochastic approximation method

In this section, the (column) vectors will be denoted by \( x, x_n, \ldots, \Theta \), and by \( \xi_i, \xi_{ni}, \ldots, \Theta_i \) (\( i = 1, \ldots, r \)) — their coordinates. The rows of a matrix \( M \), considered as row vectors, will be denoted by \( M_i \). The Euclidean norm \( ||x|| = (\sum_{i=1}^{r} \xi_i^2)^{\frac{1}{2}} \) and the corresponding norm of a matrix \( ||M|| = (\max_i (M'M)^{\frac{1}{2}}) \) will be used.

Let to each \( x \in E_r \) correspond a distribution function of \( r \) variables \( H(\eta \mid x) \). Suppose that the regression of \( \eta \) on \( x \) (i.e. the vector function, whose \( \eta \)-th coordinate is given by the integral \( \int_{-\infty}^{\infty} \eta \, dH_i(\eta \mid x) \), where \( H_i(\eta \mid x) = H(+\infty, \ldots, \eta, \ldots, +\infty) \), \( \eta \) on the \( i \)-th place) is of the type \( Mx \), where \( M \) is a matrix with constant elements. Suppose that \( M \) is nonsingular, so that, for given \( \alpha \), the system of linear equations \( Mx = \alpha \) has a unique solution \( \Theta \).

Let \( \alpha \) be a positive constant. Define the approximation procedure by taking \( x_1 \) arbitrarily, and for \( n \geq 1 \) setting recursively

\[
x_{n+1} = x_n - \frac{\alpha}{\sigma^2} y_n ,
\]

where \( y_n \) is a random vector whose distribution function, for given \( x_1, \ldots, x_n \), \( y_1^*, \ldots, y_n^* \), \( y_1, \ldots, y_n \), is \( H(y \mid y_n^* - \lambda) \), and \( y_n^* \) is a random vector whose distribution function, for given \( x_1, \ldots, x_n, y_1^*, \ldots, y_n^* \), \( y_1, \ldots, y_n \), is \( H(y \mid x_n) \).

Remark. The realization of this approximation scheme is the following: Given \( x_n \), we get first \( y_n^* \) as a result of an observation on the level \( x_n \); then we get \( y_n \) as result of an observation on the level \( y_n^* - \alpha \), and construct \( x_{n+1} \) according to (11).

We make further following two assumptions:
Assumption (I₃): There exists a constant $S₂$ such that
\[ \int_{E_r} \|y - Mx\|^2 \, dP_x \leq S₂ \quad \text{for all } x \in E_r, \]
where $P_x$ denotes the probability measure in $E_r$ induced by $H(y \mid x)$.

Assumption (II₃): The matrix $M$ is symmetrical.

Set $Kₙ = \min \lambda_i^2$, $Kₙ = 1 + \max \lambda_i^2 = 1 + \|M\|^2$, where $\lambda_i$ are the latent roots of the (symmetrical!) matrix $M$. Denote $b_n = E[\|x_n - \Theta\|^2]$.

**Theorem 7.** Suppose that the assumptions (I₃) and (II₃) are satisfied, and that $a > \frac{1}{2Kₙ}$. Then
\[ b_n = O \left( \frac{1}{n} \right), \text{ or, more precisely, } b_n \leq \frac{KₙS₂a^2}{2Kₙa - 1} \cdot \frac{1}{n} + O \left( \frac{1}{n^2} + \frac{1}{n^2Kₙa} \right). \]

**Proof.** First we observe that
\[ E[\eta_{ni} \mid y^*_n, x_n] = M_i(y^*_n - \alpha), \quad (12) \]
\[ E[\eta_{ni} \mid x_n] = E[M_i(y^*_n - \alpha) \mid x_n] = M_i M(x_n - \Theta). \quad (13) \]

Then we shall find an upper bound for $E[\|y_n\|^2 \mid x_n]$:
\[ \|y_n\|^2 = \sum_{i=1}^{r} \|\eta_{ni}\|^2 = \sum_{i=1}^{r} \{[M_i(y^*_n - \alpha)] + [M_i(y^*_n - \alpha) - M_i M(x_n - \Theta)] + + [M_i M(x_n - \Theta)] \|^2, \]

hence by (12)
\[ E[\|y_n\|^2 \mid y^*_n, x_n] = E[\|y_n - M(y^*_n - \alpha)\|^2 \mid y^*_n, x_n] + + \|M(y^*_n - Mx_n)\|^2 + \|M^2(x_n - \Theta)\|^2 + 2 \sum_{i=1}^{r} [M_i(y^*_n - Mx_n)] [M_i M(x_n - \Theta)]; \]
further, by the definition of $y^*_n$, by (I₃) and by (13),
\[ E[\|y_n\|^2 \mid x_n] \leq S^2 + E[\|M(y^*_n - Mx_n)\|^2 \mid x_n] + \|M^2(x_n - \Theta)\|^2 \leq \leq (1 + \|M\|^2) S^2 + \|M\|^4 \|x_n - \Theta\|^2. \quad (14) \]

Now, by (11) and in the next row by (13),
\[ \|x_{n+1} - \Theta\|^2 = \|x_n - \Theta\|^2 - \frac{2a}{n} (x_n - \Theta)' y_n + \frac{a^2}{n^2} \|y_n\|^2, \]
\[ E[\|x_{n+1} - \Theta\|^2 \mid x_n] = \|x_n - \Theta\|^2 - \frac{2a}{n} (x_n - \Theta)' M^2(x_n - \Theta) + \frac{a^2}{n^2} E[\|y_n\|^2 \mid x_n]. \quad (15) \]

Since $M$ is nonsingular and symmetrical, the matrix $M^2$ is positive definite, so that
\[ (x_n - \Theta)' M^2(x_n - \Theta) \geq Kₙ \|x_n - \Theta\|^2 \quad (16) \]
with $Kₙ$ equal to the smallest latent root of $M^2$. 

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Inserting (14) and (16) into (15), we obtain

$$E[||x_{n+1} - \Theta||^2 | x_n] \leq ||x_n - \Theta||^2 - \frac{2K_\alpha^a}{n} ||x_n - \Theta||^2 + \frac{\alpha^2}{n^2} [(1 + ||M||^2) S^2 + ||M||^4 ||x_n - \Theta||^2],$$

and finally,

$$b_{n+1} \leq \left(1 - \frac{2K_\alpha^a + o(1)}{n}\right) b_n + \frac{K_\alpha S^2 a^2}{n^2}.$$

Applying Chung’s lemma, we get the statement of the theorem.

REFERENCES

(AMS — Annals of Mathematical Statistics)

Резюме

ЗАМЕТКИ К СТОХАСТИЧЕСКИМ АППРОКСИМАЦИОННЫМ МЕТОДАМ

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В §§ 1-ом и 2-ом настоящей статьи показано, что эти условия можно значительно ослабить, если предположить, что искомое решение лежит в некотором заранее известном конечном промежутке.

В § 3-ем исследуется стохастический аппроксимационный метод для решения систем линейных уравнений с симметрической матрицей коэффициентов.