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ON APPROXIMATION OF CONTINUOUS FUNCTIONS IN THE METRIC $\int_a^b |x(t)|\,dt$

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The paper is devoted to the investigation of approximation of continuous functions by elements of a finite-dimensional space of continuous functions E, if the distance is measured by the norm $\int_a^b |x(t)|\,dt$. It may happen that the best approximation of a continuous function need not be unique. We investigate those spaces E which enjoy the following property: for every continuous function x, the best approximation of x by means of elements of E is always unique.

Let T be an interval of the real axis (i. e. a set of real numbers such that $t_1 \in T$, $t_2 \in T$ and $t_1 \leqq t \leqq t_2$ implies $t \in T$). We shall consider the linear space of all continuous real-valued functions defined on T and such that $\int_T |x(t)|\,dt$ exists. This space, equipped with the norm $\int_T |x(t)|\,dt$, will be denoted by B. Further, let a finite dimensional subspace E of B be given.

In the present paper we intend to examine those spaces E which enjoy the following property: for every $x_0 \in B$, the best approximation of x_0 by means of elements of E is unique.

The paper is divided into four parts. The first two sections contain some auxiliary results and remarks. In the fourth section we give a necessary and sufficient condition that the best approximation of any $x_0 \in B$ by means of elements of E be unique. In the third section we present a strengthened version of a sufficient condition due to D. Jackson [2].[1]) This, of course, could be made to follow from the general theorem of the fourth paragraph; since, however,

[1]) The work of D. Jackson treats the case where E is the set of all polynomials of degree $\leqq n$. Later, N. I. Achiezer [1] obtained the same result for any subspace E fulfilling Haar's condition. The theorem 1 of the present paper shows that a weakened form of Haar's condition is sufficient for the unicity of approximation.

there is a much simpler way of obtaining it directly we feel justified in including another paragraph devoted to this particular question. The author wishes to express his gratitude to the referee J. KRÁL for his valuable comments.

1. General remarks

In this section, let us consider a fixed normed linear space X and a fixed finite-dimensional subspace E of X.

Definition 1. Let $x_0 \in X$ be given. We say that the point $x_0 \in E$ is a best approximation of x_0 by means of elements of E if $|x_0 - e_0| \leqq |x_0 - e|$ for every $e \in E$.

(1,1) *To every $x_0 \in X$ there exists at least one best approximation of x_0 by means of elements of E.*

Proof. A simple argument based on the compactness of the unit sphere in a finite-dimensional linear space.

Definition 2. We say that a point $z \in X$ is perpendicular to E if $z \neq 0$ and $|z| \leqq |z - e|$ for every $e \in E$. If z is perpendicular to E, we shall write $z \perp E$.

(1,2) *Let $x_0 \in X$, $e_0 \in E$. The point e_0 is a best approximation of x_0 by means of elements of E if and only if $x_0 - e_0 \perp E$.*

Proof. Obvious.

(1,3) *Suppose that both e_1 and e_2 are best approximations of a point x_0. Then every point of the segment joining e_1 and e_2 is a best approximation of x_0 as well.*

Proof. Let $0 \leqq \lambda \leqq 1$ and let $e \in E$. We have $|x_0 - e_1| \leqq |x_0 - e|$ and $|x_0 - e_2| \leqq |x_0 - e|$. Hence $|x_0 - (\lambda e_1 + (1 - \lambda) e_2)| =$
$= |\lambda(x_0 - e_1) + (1 - \lambda)(x_0 - e_2)| \leqq \lambda|x_0 - e| + (1 - \lambda)|x_0 - e| = |x_0 - e|$
which proves our proposition.

Suppose now that there exists a point $v \in X$ which has two distinct best approximations e_1 and e_2. Let us put $x_0 = v - (\frac{1}{2}e_1 + \frac{1}{2}e_2)$, $e_0 = \frac{1}{2}e_1 - \frac{1}{2}e_2$. We have thus $x_0 \neq 0$, $e_0 \in E$, $e_0 \neq 0$. Clearly $x_0 + \sigma e_0 \perp E$ for every number σ fulfilling $|\sigma| \leqq 1$. We shall denote by V the set of all $x_0 \in X$ with the following property: there exists an $e_0 \in E$, $e_0 \neq 0$ such that $x_0 - \sigma e_0$ is perpendicular to E for every $\sigma \in \langle -1, +1 \rangle$. We have just seen that the set V is nonempty whenever there exists at least one point $v \in X$ with a non-unique best approximation.

2. Notation and auxiliary results

Let T be an interval of the real axis (i. e. a set of real numbers such that $t_1 \in T$, $t_2 \in T$ and $t_1 \leqq t \leqq t_2$ implies $t \in T$). We shall consider the linear space of all continuous real-valued functions defined on T and such that $\int_T |x(t)| \, dt$

exists. This space, equipped with the norm $\int_T |x(t)|\,\mathrm{d}t$, will be denoted by B. The end points of T will be denoted by a and b.

If x is a function continuous on T, we shall denote by $Z(x)$ the set of all $t \in T$ for which $x(t) = 0$. The complement of $Z(x)$ in T will be denoted by $D(x)$.

If ξ is a real number, we put sign $\xi = 1$ if $\xi > 0$ sign $\xi = -1$ if $\xi < 0$. If $\xi = 0$, the symbol sign ξ will not be defined. If M is a measurable subset of the real line, we denote by mes (M) the Lebesgue measure of M.

(2,1) *Let $x_0 \in B$, $x_0 \neq 0$. Let f be a linear functional of norm one on B which fulfills the relation $\langle x_0, f\rangle = |x_0|$. Then there exists a measurable function m defined on $Z(x_0)$ with the following properties:*

1^0 $|m(t)| \leqq 1$ *for every* $t \in Z(x_0)$,

2^0 $\langle x, f\rangle = \int_{D(x_0)} x(t) \operatorname{sign} x_0(t)\,\mathrm{d}t + \int_{Z(x_0)} x(t)\, m(t)\,\mathrm{d}t$ *for every* $x \in B$.

Proof. There exists a measurable function p defined on T, fulfilling the inequality $|p(t)| \leqq 1$ on T and such that $\langle x, f\rangle = \int_T x(t)\, p(t)\,\mathrm{d}t$ for every $x \in B$. Let us put $m(t) = p(t)$ for $t \in Z(x_0)$. Suppose now that 2^0 is not fulfilled for every $x \in B$. Then there exists a measurable subset $M \subset D(x_0)$ of positive measure such that $p(t) \neq \operatorname{sign} x_0(t)$ for $t \in M$. It follows that there exists a measurable set $A \subset M$ of positive measure and two positive numbers α and ε such that $x_0(t)$ has a constant sign on A, $|x_0(t)| \geqq \varepsilon$ on A, $|p(t) - \operatorname{sign} x_0(t)| \geqq \alpha$ on A. We have than

$$\int_{T-A} x_0(t)\, p(t)\,\mathrm{d}t \leqq \int_{T-A} |x_0(t)|\,\mathrm{d}t$$

and for $t \in A$

$$x_0(t)(\operatorname{sign} x_0(t) - p(t)) \geqq \varepsilon\alpha\,,$$

so that

$$\int_A x_0(t)\, p(t)\,\mathrm{d}t = \int_A |x_0(t)|\,\mathrm{d}t - \int_A x_0(t)\,(\operatorname{sign} x_0(t) - p(t))\,\mathrm{d}t \leqq$$
$$\leqq \int_A |x_0(t)|\,\mathrm{d}t - \varepsilon\alpha|A| < \int_A |x_0(t)|\,\mathrm{d}t\,.$$

From these two results it follows that

$$\int_T x_0(t)\, p(t)\,\mathrm{d}t < \int_T |x_0(t)|\,\mathrm{d}t\,,$$

which is a contradiction. The lemma is thus proved.

Let us consider now a fixed finite-dimensional subspace E of B. We shall investigate the best approximation of continuous functions by means of elements of E.

(2,2) *Suppose that a point $x_0 \in B$ has two distinct best approximations e_1, $e_2 \in E$. Then*

$$(x_0(t) - e_1(t))(x_0(t) - e_2(t)) \geqq 0$$

for every $t \in T$.

Proof. According to (1,3), every point of the segment joining e_1 and e_2 is a best approximation of x_0 as well. Let us consider now the points $z_1 = \frac{1}{2}(x_0 - e_1)$, $z_2 = \frac{1}{2}(x_0 - e_2)$. We have then

$$|z_1 + z_2| = |z_1| + |z_2|$$

which is impossible unless $z_1(t)\, z_2(t) \geqq 0$ for every $t \in T$.

(2,3) *Let $x_0 \in B$, $e_0 \in E$. Suppose that $x_0 \perp E$, $x_0 \pm e_0 \perp E$. Then $x_0(t) = 0$ implies $e_0(t) = 0$ for every $t \in \overline{D(x_0)}$.*

Proof. According to the preceding lemma, we have $x_0(t)(x_0(t) \pm e_0(t)) \geqq 0$ for every $t \in T$. It follows that $|x_0(t)\, e_0(t)| \leqq (x_0(t))^2$ for every $t \in T$. If $t \in D(x_0)$, we have $|e_0(t)| \leqq |x_0(t)|$. This inequality may be extended, by continuity, to all points $t \in \overline{D(x_0)}$. Now if $x_0(t) = 0$ and t belongs to $\overline{D(x_0)}$, we have $e_0(t) = 0$ by the above inequality. The lemma is established.

3. A necessary condition

Theorem 1. *Let E be an n-dimensional subspace of B and suppose that there exists a $b \in B$ the best approximation of which is not unique. Then there exists a nonzero point $e_0 \in E$ and n distinct inner points t_i of T such that $e_0(t_i) = 0$.*

Proof. Let us denote by V the set of all $x_0 \in X$ with the following property: There exists an $e_0 \in E$, $e_0 \neq 0$ such that $x_0 - \sigma e_0 \perp E$ for every $\sigma \in \langle -1, +1 \rangle$. According to our assumption, the set V is not empty.

We shall distinguish three cases:

1° For every $x \in V$, the set $Z(x)$ contains a nondegenerate interval. Let us take a point $x_0 \in V$. There exists a nonzero $e_0 \in E$ such that we have $x_0 - \sigma e_0 \perp E$ for every $\sigma \in \langle -1 +1 \rangle$. It is easy to see that $x_0 - \sigma e_0 \in V$ for every $\sigma \in (-1, +1)$. It follows that, for every $\lambda \in (-1, +1)$, there exists a nondegenerate interval $S(\lambda) \subset Z(x_0 - \lambda e_0)$. There exist, consequently, two numbers λ_1 and λ_2, different from each other, such that the intersection $S(\lambda_1) \cap S(\lambda_2)$ is a nondegenerate interval. Clearly we have then $e_0(t) = 0$ for every $t \in S(\lambda_1) \cap S(\lambda_2)$.

2° There exist points $x \in V$ such that $Z(x)$ is nondense in T. Suppose first that there exists an $x \in V$ with nondense $Z(x_0)$ such that $Z(x_0)$ contains at least n inner points of T. There exists a nonzero $e_0 \in E$ such that $x_0 \pm e_0 \perp E$ as well. According to lemma (2,3), we have then $Z(e_0) \supset Z(x_0)$ and the assertion of our theorem is proved.

3^0 Every $x \in V$ with nondense $Z(x)$ has at most $n - 1$ inner zero points. Let us choose an $x_0 \in V$ with at most $n - 1$ inner zero points. If we put $t_0 = a$, $t_n = b$, we may find points $t_0 < t_1 < \ldots < t_{n-1} < t_n$ such that $x_0(t) \neq 0$ for every t different from the t_i. Since $x_0 \perp E$ there exists a linear functional f on B such that $|f| = 1$, $\langle E, f \rangle = 0$ and $\langle x_0, f \rangle = |x_0|$. It follows from lemma (2,1) that f may be expressed in the form $f = \sum_{i=1}^{n} \varepsilon_i f_i$ where f_i are defined by $\langle x, f_i \rangle = \int_{t_{i-1}}^{t_i} x(t)\, dt$. If $e_1, \ldots, e_n$ is a basis of E, we may form the matrix $\langle e_i, f_j \rangle$. Since f vanishes on E, the columns of this matrix are linearly dependent, so that the rows are linearly dependent as well. It follows that there exists a nonzero $e \in E$ such that $\langle e, f_i \rangle = 0$ for every i. Since $\int_{t_{i-1}}^{t_i} e(t)\, dt = 0$, there exist points s_i such that $t_{i-1} < s_i < t_i$ and that $e(s_i) = 0$. This concludes the proof.

4. Sufficient and necessary conditions

Theorem 2. *Let E be a finite-dimensional subspace of B. There exists a $b \in B$ with a nonunique best approximation if and only if there exist two disjoint sets G_1 and G_2 open in T and a measurable function m defined on $T - (G_1 \cup G_2)$ with the following properties:*

1^0 $|m(t)| \leqq 1$ *for every* $t \in T - (G_1 \cup G_2)$,

2^0 $\int_{G_1} e(t)\, dt - \int_{G_2} e(t)\, dt + \int_{T-(G_1 \cup G_2)} e(t) m(t)\, dt = 0$ *for every* $e \in E$,

3^0 *there exists a nonzero* $e_0 \in E$ *vanishing on* $T - (G_1 \cup G_2)$.

Proof. Let $x_0 \in V$. There exists a linear functional f on B such that $|f| = 1$, $\langle E, f \rangle = 0$ and $\langle x_0, f \rangle = |x_0|$. Let us denote by G_1 the set of all $t \in T$ where $x_0(t) > 0$, by G_2 the set of those $t \in T$ where $x_0(t) < 0$. It follows from lemma (2,1) that there exists a measurable function m defined on $Z(x_0)$ such that $|m(t)| \leqq 1$ for $t \in Z(x_0)$ and that

$$\langle x, f \rangle = \int_{G_1} x(t)\, dt - \int_{G_2} x(t)\, dt + \int_{Z(x_0)} x(t)\, m(t)\, dt$$

for every $x \in B$. Conditions 1^0 and 2^0 are thus fulfilled for G_1, G_2 and m.

Now there exists a nonzero $e_1 \in E$ such that $x_0 - \sigma e_1 \perp E$ for every $\sigma \in \langle -1, +1 \rangle$. For $\varepsilon = \pm 1$, let us denote by $M(\varepsilon)$ the set of all numbers $\sigma \in (0, 1)$ for which mes $(D(x_0) \cap Z(x_0 + \varepsilon\sigma e_1)) > 0$. Suppose that $M(\varepsilon)$ is uncountable. Then there exist two distinct numbers $\sigma_1, \sigma_2 \in (0, 1)$ such that the set

$$M = D(x_0) \cap Z(x_0 + \varepsilon\sigma_1 e_1) \cap Z(x_0 + \varepsilon\sigma_2 e_1)$$

has positive measure. It follows that $e_1(t) = 0$ for $t \in M$. Hence $x_0(t) = 0$ for $t \in M$ which is a contradiction with $M \subset D(x_0)$. Hence both $M(+1)$ and

$M(-1)$ are countable so that there exists a $\sigma \in (0, 1)$ which does not belong to either of them. Put $e_0 = \sigma e_1$. We have thus

$$(x_0(t) - e_0(t))(x_0(t) + e_0(t)) \neq 0$$

almost everywhere on $D(x_0)$. Since $x_0 \pm e_0 \perp E$ we have, according to (2,3) $|e_0(t)| \leqq |x_0(t)|$ for every $t \in \overline{D(x_0)}$. Hence sign $(x_0(t) \pm e_0(t)) = \text{sign}\, x_0(t)$ almost everywhere on $D(x_0)$.

Hence

$$|x_0 \pm e_0| = \int_{D(x_0)} (x_0(t) \pm e(t)) \operatorname{sign} x_0(t)\, dt + \int_{Z(x_0)} |e_0(t)|\, dt =$$
$$= |x_0| \pm \int_{D(x_0)} e(t) \operatorname{sign} x_0(t)\, dt + \int_{Z(x_0)} |e_0(t)|\, dt\,.$$

Since $|x_0 \pm e_0| = |x_0|$, we have $\int_{Z(x_0)} |e_0(t)|\, dt = 0$.

Let us denote by H the interior of the set $Z(x_0)$. It follows from $\int_{Z(x_0)} |e_0(t)|\, dt = 0$ that $e_0(t) = 0$ for every $t \in H$. We know already that $|e_0(t)| \leqq |x_0(t)|$ for every $t \in \overline{D(x_0)}$. Since $Z(x_0) = (Z(x_0) \cap \overline{D(x_0)}) \cup H$, we have $e_0(t) = 0$ for every $t \in Z(x_0)$. The conditions of our theorem are thus fulfilled.

On the other hand, suppose that we have two disjoint sets G_1 and G_2 open in T, a measurable function m defined on $T - (G_1 \cup G_2)$ and a nonzero $e_0 \in E$ fulfilling the conditions of our theorem.

Let us define a function x_0 on T by the relations

$$\begin{aligned} x_0(t) &= \ \ 2|e_0(t)| \quad \text{for } t \in G_1\,,\\ &= -2|e_0(t)| \quad \text{for } t \in G_2\,,\\ &= \ \ 0 \quad\quad\quad \text{for } t \in T - (G_1 \cup G_2)\,. \end{aligned}$$

It follows that x_0 is continuous on T and different from zero. We have now $x_0(t) - e_0(t) \geqq 0$ for $t \in G_1$, $x_0(t) - e_0(t) \leqq 0$ for $t \in G_2$ and $x_0(t) = e_0(t) = 0$ for $t \in T - (G_1 \cup G_2)$.

Let us denote by f the linear functional on B defined by the relation

$$\langle x, f\rangle = \int_{G_1} x(t)\, dt - \int_{G_2} x(t)\, dt + \int_{T-(G_1 \cup G_2)} x(t)\, m(t)\, dt\,;$$

We have clearly $\langle x_0, f\rangle = |x_0|$, $|f| = 1$ and $\langle E, f\rangle = 0$. At the same time

$$|x_0 - e_0| = \int_{G_1} (x_0(t) - e_0(t))\, dt - \int_{G_2} (x_0(t) - e_0(t))\, dt =$$
$$= \langle x_0 - e_0, f\rangle = \langle x_0, f\rangle = |x_0|\,, \quad \text{and, for any } e \in E\,,$$

$|x_0 - e| \geqq \langle x_0 - e, f\rangle = |x_0|$, so that both 0 and e_0 are best approximations of x_0. The proof is complete.

In the rest of this section we shall subject the subspaces E considered to the following condition:

(F) Let $e \in E$, $e \neq 0$. Then there exists a finite set $K \subset T$ such that $e(t) \neq 0$ for every $t \in T - K$.

Under this restriction, a simple sufficient and necessary condition for the unicity of the best approximation may be given.

Let p be a natural number. A sequence $t_0 < t_1 < \ldots < t_p$ of points will be called a subdivision of T if $t_0 = a$ and $t_p = b$. Let D be a subdivision of T. We shall denote by $f(D)$ the linear functional on B defined by the relation

$$\langle x, f(D) \rangle = \sum_{i=1}^{p} (-1)^i \int_{t_{i-1}}^{t_i} x(t) \, dt \, .$$

We may state now the following corollary of theorem 2.

Theorem 3. *Let E be an n-dimensional subspace of B with property (F). There exists a $b \in B$ with a nonunique best approximation if and only if there exists a subdivision D of T and a nonzero $e_0 \in E$ with the following properties:*

1^0 $\langle E, f(D) \rangle = 0$,

2^0 $e_0(t) = 0$ *in the inner points of* D.

Proof. Suppose first that there exists a $b \in B$ with a nonunique best approximation. According to the preceding theorem, there exist disjoint sets G_1 and G_2 open in T and a nonzero $e_0 \in E$ with properties 1^0, 2^0 and 3^0.

The function e_0 vanishes on $T - (G_1 \cup G_2)$. $T - (G_1 \cup G_2)$ is finite. It follows that $\bar{G}_1 \cap \bar{G}_2 \subset T - (G_1 \cup G_2)$ is finite as well. Let D be the subdivision of T consisting of the points a, b and the points of $\bar{G}_1 \cap \bar{G}_2$. Condition 2^0 of theorem 2 reduces then to $\langle E, f(D) \rangle = 0$. The other implication being obvious, the proof is complete.

Added in proofs: For another proof of theorem 1 see the article of М. Крейн, L-проблема в абстрактном линейном нормированном пространстве, in the book on the Moment Problem by N. Achiezer and M. Krein, Moscow, 1948.

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Резюме

ОБ АППРОКСИМАЦИИ В НОРМЕ $\int_a^b |x(t)| \, dt$

Властимил Птак (Vlastimil Pták), Прага

(Поступило в редакцию 22/XI 1957 г.)

В работе исследуется приближение непрерывных функций при помощи элементов некоторого конечномерного пространства E, если расстояние измеряется нормой $\int_a^b |x(t)| \, dt$. Исследуются те пространства E, для которых наилучшая аппроксимация каждой непрерывной функции при помощи элементов пространства E всегда однозначна. Помимо прочего в работе дается усиление одного результата Н. И. Ахиезера. Метод исследования является геометрическим.