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GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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The existence and continuous dependence on a parameter theorems are proved for a special class of generalized differential equations defined in [1]; their solutions have bounded variation. Results on the continuous dependence on a parameter are then applied to classical differential equations with a disturbing term which approximates the Dirac function.

Introduction

Generalized differential equations were defined in [1], § 2 (of this paper we use § 1 and sections 1 and 2 of § 2). First we prove an existence theorem (th. 2.1) for another class of generalized differential equations, viz.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{D}F(x, t) \tag{0,1}$$

where $F(x, t)$ satisfies

$$\|F(x, t_2) - F(x, t_1)\| \leqq |h(t_2) - h(t_1)|\,, \tag{0,2}$$

$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \omega(\|x_2 - x_1\|)\, |h(t_2) - h(t_1)| \tag{0,3}$$

with $h(t)$ increasing, continuous from the left, and $\omega(\eta)$ increasing, continuous and $\omega(0) = 0$.

The solution whose existence is demonstrated has bounded variation and is continuous from the left. Theorem 2,1 differs from customary existence theorems for differential equations in that the existence domain for a solution $x(\tau)$ satisfying given initial conditions $x(t_0) = x_0$ is an interval $t_0 \leqq \tau < t_0 + \zeta$ ($\zeta > 0$) — it need not exist for $\tau < t_0$. This is a consequence of assuming $h(t)$ to be merely continuous from the left: if $h(t)$ is continuous, we obtain this solution $x(\tau)$ on a neighbourhood of t_0.

The study of these generalized equations was suggested by the following problem: Take a sequence of classical differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + \varphi_k(t)\,, \quad t \in \langle -T, T\rangle \tag{0,4}$$

with

$$\varphi_k(t) \geqq 0\,, \lim_{k\to\infty} \int_{-T}^{t} \varphi_k(\tau)\, d\tau = 0 \text{ or } 1 \text{ as } -T \leqq t < 0 \text{ or } 0 < t \leqq T \text{ resp.} \tag{0,5}$$

Let $x_k(t)$ be a solution of (0,4) with $x_k(-T) = 1$. Then obviously $x_k(t) \to x(t)$ for $t \neq 0$, where

$$x(t) = \begin{cases} e^{t+T} & \text{for} \quad -T \leqq t \leqq 0\,, \\ (e^T + 1)\, e^t & \text{for} \quad 0 < t \leqq T\,. \end{cases}$$

The problem is to examine such situations in a more general setting; i. e., to define a class of generalized differential equations such that the function $x(t)$ would appear as a solution of some such equation, and that $x_k(t) \to x(t)$ would be a consequence of a theorem on the continuous dependence on a parameter.

If $f(x, t)$ satisfies the Carathéodory conditions, viz.

$$\|f(x, t)\| \leqq m(t) \tag{0,6}$$

with $m(t)$ integrable and

$$f(x, t) \text{ with fixed } t \text{ is a continuous function in } x\,, \tag{0,7}$$

then every (absolutely continuous) solution of

$$\frac{dx}{dt} = f(x, t) \tag{0,8}$$

is a solution of (0,1) and conversely, provided that

$$F(x, t) = \int_0^t f(x, \tau)\, d\tau\,. \tag{0,9}$$

The proof is quite similar to that for the case of continuous $f(x, t)$, cf. [1], 2,2.

Our theorem 2,1 contains the Carathéodory existence theorem as a special case; if we put $h(t) = \int_0^t m(\tau)\, d\tau$, then (0,6), (0,7), and (0,9) imply (0,2), but (0,3) need not be fulfilled for any $\omega(\eta)$. Nevertheless it can be proved, that there always exist such functions $\omega(\eta)$, $h(t)$ ($h(t)$ different from $\int_0^t m(\tau) d\tau$), that (0,2) and (0,3) are fulfilled (using the $F(x, t)$ of (0,9)), provided that $f(x,t)$ is defined and fulfils (0,6) and (0,7) on $Kx \langle -T,T \rangle$, where K is compact.

In § 3 we give a unicity theorem for the case $\omega(\eta) = K\eta$; the proof is an analogue of the classical case, using the notion of distance of two solutions.

In § 4, Theorem 4,2 treats a general case of the continuous dependence on a parameter. Our assumptions become rather complex; the main source of difficulties is that

i) the classical equation (0,4) is equivalent to the generalized equation

$$\frac{dx}{dt} = D[xt + \int_{-T}^{t} \varphi_k(\tau)\, d\tau]\,; \tag{0,10}$$

however, taking $F_k(x, t) = xt + \int_{-T}^{t} \varphi_k(\tau)\,\mathrm{d}\tau$ for $k = 1, 2, \ldots$, there obviously exists no function $h(t)$ to satisfy (0,2) with $F(x, t) = F_k(x, t)$, $k = 1, 2, \ldots$;

ii) the convergence effect in equation (0,4) apparently depends on the behaviour of the right hand sides of these equations only in a (suitably defined) neighbourhood of the discontinuous limit solution $x(t)$.

In § 5, Theorem 4,2 is applied to the convergence effect of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t) + g(x)\,\varphi_k(t) \tag{0,11}$$

with $f(x, t)$, $g(x)$ continuous, $\varphi_k(t)$ satisfying (0,5) (or more general conditions) and assuming some unicity conditions in (0.11).

Similar questions are treated in the author's paper [2]; the results of §§ 1—2 of this paper are contained in [2], but the results on the continuous dependence on a parameter are more general and complete here.

1. The Existence of $\int_{\sigma_1}^{\sigma_2} \mathrm{D}F(x, (\tau), t)$

We make the following assumptions: T is a positive number; $h_1(t)$, $h_2(t)$ are functions on the closed interval $\langle -T, T\rangle$, and are increasing, bounded and continuous from the left; $\omega(\eta)$ is continuous and increasing for $\eta \geqq 0$, $\omega(0) = 0$. G is an open set in the product space $E_n \times \langle -T, T\rangle$ (E_n is the Euclidean n-space).

$$\mathbf{F} = \mathbf{F}(G, h_1(t), \omega(\eta))$$

denotes the set of all functions $F(x, t)$ on G with values in E_n such that

$$\|F(x, t_2) - F(x, t_1)\| \leqq |h_1(t_2) - h_1(t_1)| \tag{1,1}$$

for all (x, t_1), $(x, t_2) \in G$, and

$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq |h_1(t_2) - h_1(t_1)| \,.\, \omega(\|x_2 - x_1\|) \tag{1,2}$$

for all (x_1, t_1), (x_1, t_2), (x_2, t_1), $(x_2, t_2) \in G$.

Let $x(\tau)$ be a function defined for $\tau \in \langle \sigma_1, \sigma_2\rangle \subset \langle -T, T\rangle$ with values in E_n and such that $(x(\tau), \tau) \in G$ if $\tau \in \langle \sigma_1, \sigma_2\rangle$ and

$$\|x(\tau_2) - x(\tau_1)\| \leqq |h_2(\tau_2) - h_2(\tau_1)|$$

if $\tau_1, \tau_2 \in \langle \sigma_1, \sigma_2\rangle$.

For $\xi > 0$ let $N = N(\xi, \sigma_1, \sigma_2, h_2, \omega)$ be the set of all $t \in \langle \sigma_1, \sigma_2)$ with $\omega(h_2(t+) - h_2(t)) \geqq \xi$. Here, of course, $h_2(t+)$ means $\lim_{\tau \to t+} h_2(\tau)$.

Finally, let $\tilde{A} = \tilde{A}(\xi, \sigma_1, \sigma_2, h_2, \omega)$ denote the set of all the decompositions $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ of the interval $\langle\sigma_1, \sigma_2\rangle$ with the following properties:

$$\sigma_1 = \alpha_0 \leqq \tau_1 \leqq \alpha_1 \leqq \ldots \leqq \tau_s \leqq \alpha_s = \sigma_2 \,, \quad \alpha_0 < \alpha_1 < \ldots < \alpha_s \,,$$

if τ_j is not an element of N, then $\omega(h_2(\alpha_j) - h_2(\alpha_{j-1})) < \xi$, if τ_j is an element of N, then $\omega(h_2(\tau_j) - h_2(\alpha_{j-1})) < \xi$, $\omega(h_2(\alpha_j) - h_2(\tau_j+)) < \xi$ and $\alpha_j > \tau_j$.

Lemma 1,1. *$\tilde{A}$ is non-empty.*

Proof. If τ non $\in N$, then choose a $\delta(\tau) > 0$ such that

$$\omega(h_2(\tau + \delta(\tau)) - h_2(\tau - \delta(\tau))) < \xi$$

(if $\sigma_1 < \tau < \sigma_2$; $\omega(h_2(\sigma_2) - h_2(\sigma_2 - \delta(\sigma_2)) < \xi$ or $\omega(h_2(\sigma_1 + \delta(\sigma_1)) - h_2(\sigma_1)) < \xi$ in the remaining cases), and

$$\langle\tau - \delta(\tau), \tau + \delta(\tau)\rangle \cap N = \varnothing \,.$$

If $\tau \in N$, then choose a $\delta(\tau) > 0$ such that

$$\omega(h_2(\tau + \delta(\tau)) - h_2(\tau+)) < \xi \,, \quad \omega(h_2(\tau) - h_2(\tau - \delta(\tau))) < \xi \,,$$

$$\langle\tau - \delta(\tau), \tau) \cap N = \varnothing = N \cap (\tau, \tau + \delta(\tau)\rangle$$

for $\tau > \sigma_1$). Let S be the set of pairs (τ, t) with

$$\sigma_1 \leqq \tau \leqq \sigma_2 \,, \quad \sigma_1 \leqq t \leqq \sigma_2 \,, \quad \tau - \delta(\tau) \leqq t \leqq \tau + \delta(\tau) \,.$$

Using Lemma 1,1,1 in [1], there exists a decomposition $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ subordinate to S, which is easily shown to be an element of $\tilde{A}$.

Theorem 1,1. *The integral* $\int_{\sigma_1}^{\sigma_2} \mathrm{D}F(x(\tau), t)$ *exists. If* $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\} \in \tilde{A}$, *then*

$$\|\int_{\sigma_1}^{\sigma_2} \mathrm{D}F(x(\tau), t) - \sum_{i=1}^{s} \Delta_i\| \leqq \xi n[h_1(\sigma_2) + h_2(\sigma_2) - h_1(\sigma_1) - h_2(\sigma_1)]$$

with $\Delta_i = F(x(\tau_i), \alpha_i) - F(x(\tau_i), \alpha_{i-1})$ *for* τ_i non $\in N$ *and*

$$\Delta_i = F(x(\tau_i+), \alpha_i) - F(x(\tau_i+), \tau_i+) + F(x(\tau_i), \tau_i+) - F(x(\tau_i), \alpha_{i-1})$$

for $\tau_i \in N$.

Proof. Let $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\} \in \tilde{A}$ and let $[F(x, t)]_j$, $[\Delta_i]_j$ be the j-th coordinates of the vectors $F(x, t)$, Δ_i ($j = 1, 2, \ldots, n$). Define functions $M_j(\tau)$, $m_j(\tau)$ thus: $M_j(\sigma_1) = 0$; if $\alpha_{k-1} < \tau \leqq \alpha_k$, τ_k non $\in N$ or $\alpha_{k-1} < \tau \leqq \tau_k$, $\tau_k \in N$, then

$$M_j(\tau) = \sum_{i=1}^{k-1} [\Delta_i]_j + [F(x(\tau_k), \tau) - F(x(\tau_k), \alpha_{k-1})]_j +$$
$$+ \xi \,.\, (h_1(\tau) + h_2(\tau) - h_1(\sigma_1) - h_2(\sigma_1)) \,;$$

if $\tau_k < \tau \leqq \alpha_k$, $\tau_k \in N$, then

$$M_j(\tau) = \sum_{i=1}^{k-1} [\Delta_i]_j + [F(x(\tau_k), \tau_k+) - F(x(\tau_k), \alpha_{k-1})]_j +$$
$$+ [F(x(\tau_k+), \tau) - F(x(\tau_k+), \tau_k+)]_j + \xi \, . \, (h_1(\tau) + h_2(\tau) - h_1(\sigma_1) - h_2(\sigma_1)) \, ,$$
$$k = 1, 2, \ldots, s \, ,$$
$$m_j(\tau) = M_j(\tau) - 2\xi(h_1(\tau) + h_2(\tau) - h_1(\sigma_1) - h_2(\sigma_1)) \, .$$

We will show that $M_j(\tau)$ is a major function to $[F(x(\tau), t)]_j$.

Let $t_0 \in \langle \sigma_1, \sigma_2 \rangle$, t_0 non $\in N$, $t_0 < \sigma_2$, and let k be the index with $\alpha_{k-1} \leqq t_0 < \alpha_k$. Then for $t \geqq t_0$ sufficiently near t_0 (more precisely, for $t_0 \leqq t < \alpha_k$ if τ_k non $\in N$ or $\tau_k \in N$, $\tau_k < t_0$, and for $t_0 \leqq t < \tau_k$ if $\tau_k \in N$, $\tau_k > t_0$), we have

$$M_j(t) - M_j(t_0) = \xi \, . \, (h_1(t) + h_2(t) - h_1(t_0) - h_2(t_0)) + [F(z, t) - F(z, t_0)]_j , \quad (1,3)$$

where $z = x(\tau_k)$ if τ_k non $\in N$ or $\tau_k \in N$, $t_0 \leqq \tau_k$, and $z = x(\tau_k+)$ if $\tau_k \in N$, $\tau_k < t_0$.

Since $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\} \in \tilde{A}$, in every case

$$\omega(\|z - x(t_0)\|) < \xi \, , \quad (1,4)$$

$$|[F(z, t) - F(z, t_0) - F(x(t_0), t) + F(x(t_0), t_0)]_j| \leqq$$
$$\leqq (h_1(t) - h_1(t_0)) \, . \, \omega(\|z - x(t_0)\|) \leqq \xi(h_1(t) - h_1(t_0)) \, . \quad (1,5)$$

Then (1,3) and (1,5) imply

$$M_j(t) - M_j(t_0) - [F(x(t_0), t) - F(x(t_0), t_0)]_j \geqq 0 \quad (1,6)$$

(with $t_0 \in \langle \sigma_1, \sigma_2 \rangle$, t_0 non $\in N$, and t sufficiently near t_0).

Let $t_0 \in (\sigma_1, \sigma_2\rangle$, and let k be the index with $\alpha_{k-1} < t_0 \leqq \alpha_k$. Then for $t < t_0$ sufficiently near t_0 (more precisely, for $\alpha_{k-1} < t < t_0$ if τ_k non $\in N$ or $\tau_k \in N$, $t_0 \leqq \tau_k$, and for $\tau_k < t < t_0$ if $\tau_k \in N$, $\tau_k < t_0$) we have (1,3), (1,4) and (1,5) again with the same z, so that

$$M_j(t) - M_j(t_0) - [F(x(t_0), t) - F(x(t_0,) \, t_0)]_j \leqq 0 \, . \quad (1,7)$$

The remaining case is $t_0 = \tau_k \in N$, $t > t_0$. If $t < \alpha_k$, then

$$M_j(t) - M_j(t_0) = \xi(h_1(t) + h_2(t) - h_1(t_0) - h_2(t_0)) +$$
$$+ [F(x(\tau_k), \tau_k+) - F(x(\tau_k), \tau_k) + F(x(\tau_k+), t) - F(x(\tau_k+), \tau_k+)]_j \, ,$$
$$h_2(t) - h_2(t_0) \geqq \lambda \, , \quad \text{where } \omega(\lambda) = \xi,$$

$$\left. \begin{aligned} & M_j(t) - M_j(t_0) - [F(x(t_0), t) - F(x(t_0), t_0)]_j \geqq \xi\lambda - \\ & - |[F(x(\tau_k+), t) - F(x(\tau_k+), \tau_k+) - F(x(t_0), t) + F(x(t_0), t_0+)]_j| \geqq 0 \end{aligned} \right\} \quad (1,8)$$

for t sufficiently near t_0, since

$$|[F(x(\tau_k+), t) - F(x(\tau_k+), \tau_k+) - F(x(t_0), t) + F(x(t_0), t_0)]_j| \leqq$$
$$\leqq \omega(\|x(\tau_k+) - x(t_0)\|) \, . \, (h_1(t) - h_1(t_0+)) \, .$$

From (1,6), (1,8) and (1,7) it follows that $M_j(\tau)$ is a major function to $[F(x(\tau), t)]_j$ for $j = 1, 2, \ldots, n$; similarly, $m_j(\tau)$ is a minor function. Since ξ has been arbitrary, the integral $\int_{\sigma_1}^{\sigma_2} \mathrm{D}F(x(\tau), t)$ exists; finally, the inequality in the statement of our theorem is a simple consequence of the fact that $M_j(\tau)$ and $m_j(\tau)$ are respectively major and minor functions to $[F(x, t)]_j$, $j = 1, 2, \ldots, n$.

2. The Existence Theorem for $\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{D}F(x, t)$

Lemma 2,1. *Let $F(x, t)$ be defined on G and satisfy*

$$\|F(x, t_2) - F(x, t_1)\| \leqq |h(t_2) - h(t_1)|$$

for (x, t_1), $(x, t_2) \in G$. Let $y(\tau)$ be a function on $\langle\tau_3, \tau_4\rangle$ with $(y(\tau), \tau) \in G$ when $\tau \in \langle\tau_3, \tau_4\rangle$; let $\int_{\tau_3}^{\tau_4} \mathrm{D}F(y(\tau), t)$ exist. Then

$$\|\int_{\tau_3}^{\tau_4} \mathrm{D}F(y(\tau), t)\| \leqq h(\tau_4) - h(\tau_3)\,.$$

Proof. Using Theorem 1,2,1 of [1], to every $\xi > 0$ there is a decomposition $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ of the interval $\langle\tau_3, \tau_4\rangle$ such that

$$\|\int_{\tau_3}^{\tau_4} \mathrm{D}F(y(\tau), t) - \sum_{i=1}^{s} [F(y(\tau_i), \alpha_i) - F(y(\tau_i), \alpha_{i-1})]\| < \xi\,,$$

so that

$$\|\int_{\tau_3}^{\tau_4} \mathrm{D}F(y(\tau), t)\| \leqq \sum_{i=1}^{s} \|F(y(\tau_i), \alpha_i) - F(y(\tau_i), \alpha_{i-1})\| + \xi \leqq h(\tau_4) - h(\tau_3) + \xi\,;$$

since ξ is arbitrary, this implies our lemma.

If $x(\tau)$ is a function on $\langle\tau_1, \tau_2\rangle$, which is a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{2,1}$$

in the sense of definition 2,1,1 in [1], then lemma 2,1 implies

$$\|x(\tau_4) - x(\tau_3)\| \leqq |h(\tau_4) - h(\tau_3)|$$

for all $\tau_3, \tau_4 \in \langle\tau_1, \tau_2\rangle$. Thus $x(\tau)$ has finite variation and is continuous from the left. Theorem 1,3,6 of [1] yields

$$x(\tau+) - x(\tau) = F(x(\tau), \tau+) - F(x(\tau), \tau) \tag{2,2}$$

for every $\tau \in \langle\tau_1, \tau_2)$.

For the existence theorem, we assume that G_F is the set of all $(x, t) \in G$ with $(x + F(x, t+) - F(x, t), t) \in G$. Choose a point $(x_0, t_0) \in G_F$. Also choose a $\sigma > 0$ such that $(x, \tau) \in G_F$ whenever $t_0 \leqq \tau \leqq t_0 + \sigma$, $\|x - x_1\| \leqq h(\tau) - h(t_0+)$, $x_1 = x_0 + F(x_0, t_0+) - F(x_0, t_0)$ (such σ exists).

Theorem 2,1. *Let $F(x, t) \in \mathbf{F}(G, h, \omega)$. Then there exists a solution $x(\tau)$ of the differential equation* (2,1), *defined on $\langle \tau_0, \tau_0 + \sigma \rangle$ and such that $x(t_0) = x_0$.*

Note 2,1. If $(x(t_0 + \sigma), t_0 + \sigma)$ also belongs to G_F, we can apply Theorem 2,1 again, thus continuing the solution to the right; in complete analogy with classical theorems, to every solution $x(\tau)$ of equation (2,1) defined on $\langle t_0, t_1 \rangle$ there exists a solution $y(\tau)$ defined on $\langle t_0, t_2) \supset \langle t_0, t_1)$ such that they coincide on the smaller interval and that the point $(y(t_2-), t_2)$ does not belong to G_F.

As the following example shows, a completely different situation arises on continuing the solution to the left.

Let $n = 1$ (so that x is a real number) and set

$$F(x, t) = x \text{ for } |x| < 1,\ -1 < t \leqq 0, \quad F(x, t) = 0 \text{ for } |x| < 1,\ 0 < t < 1 .$$

Obviously

$$F(x, t) \in \mathbf{F}(G, h(t), \eta)$$

with G the set of points $(x, t) : |x| < 1$, $|t| < 1$, and $h(t) = 0$ for $-1 \leqq t \leqq 0$, $h(t) = 1$ for $0 < t \leqq 1$. Obviously $G_F = G$. The solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{*}$$

that passes through $(x_0, t_0) \in G$ with $t_0 \leqq 0$ is easily shown to be $x(\tau)$,

$$x(\tau) = x_0 \text{ if } -1 < \tau \leqq 0 , \quad x(\tau) = 0 \text{ if } 0 < \tau < 1 .$$

The solution $z(\tau)$ of (*) passing through $(x_0, t_0) \in G$ with $t_0 > 0$, $x_0 \neq 0$ is

$$z(\tau) = x_0 \quad \text{for} \quad 0 < \tau < 1 ,$$

and cannot be continued to $\tau \leqq 0$.

Note 2,2. If the function $h(t)$ is continuous, theorem 2,1 implies the existence of a solution $x(\tau)$ of (2,1) passing through a given point, $x(t_0) = x_0$, and defined on an interval $\langle t_0 - \sigma_1, t_0 + \sigma \rangle$ with $\sigma_1 > 0$.

Proof of Theorem 2,1. For $i = 1, 2, 3, \ldots$ and $\eta \in \langle t_0, t_0 + \sigma \rangle$ define functions $x_i(\eta)$ thus:

$$x_i(\eta) = x_0 + \int_{t_0}^{\eta} \mathrm{D}F(x_0, t) \quad \text{for} \quad t_0 \leqq \eta \leqq t_0 + \frac{\sigma}{i} ,$$

$$x_i(\eta) = x_i\left(t_0 + \frac{j-1}{i}\sigma\right) + \int_{t_0 + \frac{j-1}{i}\sigma}^{\eta} \mathrm{D}F\left(x_i\left(\tau - \frac{\sigma}{i}\right), t\right)$$

for

$$t_0 + \frac{j-1}{i}\sigma < \eta \leqq t_0 + \frac{j}{i}\sigma , \quad j = 2, 3, \ldots, i .$$

Lemma 2,1 and $\int\limits_{t_0}^{\eta} \mathrm{D}F(x_0, t) = F(x_0, \eta) - F(x_0, t_0)$ imply that

$$\left.\begin{array}{c} \|x_i(\tau_2) - x_i(\tau_1)\| \leqq |h(\tau_2) - h(\tau_1)| , \\ \|x_i(\tau_3) - x_0 - F(x_0, t_0+) + F(x_0, t_0)\| \leqq h(\tau_3) - h(\tau_0+) , \\ \tau_1, \tau_2 \in \langle t_0, t_0 + \sigma\rangle , \quad \tau_3 \in (t_0, t_0 + \sigma\rangle , \quad i = 1, 2, 3, \ldots . \end{array}\right\} \qquad (2,3)$$

The functions $x_i(\tau)$ obviously satisfy

$$x_i(\tau_5) - x_i(\tau_4) = \int\limits_{\tau_4}^{\tau_5} \mathrm{D}F\left(x_i\left(\tau - \frac{\sigma}{i}\right), t\right), \quad t_0 \leqq \tau_4 \leqq \tau_5 \leqq t_0 + \sigma \qquad (2,4)$$

$\left(x_i(\tau) = x_0 \text{ for } t_0 - \frac{\sigma}{i} \leqq \tau \leqq t_0\right)$. It is a matter of routine to show that the sequence $x_i(\tau)$ contains a uniformly convergent subsequence $x_{i_j}(\tau) = y_j(\tau)$; set

$$x(\tau) = \lim_{j \to \infty} y_j(\tau) \quad \text{for} \quad \tau \in \langle t_0, t_0 + \sigma\rangle .$$

From (2,3) we have $(x(\tau), \tau) \in G$ for $\tau \in \langle t_0, t_0 + \sigma\rangle$. We shall prove that

$$\int\limits_{\tau_4}^{\tau_5} \mathrm{D}F\left(y_j\left(\tau - \frac{\sigma}{i_j}\right), t\right) \to \int\limits_{\tau_4}^{\tau_5} \mathrm{D}F(x, (\tau), t) \qquad (2,6)$$

for $j \to \infty$, $t_0 < \tau_4 \leqq \tau_5 \leqq t_0 + \sigma$.

Choose a $\xi > 0$ and

$$\{\alpha_0, \tau_1', \alpha_1, \ldots, \tau_s', \alpha_s\} \in \tilde{A}(\xi, \tau_4, \tau_5, h, \omega) , \quad \tau_k' < \alpha_k , \quad k = 1, 2, \ldots, s .$$

From Theorem 1,1 we have

$$\left\| \int\limits_{\tau_4}^{\tau_5} \mathrm{D}F(x(\tau), t) - \sum_{k=1}^{s} \Delta_k \right\| \leqq \xi\, 2n[h(\tau_5) - h(\tau_4)] \qquad (2,7)$$

where $\Delta_k = F(x(\tau_k'), \alpha_k) - F(x(\tau_k'), \alpha_{k-1})$ if τ_k' non $\in N(\xi, \tau_4, \tau_5, h, \omega)$ and

$$\Delta_k = F(x(\tau_k'), \tau_k'+) - F(x(\tau_k'), \alpha_{k-1}) + F(x(\tau_k'+), \alpha_k) - F(x(\tau_k'+), \tau_k'+)$$

if $\tau_k' \in N(\xi, \tau_4, \tau_5, h, \omega)$. For sufficiently large j

$$\left\{\alpha_0, \tau_1' + \frac{\sigma}{i_j}, \alpha_1, \ldots, \tau_s' + \frac{\sigma}{i_j}, \alpha_s\right\} \in \tilde{A}\left(\xi, \tau_4, \tau_5, h\left(t - \frac{\sigma}{i_j}\right), \omega\right),$$

and by theorem 1,1

$$\left\| \int\limits_{\tau_4}^{\tau_5} \mathrm{D}F\left(y_j\left(\tau - \frac{\sigma}{i_j}\right), t\right) - \sum_{k=1}^{s} \Delta_k^* \right\| \leqq \xi\, 2n[h(\tau_5) - h(\tau_4)] \qquad (2,8)$$

with $\Delta_k^* = F(y_j(\tau_k'), \alpha_k) - F(y_j(\tau_k'), \alpha_{k-1})$ if τ_k' non $\in N(\xi, \tau_4, \tau_5, h, \omega)$ and

$$\Delta_k^* = F\left(y_j(\tau_k'), \left(\tau_k' + \frac{\sigma}{i_j}\right)+\right) - F(y_j(\tau_k'), \alpha_{k-1}) +$$
$$+ F(y_j(\tau_k'+), \alpha_k) - F\left(y_j(\tau_k'+), \left(\tau_k' + \frac{\sigma}{i_j}\right)+\right)$$

if $\tau_k' \in N(\xi, \tau_4, \tau_5, h, \omega)$. Since

$$\left\|\sum_{k=1}^{s}(\Delta_k - \Delta_k^*)\right\| \leqq \sup_{t_0 \leqq \tau \leqq t_0 + \sigma} \|x(\tau) - y_j(\tau)\| \cdot (h(t_0 + \sigma) - h(t_0)) + \xi$$

holds for sufficiently large j, (2,7) and (2,8) imply

$$\left\| \int_{\tau_4}^{\tau_5} \mathrm{D}F(x(\tau), t) - \int_{\tau_4}^{\tau_5} \mathrm{D}F\left(y_j\left(\tau - \frac{\sigma}{i_j}\right), t\right) \right\| \leqq$$
$$\leqq [\sup_{t_0 \leqq \tau \leqq t_0 + \sigma} \|x(\tau) - y_j(\tau)\| + \xi\, 4n] \,.\, [h(t_0 + \sigma) - h(t_0)] + \xi$$

for large j; thus we obtain (2,6). From (2,4), (2,5) and (2,6)

$$x(\tau_5) - x(\tau_4) = \int_{\tau_4}^{\tau_5} \mathrm{D}F(x(\tau), t)\,, \quad x(t_0) = x_0$$

and according to Definition 2,1,1 of [1] $x(\tau)$ is the solution of equation (2,1).

3. The Distance of Two Solutions When a Lipschitz Condition Is Fulfilled

Taking $\omega(\eta) = K\eta$ with K a positive constant, we give an estimate whose special case is the so called fundamental inequality for classical equations $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)$ (here $f(x, t)$ is continuous in (x, t), satisfies a Lipschitz condition for the variable x; cf. [3], ch. I, § 2, p. 21 or [4], ch. IV, § 17, Theorem 5).

Since our aim is an estimate, it seems worth while to have finer assumptions. ($h_j(t)$, G retain their meaning, K is a positive constant.)

Let $F(x, t)$, $F_1(x, t)$, $F_2(x, t)$ be functions on G with values in E_n, such that

$$\|F(x, t_2) - F(x, t_1)\| \leqq |h_1(t_2) - h_1(t_1)| \quad \text{for} \quad (x, t_2), (x, t_1) \in G\,,$$

$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq K\|x_2 - x_1\| \,.\, |h_2(t_2) - h_2(t_1)| \quad \mathbf{(3,1)}$$

for (x_1, t_1), (x_1, t_2), (x_2, t_1), $(x_2, t_2) \in G$ and

$\|F_1(x, t_2) - F_1(x, t_1)\| \leqq |h_3(t_2) - h_3(t_1)|$, $\|F_2(x, t_2) - F_2(x, t_1)\| \leqq |h_3(t_2) - h_3(t_1)|$, whenever (x, t_1), $(x, t_2) \in G$. Finally, let $x(\tau)$ be a function on $\langle t_1, t_2 \rangle$ which satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[F(x, t) + F_1(x, t)]\,, \quad \mathbf{(3,2)}$$

and $y(\tau)$ a function on $\langle t_1, t_2 \rangle$ which satisfies

$$\frac{dy}{d\tau} = D[F(y, t) + F_2(y, t)]. \tag{3,3}$$

We shall prove that the following inequality holds:

$$\|x(\xi) - y(\xi)\| \leqq \|x(t_1) - y(t_1)\| \,.\, \exp\{K[h_2(\xi) - h_2(t_1)]\} + \\ + 2\int_{t_1}^{\xi} \exp\{K[h_2(\xi) - h_2(\tau)]\}\, dh_3(\tau)\,. \tag{3,4}$$

Note 3,1. If $v(t)$ is a real-valued function, we write $\int_{\sigma_1}^{\sigma_2} u(\tau)\, dv(\tau)$ instead of $\int_{\sigma_1}^{\sigma_2} Du(\tau)\, v(t)$ (see also Note 1,2,1 in [1]).

Note 3,2. Let us examine (3,4) in the classical case. If $x(t)$, $y(t)$ are solutions of

$$\frac{dx}{dt} = f(x, t) + f_1(x, t)\,, \tag{3,2'}$$

$$\frac{dy}{dt} = f(y, t) + f_2(y, t)\,, \tag{3,3'}$$

respectively on $\langle t_1, t_2 \rangle \subset (0, T)$, with f, f_1, f_2 continuous on the region

$$G = \underset{x,t}{E}[\|x\| < c,\ 0 < t < T]\,,$$

and such that

$$\|f_1(x, t)\| < \varepsilon, \quad \|f_2(x, t)\| < \varepsilon, \quad \|f(x_1, t) - f(x_2, t)\| \leqq L\|x_1 - x_2\|,$$

put, as usual,

$$F(x, t) = \int_{\frac{T}{2}}^{t} f(x, \tau)\, d\tau, \quad F_1(x, t) = \int_{\frac{T}{2}}^{t} f_1(x, \tau)\, d\tau, \quad F_2(x, t) = \int_{\frac{T}{2}}^{t} f_2(x, \tau)\, d\tau\,.$$

Then 3,4 can be applied with $Kh_2(t) = Lt$, $h_3(t) = \varepsilon t$, giving

$$\left.\begin{aligned} \|x(\xi) - y(\xi)\| \leqq \|x(t_1) - y(t_1)\| \exp\{L(\xi - t_1)\} + 2\varepsilon \int_{t_1}^{\xi} \exp\{L(\xi - \eta)\}\, d\eta = \\ = \|x(t_1) - y(t_1)\| \exp\{L(\xi - t_1)\} + \frac{2\varepsilon}{L}[\exp\{L(\xi - t_1)\} - 1] \leqq \\ \leqq \|x(t_1) - y(t_1)\| \exp\{L(\xi - t_1)\} + 2\varepsilon(\xi - t_1) \exp L(\xi - t_1)\,. \end{aligned}\right\} \tag{3,4'}$$

Here $\|x\|$ can mean any norm in E_n; e. g. taking $\|x\| = \max_{i=1,2,\ldots,n} |x_i|$, (3,4') yields the inequality mentioned above.

Inequality (3,4) will follow from a number of lemmas which we now proceed to prove.

Lemma 3,1. *Let* $U(\tau, t)$ *be a function with values in* E_n *and such that* $\int_{\sigma_1}^{\sigma_2} \mathrm{D}U(\tau, t)$ *exists* $(\sigma_1 < \sigma_2)$. *Also let* $V(\tau, t)$ *be a real-valued function such that* $\int_{\sigma_1}^{\sigma_2} \mathrm{D}V(\tau, t)$ *exists and that to every* $\tau \in \langle\sigma_1, \sigma_2\rangle$ *there is a* $\delta(\tau) > 0$ *with*

$$|\tau_1 - \tau| \,.\, \|U(\tau, \tau_1) - U(\tau, \tau)\| \leqq (\tau_1 - \tau) \,.\, (V(\tau, \tau_1) - V(\tau, \tau)) \qquad (3,5)$$

for $\tau_1 \in \langle\sigma_1, \sigma_2\rangle \cap \langle\tau - \delta(\tau), \tau + \delta(\tau)\rangle$. *Then* $\|\int_{\sigma_1}^{\sigma_2} \mathrm{D}U(\tau, t)\| \leqq \int_{\sigma_1}^{\sigma_2} \mathrm{D}V(\tau, t)$.

Proof. From Theorem 1,2,1 and lemma 1,1,1 of [1] it follows that to any $\varepsilon > 0$ there is a decomposition $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ of the interval $\langle\sigma_1, \sigma_2\rangle$ such that $\alpha_i - \tau_i < \delta(\tau_i)$, $\tau_i - \alpha_{i-1} < \delta(\tau_i)$ for $i = 1, 2, \ldots, s$,

$$\|\int_{\sigma_1}^{\sigma_2} \mathrm{D}U(\tau, t) - \sum_{i=1}^{s} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})]\| < \varepsilon\,, \qquad (3,6)$$

$$|\int_{\sigma_1}^{\sigma_2} \mathrm{D}V(\tau, t) - \sum_{i=1}^{s} [V(\tau_i, \alpha_i) - V(\tau_i, \alpha_{i-1})]| < \varepsilon\,. \qquad (3,7)$$

Since ε is arbitrary, (3,5), (3,6) and (3,7) imply our lemma.

Lemma 3,2. *If* $v(t)$ *is real-valued non-decreasing, if* $u_1(t)$, $u_2(t)$ *are real-valued,* $u_1(t) \leqq u_2(t)$, *and if the integrals* $\int_{\sigma_1}^{\sigma_2} u_1(\tau)\,\mathrm{d}v(\tau)$, $\int_{\sigma_1}^{\sigma_2} u_2(\tau)\,\mathrm{d}v(\tau)$ *exist* $(\sigma_1 < \sigma_2)$, *then*

$$\int_{\sigma_1}^{\sigma_2} u_1(\tau)\,\mathrm{d}v(\tau) \leqq \int_{\sigma_1}^{\sigma_2} u_2(\tau)\,\mathrm{d}v(\tau)\,.$$

The proof is similar to that of Lemma 3,1.

Lemma 3,3. *If the function* $h(t)$ *is real-valued, non-decreasing, non-negative and continuous from the left in* $\langle\sigma_1, \sigma_2\rangle$, *then*

$$\int_{\sigma_1}^{\sigma_2} h^k(\tau)\,\mathrm{d}h(\tau) \leqq \frac{1}{k+1}\,[h^{k+1}(\sigma_2) - h^{k+1}(\sigma_1)]$$

for $k = 0, 1, 2, \ldots$.

Proof. For every $\varepsilon > 0$, $\frac{1}{k+1}\,h^{k+1}(\tau) + \varepsilon h(\tau)$ is a major function to $h^k(\tau)\,h(t)$ (see Definition 1,1,1 in [1]), since our assumptions imply

$$(t - \tau)\left[\frac{1}{k+1}\,h^{k+1}(t) - \frac{1}{k+1}\,h^{k+1}(\tau) + \varepsilon h(t) - \varepsilon h(\tau)\right] =$$

$$= (t - \tau)(h(t) - h(\tau))\,\frac{1}{k+1}\,[\sum_{i=0}^{k} h^{k-i}(t)\,h^i(\tau) + \varepsilon] \geqq (t - \tau)(h(t) - h(\tau))\,h^k(\tau)$$

for t sufficiently near τ; this implies our lemma.

Lemma 3,4. *If the functions $h_2(t)$, $h_3(t)$ are real-valued, non-decreasing and continuous from the left, then*

$$\int_{t_1}^{\xi} \dots \int_{t_1}^{\tau_1} \mathrm{d}h_3(\tau_0)\, \mathrm{d}h_2(\tau_1) \dots \mathrm{d}h_2(\tau_l) \leqq \int_{t_1}^{\xi} \frac{1}{l!} [h_2(\xi) - h_2(\tau)]^l \, \mathrm{d}h_3(\tau)$$

$$\text{for} \quad t_1 < \xi, l = 0, 1, 2, \dots .$$

Proof. The existence of the integrals of lemmas 3,4 and 3,3 follows from theorem 1,1 and Lemma 2,1. Now, Lemma 3,4 holds if $l = 0$; assume it holds for $l = k$, an integer. Then, using Lemma 3,2

$$\int_{t_1}^{\xi} \int_{t_1}^{\tau_{k+1}} \dots \int_{t_1}^{\tau_1} \mathrm{d}h_3(\tau_0)\, \mathrm{d}h_2(\tau_1) \dots \mathrm{d}h_2(\tau_k)\, \mathrm{d}h_2(\tau_{k+1}) \leqq$$

$$\leqq \int_{t_1}^{\xi} \frac{1}{k!} \int_{t_1}^{\tau_{k+1}} [h_2(\tau_{k+1}) - h_2(\tau)]^k \, \mathrm{d}h_3(\tau)\, \mathrm{d}h_2(\tau_{k+1}) .$$

Thus we must show that

$$(k + 1) \int_{t_1}^{\xi} \mathrm{D} \int_{t_1}^{\tau} [h_2(\tau) - h_2(\sigma)]^k \, \mathrm{d}h_3(\sigma)\, h_2(t) \leqq \int_{t_1}^{\xi} [h_2(\xi) - h_2(\tau)]^{k+1} \, \mathrm{d}h_3(\tau) . \quad (3,8)$$

For this inequality, it is sufficient to prove that for every $\varepsilon > 0$ the function

$$M(\tau) = \int_{t_1}^{\tau} [h_2(\tau) - h_2(\sigma)]^{k+1} \, \mathrm{d}h_3(\sigma) + \varepsilon[h_2(\tau) + h_3(\tau)]$$

is a major function to

$$U(\tau, t) = (k + 1) \int_{t_1}^{\tau} [h_2(\tau) - h_2(\sigma)]^k \, \mathrm{d}h_3(\sigma)\, h_2(t) .$$

Obviously

$$(t - \tau)(U(\tau, t) - U(\tau, \tau)) = (t - \tau)(h_2(t) - h_2(\tau))(k + 1) \int_{t_1}^{\tau} [h_2(\tau) - h_2(\sigma)]^k \, \mathrm{d}h_3(\sigma) \quad (3,9)$$

and

$$(t - \tau)[M(t) - M(\tau)] = \varepsilon(t - \tau)[h_2(t) + h_3(t) - h_2(\tau) - h_3(\tau)] +$$

$$+ (t - \tau)\Big[\int_{t_1}^{t} [h_2(t) - h_2(\sigma)]^{k+1} \, \mathrm{d}h_3(\sigma) - \int_{t_1}^{\tau} [h_2(\tau) - h_2(\sigma)]^{k+1} \, \mathrm{d}h_3(\sigma)\Big] =$$

$$= \varepsilon(t - \tau)[h_2(t) + h_3(t) - h_2(\tau) - h_3(\tau)] + (t - \tau) \int_{\tau}^{t} [h_2(t) - h_2(\sigma)]^{k+1} \, \mathrm{d}h_3(\sigma) +$$

$$+ (t - \tau)(h_2(t) - h_2(\tau)) \int_{t_1}^{\tau} \Big[\sum_{i=0}^{k} (h_2(t) - h_2(\sigma))^{k-i} (h_2(\tau) - h_2(\sigma))^i\Big] \mathrm{d}h_3(\sigma) . \quad (3,10)$$

If $t < \tau$ is sufficiently near τ,

$$\Big|\int_{\tau}^{t} (h_2(t) - h_2(\sigma))^{k+1} \, \mathrm{d}h_3(\sigma)\Big| \leqq \varepsilon |h_3(t) - h_3(\tau)|$$

and

$$|(k+1)\int_{t_1}^{\tau}(h_2(\tau)-h_2(\sigma))^{k+1}\,\mathrm{d}h_3(\sigma)-$$
$$-\int_{t_1}^{\tau}[\sum_{i=0}^{k}(h_2(t)-h_2(\sigma)^{k-i}(h_2(\tau)-h_2(\sigma))i]\,\mathrm{d}h_3(\sigma)|\leqq\varepsilon|h_2(t)-h_2(\tau)|\,.$$

This together with (3,9) and (3,10) implies that $M(\tau)$ is in fact a major function to $U(\tau, t)$; thus proving Lemma 3,4.

Now we prove the inequality 3,4. According to Lemma 2,1 or 3,1 the solutions $x(\tau)$, $y(\tau)$ are bounded, so that

$$\|x(\xi)-y(\xi)\|\leqq c \quad \text{for} \quad \xi\in\langle t_1, t_2\rangle\,.$$

Also,

$$\left.\begin{aligned}x(\xi)-y(\xi)=x(t_1)-y(t_1)+\\+\int_{t_1}^{\xi}\mathrm{D}[F(x(\tau),t)-F(y(\tau),t)]+\int_{t_1}^{\xi}\mathrm{D}F_1(x(\tau),t)-\int_{t_1}^{\xi}\mathrm{D}F_2(y(\tau),t)\,.\end{aligned}\right\}\quad(3,11)$$

Since

$$\|F(x(\tau),t_2)-F(y(\tau),t_2)-F(x(\tau),t_1)+F(y(\tau),t_1)\|\leqq Kc|h_2(t_2)-h_2(t_1)|\,,$$

Lemma 3,1 and (3,11) give

$$\|x(\xi)-y(\xi)\|\leqq\|x(t_1)-y(t_1)\|+Kc\int_{t_1}^{\xi}\mathrm{d}h_2(\tau_0)+2\int_{t_1}^{\xi}\mathrm{d}h_3(\tau_0)$$

for $\xi\in\langle t_1, t_2\rangle$. Now, assume

$$\left.\begin{aligned}\|x(\xi)-y(\xi)\|\leqq\|x(t_1)-y(t_1)\|[1+\sum_{i=1}^{l}K^i\int_{t_1}^{\xi}\dots\int_{t_1}^{\tau_2}\mathrm{d}h_2(\tau_1)\dots\mathrm{d}h_2(\tau_i)]+\\+K^{l+1}c\int_{t_1}^{\xi}\dots\int_{t_1}^{\tau_1}\mathrm{d}h_2(\tau_0)\,\mathrm{d}h_2(\tau_1)\dots\mathrm{d}h_2(\tau_l)+\\+2\sum_{i=0}^{l}K^i\int_{t_1}^{\xi}\dots\int_{t_1}^{\tau_1}\mathrm{d}h_3(\tau_0)\,\mathrm{d}h_2(\tau_1)\dots\mathrm{d}h_2(\tau_i)\end{aligned}\right\}\quad(3,12)$$

holds for $l=k$, an integer. Using (3,11), Lemma 3,1, (3,12) and (3,1), we obtain that (3,12) also holds for $l=k+1$. Since it does hold when $l=0$, we have (3,12) for all l by induction.

Applying Lemmas 3,3 and 3,4 to (3,12),

$$\left.\begin{aligned}\|x(\xi)-y(\xi)\|\leqq\|x(t_1)-y(t_1)\|\left[1+\sum_{i=1}^{l}\frac{1}{i!}K^i(h_2(\xi)-h_2(t_1))^i\right]+\\+\frac{1}{(l+1)!}K^{l+1}c(h_2(\xi)-h_2(t_1))^{l+1}+2\int_{t_1}^{\xi}\left[\sum_{i=0}^{l}\frac{1}{i!}K^i(h_2(\xi)-h_2(\tau))^i\right]\mathrm{d}h_3(\tau)\,.\end{aligned}\right\}\quad(3,13)$$

To deduce inequality (3,4), it suffices to apply Lemma 3,2 and take $l\to\infty$.

Some consequences of (3,4) will be formulated as theorems.

Theorem 3,1. *If* $F(x, t) \in F(G, h(t), K\eta)$ *and* $x(\tau), y(\tau)$ *are solutions of*

$$\frac{dx}{d\tau} = DF(x, t) \tag{3,14}$$

on the interval $\langle t_1, t_2 \rangle$, *then*

$$\|x(\tau) - y(\tau)\| \leqq \|x(t_1) - y(t_1)\| \exp \{K[h(\tau) - h(t_1)]\}$$

for every $\tau \in \langle t_1, t_2 \rangle$. *Thus, if* $x(t_1) = y(t_1)$, *then* $x(\tau) = y(\tau)$ *for all* $\tau \in \langle t_1, t_2 \rangle$.

Note 3,3. Thus Theorem 3,1 yields a unicity theorem for the equation (3,14), but only for increasing τ. There is no corresponding unicity property for τ decreasing, as has been shown in the example of note 2,1.

Theorem 3,2. *Let* $F(x, t) \in \mathbf{F}(G, h(t), K\eta)$; *let the functions* $h_k(t)$ $(k = 1, 2, 3, \ldots)$ *be increasing and continuous from the left on* $\langle 0, T \rangle$, $h_k(T) - h_k(0) \to 0$ *with* $k \to \infty$. *Let the functions* $F_k(x, t)$ *be defined on* G *and satisfy*

$$\|F_k(x, t_2) - F_k(x, t_1)\| \leqq |h_k(t_2) - h_k(t_1)|$$

whenever (x, t_1), $(x, t_2) \in G$, $k = 1, 2, 3, \ldots$. *Let* $x(\tau)$ *be a solution of* $\dfrac{dx}{d\tau} = DF(x, t)$ *on* $\langle t_1, t_2 \rangle$, *and* $x_k(t)$ *be solutions of* $\dfrac{dx}{d\tau} = D[F(x, t) + F_k(x, t)]$ *on the same interval* $\langle t_1, t_2 \rangle$, $\lim_{k \to \infty} x_k(t_1) = x(t_1)$.

Then $x_k(\tau) \to x(\tau)$ *with* $k \to \infty$ *uniformly for* $\tau \in \langle t_1, t_2 \rangle$.

4. The Continuous Dependence on a Parameter

In this paragraph we prove a general theorem on the continuous dependence on a parameter of solutions of our equations (without supposing $\omega(\eta) = K\eta$). First, a definition.

Definition 4,1. *Let* $F_k(x, t)$ $(k = 0, 1, 2, \ldots)$ *be a sequence of functions defined on* G *and with values in* E_n; *let* $R \subset G$.

We will say that the sequence $F_k(x, t)$ *converges R-emphatically to* $F_0(x, t)$ *with* $k \to \infty$, *if the following conditions are fulfilled:*

there exist functions $\omega(\eta)$, $h_k(t)$ *(with properties as described in* § 1);

$$F_k(x, t) \in \mathbf{F}(G, h_k, \omega) \quad \text{for} \quad k = 0, 1, 2, \ldots;$$

$$\limsup_{k \to \infty} [h_k(t_2) - h_k(t_1)] \leqq h_0(t_2) - h_0(t_1), \text{ if } h_0(t) \text{ is continuous at } t_1 \text{ and } t_2,\ t_1 < t_2; \tag{4,01}$$

$$F_k(x_0, t_0) \to F_0^*(x_0, t_0) = F_0(x_0, t_0) + F_0''(x_0, t_0)$$

if $(x_0, t_0) \in G$ *and* $h_0(t)$ *is continuous at* t_0, *and* $F_0''(x, t)$ *satisfies*

$$\|F_0''(x, t_2) - F_0''(x, t_1)\| \leqq |h_0''(t_2) - h_0''(t_1)|$$

with $h_0''(t)$ *the saltus-function of* $h_0(t)$ *and* (x, t_1), $(x, t_2) \in G$; $R \subset G_{F_0}$;

(Ω) *if* $(x_0, t_0) \in R$, $h_0(t_0+) > h_0(t_0)$, *then to every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that to each* δ', $0 < \delta' < \delta$ *there corresponds a* k_0 *with the following property: If* $y(\tau)$ *is a solution of* $\frac{\mathrm{d}y}{\mathrm{d}\tau} = \mathrm{D}F_k(y, t)$ *on the interval* $\langle t_0 - \delta', t_0 + \delta' \rangle$ *and* $k > k_0$, $\|y(t_0 - \delta') - x_0\| \leqq \delta$, *then*

$$\|y(t_0 + \delta') - y(t_0 - \delta') - F_0(x_0, t_0+) + F_0(x_0, t_0)\| < \varepsilon .$$

Theorem 4.1. *Let the sequence* $F_k(x, t)$, $k = 1, 2, \ldots$, *converge* R-*emphatically to* $F_0(x, t)$. *Let* $x_k(\tau)$ *be solutions of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_k(x, t) \tag{4,1}$$

on $\langle t_1, t_2 \rangle$, *such that* $\lim_{k \to \infty} x_k(\tau) = z(\tau)$ *exists whenever* $\tau \in \langle t_1, t_2 \rangle$, $h_0(\tau+) = h_0(\tau)$. (*Obviously* $\|z(\sigma_2) - z(\sigma_1)\| \leqq |h_0(\sigma_2) - h_0(\sigma_1)|$ *whenever* $\sigma_1, \sigma_2 \in \langle t_1, t_2 \rangle$, $h_0(\sigma_1+) = h_0(\sigma_1)$, $h_0(\sigma_2+) = h_0(\sigma_2)$). *Let* $x(\tau)$ *be a function on* $\langle t_1, t_2 \rangle$, *continuous from the left*, $x(\tau) = z(\tau)$ *whenever* $\tau \in \langle t_1, t_2 \rangle$, $h_0(\tau+) = h_0(\tau)$, *and with* $(x(\tau), \tau) \in R$ *for* $\tau \in \langle t_1, t_2)$, $(x(t_2), t_2) \in G$. *Finally, let* $h_0(t_1+) = h_0(t_1)$.

Then $x(\tau)$ *is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_0(x, t) \qquad \text{on } \langle t_1, t_2 \rangle . \tag{4,2}$$

Proof. We may assume $\omega(\eta) \geqq \eta$. Choose $\sigma_1, \sigma_2 \in \langle t_1, t_2 \rangle$ with $\sigma_1 < \sigma_2$, $h_0(\sigma_1+) = h_0(\sigma_1)$, $h_0(\sigma_2+) = h_0(\sigma_2)$; also a $\xi > 0$ and a decomposition $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\} \in \tilde{A}(\xi, \sigma_1, \sigma_2, h_0, \omega)$ such that $h_0(\alpha_j+) = h_0(\alpha_j)$ and $h_0(\tau_j+) = h_0(\tau_j)$ for τ_j non $\in N$ ($j = 0, 1, 2, \ldots, s$). (Such decompositions exist, since $\tilde{A}$ is not empty, $h_0(\alpha_0+) = h_0(\alpha_0)$, $h_0(\alpha_s+) = h_0(\alpha_s)$, and the set of points of continuity of $h_0(t)$ is dense in $\langle \sigma_1, \sigma_2 \rangle$.) Let $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_r}$, $i_1 < i_2 < \ldots < i_r$, be the elements of $(\tau_1, \tau_2, \ldots, \tau_s) \cap N$ (possibly empty).

Taking $(x(\tau_{i_j}), \tau_{i_j})$ and $\varepsilon = \frac{\xi}{r}$, there exist (according to definition 4,1) δ_j, δ'_j, $0 < \delta'_j < \delta_j$, such that

$$\alpha_{i_j - 1} < \tau_{i_j} - \delta'_j , \quad \tau_{i_j} + \delta'_j < \alpha_{i_j} , \quad \delta'_j < \varepsilon ,$$

$$h_0(\tau_{i_j} + \delta'_j) - h_0(\tau_{i_j}+) < \varepsilon , \quad h_0(\tau_{i_j}) - h_0(\tau_{i_j} - \delta'_j) < \min\left(\frac{\delta_j}{2}, \varepsilon\right) ,$$

$$h_0((\tau_{i_j} - \delta'_j)+) = h_0(\tau_{i_j} - \delta'_j) , \quad h_0((\tau_{i_j} + \delta'_j)+) = h_0(\tau_{i_j} + \delta'_j) .$$

Also, there is a k_0 such that $\|x_k(\tau_{i_j} - \delta'_j) - x(\tau_{i_j} - \delta'_j)\| < \frac{\delta_j}{2}$ for all $k > k_0$. Since we have sharp inequalities in the definition of $\tilde{A}$, there exists a $k_1 > k_0$ such that

$$\{\alpha_{0,0}, \tau_{1,0}, \alpha_{1,0}, \ldots, \tau_{s_0 - 1,0}, \alpha_{s_0 - 1,0}, \tau_{s_0,0}, \alpha_{s_0,0}\} =$$
$$= \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_{i_1 - 1}, \alpha_{i_1 - 1}, \alpha_{i_1 - 1}, \tau_{i_1} - \delta'_1\} \in$$
$$\in \tilde{A}(\xi, \alpha_0, \tau_{i_1} - \delta'_1, h_0, \omega) \cap \tilde{A}(\xi, \alpha_0, \tau_{i_1} - \delta'_1, h_k, \omega)$$

for $k > k_1$,

$$\{\alpha_{0,j}, \tau_{1,j}, \alpha_{1,j}, \tau_{2,j}, \alpha_{2,j}, \ldots, \tau_{s_j-1,j}, \alpha_{s_j-1,j}, \tau_{s_j,j}, \alpha_{s_j,j}\} =$$
$$= \{\tau_{i_j} + \delta'_j, \alpha_{i_j}, \alpha_{i_j}, \tau_{i_j+1}, \alpha_{i_j+1}, \ldots, \tau_{i_{j+1}-1}, \alpha_{i_{j+1}-1}, \alpha_{i_{j+1}-1}, \tau_{i_{j+1}} - \delta'_{j+1}\} \in$$
$$\in \tilde{A}(\xi, \tau_{i_j} + \delta'_j, \tau_{i_{j+1}} - \delta'_{j+1}, h_0, \omega) \cap \tilde{A}(\xi, \tau_{i_j} + \delta'_j, \tau_{i_{j+1}} - \delta'_{j+1}, h_k, \omega)$$

for $k > k_1$, $j = 1, 2, \ldots, r-1$,

$$\{\alpha_{0,r}, \tau_{1,r}, \alpha_{1,r}, \tau_{2,r}, \alpha_{2,r}, \ldots, \alpha_{s_r-1,r}, \tau_{s_r,r}, \alpha_{s_r,r}\} =$$
$$= \{\tau_{i_r} + \delta'_r, \alpha_{i_r}, \alpha_{i_r}, \tau_{i_r+1}, \alpha_{i_r+1}, \ldots, \alpha_{s-1}, \tau_s, \alpha_s\} \in$$
$$\in \tilde{A}(\xi, \tau_{i_r} + \delta'_r, \alpha_s, h_0, \omega) \cap \tilde{A}(\xi, \tau_{i_r} + \delta'_r, \alpha_s, h_k, \omega)$$

for $k > k_1$. Now, choose a $k_2 > k_1$ in such a manner that

$$\|F_k(x(\tau_{i,j}), \alpha_{i,j}) - F_0^*(x(\tau_{i,j}), \alpha_{i,j})\| \leqq \frac{\varepsilon}{s_j},$$
$$\|F_k(x(\tau_{i,j}), \alpha_{i-1,j}) - F_0^*(x(\tau_{i,j}), \alpha_{i-1,j})\| \leqq \frac{\varepsilon}{s_j},$$
$$\omega(\|x_k(\tau_{i,j}) - x(\tau_{i,j})\|) \leqq \xi$$

for $k > k_2$, $j = 0, 1, \ldots, r$, $i = 1, 2, \ldots, s_j$.

From Theorem 1,1 we then have

$$\|\int_{\alpha_{0,j}}^{\alpha_{s_j,j}} DF_k(x_k(\tau), t) - \sum_{i=1}^{s_j} [F_k(x_k(\tau_{i,j}), \alpha_{i,j}) - F_k(x_k(\tau_{i,j}), \alpha_{i-1,j})]\| \leqq$$
$$\leqq 2\xi[h_k(\alpha_{s_j,j}) - h_k(\alpha_{0,j})]$$

for $j = 0, 1, \ldots, r$, $k > k_1$, or $k = 0$. Thence

$$\|\int_{\alpha_{0,j}}^{\alpha_{s_j,j}} DF_k(x_k(\tau), t) - \int_{\alpha_{0,j}}^{\alpha_{s_j,j}} DF_0(x(\tau), t)\| \leqq$$
$$\leqq \sum_{i=1}^{s_j} \|F_k(x_k(\tau_{i,j}), \alpha_{i,j}) - F_k(x_k(\tau_{i,j}), \alpha_{i-1,j}) - F_0(x(\tau_{i,j}), \alpha_{i,j}) +$$
$$+ F_0(x(\tau_{i,j}), \alpha_{i-1,j})\| + 2\xi[h_k(\alpha_{s_j,j}) - h_k(\alpha_{0,j}) + h_0(\alpha_{s_j,j}) - h_0(\alpha_{0,j})] \leqq$$
$$\leqq \sum_{i=1}^{s_j} \|F_k(x_k(\tau_{i,j}), \alpha_{i,j}) - F_k(x_k(\tau_{i,j}), \alpha_{i-1,j}) - F_k(x(\tau_{i,j}), \alpha_{i,j}) +$$
$$+ F_k(x(\tau_{i,j}), \alpha_{i-1,j})\| + \sum_{i=1}^{s_j} \|F_k(x(\tau_{i,j}), x_{i,j}) - F_0^*(x(\tau_{i,j}), \alpha_{i,j})\| +$$
$$+ \sum_{i=1}^{s_j} \|F_k(x(\tau_{i,j}), \alpha_{i-1,j}) - F_0^*(x(\tau_{i,j}), \alpha_{i-1,j})\| + \sum_{i=1}^{s_j} \|F''_0(x(\tau_{i,j}), \alpha_{i,j}) -$$
$$- F''_0(x(\tau_{i,j}), \alpha_{i-1,j})\| + 2\xi[h_k(\alpha_{s_j,j}) - h_k(\alpha_{0,j}) + h_0(\alpha_{s_j,j}) - h_0(\alpha_{0,j})] \leqq$$
$$\leqq \sum_{i=1}^{s_j} \omega(\|x_k(\tau_{i,j}) - x(\tau_{i,j})\|)(h_k(\alpha_{i,j}) - h_k(\alpha_{i-1,j})) + 2\varepsilon + h''_0(\alpha_{s_j,j}) - h''_0(\alpha_{0,j}) +$$
$$+ 2\xi[h_k(\alpha_{s_j,j}) - h_k(\alpha_{0,j}) + h_0(\alpha_{s_j,j}) - h_0(\alpha_{0,j})] \leqq$$
$$\leqq 2\xi\left[\frac{1}{r} + 2h_k(\alpha_{s_j,j}) - 2h_k(\alpha_{0,j}) + h_0(\alpha_{s_j,j}) - h_0(\alpha_{0,j})\right] + h''_0(\alpha_{s_j,j}) - h''_0(\alpha_{0,j}). \quad (4,3)$$

Further

$$\int_{\tau_{i_j}-\delta_j'}^{\tau_{i_j}+\delta_j'} \mathrm{D}F_k(x_k(\tau), t) = x_k(\tau_{i_j} + \delta_j') - x_k(\tau_{i_j} - \delta_j'),$$

$$\left\| \int_{\tau_{i_j}-\delta_j'}^{\tau_{i_j}+\delta_j'} \mathrm{D}F_0(x(\tau), t) - F_0(x(\tau_{i_j}), \tau_{i_j}+) + F_0(x(\tau_{i_j}), \tau_{i_j}) \right\| \leqq$$

$$\leqq h_0(\tau_{i_j} + \delta_j') - h_0(\tau_{i_j}+) + h_0(\tau_{i_j}) - h_0(\tau_{i_j} - \delta_j') \leqq 2\varepsilon = \frac{2}{r}\xi$$

(on taking $\xi \to \tau_{i_j}+$ in $\int_{\tau_{i_j}-\delta_j'}^{\tau_{i_j}+\delta_j'} = \int_{\tau_{i_j}-\delta_j'}^{\tau_{i_j}} + \int_{\tau_{i_j}}^{\xi} + \int_{\xi}^{\tau_{i_j}+\delta_j'}$, using Theorem 1,3,6 of [1] on the middle term and Lemma 2,1 on the remaining terms). Since

$$\|x_k(\tau_{i_j} - \delta_j') - x(\tau_{i_j})\| \leqq \|x_k(\tau_{i_j} - \delta_j') - x(\tau_{i_j} - \delta_j')\| + \|x(\tau_{i_j} - \delta_j') - x(\tau_{i_j})\| \leqq$$

$$\leqq \frac{\delta_j}{2} + h_0(\tau_{i_j}) - h_0(\tau_{i_j} - \delta_j') < \delta_j,$$

the R-emphatic convergence of $F_k(x, t)$ to $F_0(x, t)$ implies

$$\left.\begin{aligned} &\left\| \int_{\tau_{i_j}-\delta_j'}^{\tau_{i_j}+\delta_j'} \mathrm{D}F_k(x_k(\tau), t) - \int_{\tau_{i_j}-\delta_j'}^{\tau_{i_j}+\delta_j'} \mathrm{D}F_0(x(\tau), t) \right\| \leqq \\ &\leqq \|x_k(\tau_{i_j} + \delta_j') - x_k(\tau_{i_j} - \delta_j') - F_0(x(\tau_{i_j}), \tau_{i_j}+) + F_0(x(\tau_{i_j}), \tau_{i_j})\| + \\ &+ \frac{2}{r}\xi \leqq \varepsilon + \frac{2}{r}\xi = \frac{3}{r}\xi \end{aligned}\right\} \quad (4,4)$$

for $k > k_2$.

Adding the (4,3) with $j = 0, 1, 2, \ldots, r$ and the (4,4) with $j = 1, 2, \ldots, r$,

$$\left.\begin{aligned} &\left\| \int_{\sigma_1}^{\sigma_2} \mathrm{D}F_k(x_k(\tau), t) - \int_{\sigma_1}^{\sigma_2} \mathrm{D}F_0(x(\tau), t) \right\| \leqq \\ &\leqq 2\xi[2 + 2h_k(\sigma_2) - 2h_k(\sigma_1) + h_0(\sigma_2) - h_0(\sigma_1)] + \\ &+ \{h_0''(\sigma_2) - h_0''(\sigma_1) - \sum_{\eta \in N} (h_0''(\eta+) - h_0''(\eta))\} + 3\xi \end{aligned}\right\} \quad (4,5)$$

Now $\limsup_{k\to\infty} [h_k(\sigma_2) - h_k(\sigma_1)] \leqq h_0(\sigma_2) - h_0(\sigma_1)$. The brace approaches zero with $\xi \to 0$ as $N(\xi_1) \supset N(\xi_2)$ for $\xi_1 \leqq \xi_2$ and $\bigcup_{\xi>0} N(\xi)$ is the set of all points of $\langle\sigma_1, \sigma_2\rangle$ where $h_0(t)$ is discontinuous. As ξ has been arbitrary, $\int_{\sigma_1}^{\sigma_2} \mathrm{D}F_k(x_k(\tau), t) \to$ $\to \int_{\sigma_1}^{\sigma_2} \mathrm{D}F_0(x(\tau), t)$ with $k \to \infty$. Since $x_k(\sigma_1) \to x(\sigma_1)$, $x_k(\sigma_2) \to x(\sigma_2)$ with $k \to \infty$, this implies

$$x(\sigma_2) - x(\sigma_1) = \int_{\sigma_1}^{\sigma_2} \mathrm{D}F_0(x(\tau), t) \quad (4,6)$$

whenever $\sigma_1, \sigma_2 \in \langle t_1, t_2\rangle$, $h_0(\sigma_1+) = h_0(\sigma_1)$, $h_0(\sigma_2+) = h_0(\sigma_2)$. Finally, $x(\sigma)$ is continuous from the left, and so is $\int\limits_{t_1}^{\sigma} \mathrm{D}F_0(x(\tau), t)$ as a function of σ; this, together with $h_0(t_1+) = h_0(t_1)$ implies (4,6) holds for all $\sigma_1, \sigma_2 \in \langle t_1, t_2\rangle$, thus proving Theorem 4,1.

Theorem 4,2. *Let the functions $F_k(x, t)$ ($k = 1, 2, \ldots$) converge R-emphatically to $F_0(x, t)$. Let $x(\tau)$ be a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_0(x, t) \tag{4,7}$$

on $\langle t_1, t_2\rangle$ and satisfy this unicity condition: if $z(\tau)$ is any solution of (4,7) on $\langle t_1, t_3\rangle \subset \langle t_1, t_2\rangle$ with $z(t_1) = x(t_1)$, then $z(\tau) = x(\tau)$ for all $\tau \in \langle t_1, t_3\rangle$.

Assume that to each $t_0 \in \langle t_1, t_2\rangle$ there corresponds a $\varrho = \varrho(t_0) > 0$ such that:

if $h_0(t_0+) = h_0(t_0)$, then $(y, t) \in R$ whenever $|t - t_0| < \varrho$, $\|y - x(t_0)\| < \varrho$;

if $h_0(t_0+) > h_0(t_0)$, then $(x(t_0), t_0) \in R$ and $(y, t) \in R$ whenever either $t_0 - \varrho < t < t_0$, $\|y - x(t_0)\| < \varrho$ or

$$t_0 < t < t_0 + \varrho\,, \quad \|y - x(t_0) - F_0(x(t_0), t_0+) + F_0(x(t_0), t_0)\| < \varrho\,. \tag{4,8}$$

Assume also that $h_0(t_1+) = h_0(t_1)$, $h_0(t_2+) = h_0(t_2)$, and that

(Λ) to every $t_4 \in (t_1, t_2)$ with $h_0(t_4+) > h_0(t_4)$ and every sufficiently small $\lambda > 0$ there exists an index k_3 such that to any y with $\|y - x(t_4)\| < \lambda$ there corresponds a solution $x_k(\tau)$ of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_k(x, t) \quad (k > k_3) \tag{4,9}$$

with $x_k(t_4 - \lambda) = y$ (supposing $t_1 < t_4 - \lambda$), defined on $\langle t_4 - \lambda, t_4 + \lambda\rangle \cap \langle t_1, t_2\rangle$. Finally, let $y_k \to x(t_1)$ with $k \to \infty$.

Then for sufficiently large k there exist solutions $x_k(\tau)$ of the equations (4,9), defined on $\langle t_1, t_2\rangle$ and such that $x_k(t_1) = y_k$ and $x_k(\tau) \to x(\tau)$ with $k \to \infty$ whenever $h_0(\tau+) = h_0(\tau)$, $\tau \in \langle t_1, t_2\rangle$.

The connection between Theorems 4,2 and 4,1 is similar to that between Theorems 4,2,1 and 4,1,1 in [1].

Proof. Since $h_0(t_1+) = h_0(t_1)$ and $h_k(t) \to h_0(t)$ with $k \to \infty$ at points of continuity of $h_0(t)$, there exist $t_7 \in (t_1, t_2\rangle$ and $\varrho_1 > 0$ such that for sufficiently large k one may determine solutions $x_k(\tau)$ of (4,9), defined on $\langle t_1, t_7\rangle$ and having $x_k(t_1) = y_k$, and such that the distance of all points $(x_k(\tau), \tau)$ (with $\tau \in \langle t_1, t_7\rangle$) from the complement of R is larger than ϱ_1. Using Theorem 4,1 and our unicity condition, if, for some subsequence the $\lim\limits_{j\to\infty} x_{k_j}(\tau)$ exists whenever $\tau \in \langle t_1, t_7\rangle$, $h_0(\tau+) = h_0(\tau)$, necessarily $\lim\limits_{j\to\infty} x_{k_j}(\tau) = x(\tau)$ for these τ. As every subsequence of $x_k(\tau)$ contains (according to (4,01), [1] and Lemma 2,1) such a subsequence

$x_{k_j}(\tau)$ that $\lim_{j\to\infty} x_{k_j}(\tau)$ exists whenever $\tau \in \langle t_1, t_7\rangle$, $h_0(\tau+) = h_0(\tau)$, this implies $\lim_{k\to\infty} x_k(\tau) = x(\tau)$ whenever $\tau \in \langle t_1, t_7\rangle$ and $h_0(\tau+) = h_0(\tau)$; i. e., our theorem restricted to the interval $\langle t_1, t_7\rangle$.

Now assume theorem 4,2 does not hold for the whole interval $\langle t_1, t_2\rangle$. Then there exists an $t_8 \in (t_1, t_2\rangle$ such that the theorem holds on every interval $\langle t_1, t_9\rangle$ with $t_9 < t_8$ and $h_0(t_9+) = h_0(t_9)$, but fails to hold on any $\langle t_1, t_{10}\rangle$ with $t_{10} > t_8$ (or, if $t_8 = t_2$, does not hold on $\langle t_1, t_2\rangle$).

If $h_0(t_8+) = h_0(t_8)$, then there is a $\zeta > 0$ with $h_0((t_8 - \zeta)+) = h_0(t_8 - \zeta)$ such that, for sufficiently large k, solutions $w_k(\tau)$ of (4,9) are defined on $\langle t_8 - \zeta, t_8 + \zeta\rangle$ (or on $\langle t_2 - \zeta, t_2\rangle$) satisfying any initial conditions $w_k(t_8 - \zeta) = \tilde{w}_k$ with $\|\tilde{w}_k - x(t_8 - \zeta)\| < \zeta$, and such that all the points $(w_k, (\tau), \tau)$ with $\tau \in \langle t_8 - \zeta, t_8 + \zeta\rangle$ have a distance from the complement of R larger than ζ. Just as we proved Theorem 4,2 on the interval $\langle t_1, t_7\rangle$, we see that it also holds on $\langle t_1, t_8 + \zeta\rangle$ (or on $\langle t_1, t_2\rangle$).

It remains to examine the case of $h(t_8+) > h(t_8)$, $t_8 < t_2$. Using condition (Λ), we take a sufficiently small λ with $h_0((t_8 - \lambda)+) = h_0(t_8 - \lambda)$. From the definition of t_8 it follows that the solutions $x_k(\tau)$ of (4,9) with $x_k(t_1) = y_k$ can be continued over the whole interval $\langle t_1, t_8 - \lambda\rangle$ (at least for sufficiently large k); According to (Λ) they can be continued over $\langle t_1, t_8 + \lambda\rangle$.

Let $x_{k_j}(\tau)$ be a subsequence of these continued solutions, such that $\lim_{j\to\infty} x_{k_j}(\tau) = u(\tau)$ whenever $\tau \in \langle t_1, t_8 + \lambda\rangle$, $h_0(\tau+) = h_0(\tau)$, and take $u(\tau)$ continuous from the left. From the definition of t_8 it follows that $u(\tau) = x(\tau)$ for $\tau \in \langle t_1, t_8 - \lambda\rangle$. If λ is sufficiently small, then (4,8), $x_k(t_8 - \lambda) \to x(t_8 - \lambda)$ (and $h_k(t) \to h_0(t)$ at points of continuity of $h_0(t)$) imply:

If $t_8 - \lambda < \xi < t_8$, then there exist $\eta_1 > 0$ and an index k_4 such that for $\tau \in \langle t_8 - \lambda, \xi\rangle$ and $k > k_4$ the distance from $(x_k(\tau), \tau)$ to the complement of R is larger than η_1; thus $(u(\tau), \tau) \in R$ for $\tau \in \langle t_1, t_8)$. Thence Theorem 4,1 gives $u(\tau) = x(\tau)$ for $\tau \in \langle t_1, t_8)$ and thus for $\tau = t_8$ also.

Similarly we may use condition (Ω) of Definition 4,1 to obtain:

If $t_8 < \xi_2 < t_8 + \lambda$, then there exists $\eta_2 > 0$ and an index k_5 such that for $\tau \in \langle \xi_2, t_8 + \lambda\rangle$ and $k > k_5$ the distance from $(x_k(\tau), \tau)$ to the complement of R is larger than η_2; thus $(u(\tau), \tau) \in R$ when $\tau \in \langle t_8 - \lambda, t_8 + \lambda\rangle$, and Theorem 4,1 implies that Theorem 4,2 holds on the interval $\langle t_1, t_8 + \lambda\rangle$.

In every case the point t_8 cannot exist; this proves Theorem 4,2 completely.

Note 4,1. It is simple to show that a sequence $F_k(x, t)$ converges G_{F_0}-emphatically to $F_0(x, t)$ if $h_k(t) \to h_0(t)$ and $F_k(x, t) \to F_0(x, t)$ uniformly; and that this also implies condition (Λ) for every $(x_0, t_0) \in G_{F_0}$ with $h_0(t_0+) > h_0(t_0)$. In this manner more special theorems on continuous dependence on a parameter may be deduced from Theorems 4,1 and 4,2.

5. The Dirac Function in Non-linear Differential Equations

We shall use Theorem 4,2 to examine the behaviour of solutions of the sequence of classical differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t) + g(x)\, \varphi_k(t)$ where $\varphi_k(t)$ tends to the Dirac function.

Let the functions $\varphi_k(t)$, $k = 1, 2, 3, \ldots$ be defined, real-valued and continuous for $t \in \langle -T_1, T_1 \rangle$ and let $\Phi_k(t) = \int_{-T_1}^{t} \varphi_k(\tau)\, \mathrm{d}\tau$.

Definition 5.1. *The sequence $\varphi_k(t)$ tends to the Dirac function, if*

$$\limsup_{k\to\infty} \int_{-T_1}^{T_1} |\varphi_k(t)|\, \mathrm{d}t = L < \infty\,, \tag{5,1}$$

$$\left.\begin{aligned} &\int_{-T_1}^{t} |\varphi_k(\tau)|\, \mathrm{d}\tau \to 0, \quad \textit{for} \quad -T_1 \leqq t < 0, \\ &\int_{t}^{T_1} |\varphi_k(\tau)|\, \mathrm{d}\tau \to 0 \quad \textit{for} \quad 0 < t \leqq T_1, \quad k \to \infty \end{aligned}\right\} \tag{5,2}$$

and if

$$\Phi_k(t) \to 0 \text{ for } -T_1 \leqq t < 0,\ \Phi_k(t) \to 1 \text{ for } 0 < t \leqq T_1,\ k \to \infty\,. \tag{5,3}$$

(Obviously $L \geqq 1$.)

Definition 5,2. *The sequence $\varphi_k(t)$ tends to the Dirac function positively, if $\varphi_k(t) \geqq 0$ and if* (5,3) *holds.*

Note 5,1. If $\varphi_k(t)$ tends to the Dirac function positively, then $\varphi_k(t)$ tends to the Dirac function.

Definition 5,3. *The solution $x(t)$, $t \in \langle t_1, t_2 \rangle$ of a differential equation is positively unique, if the following condition holds: if $y(t)$, $t \in \langle t_1, t_3 \rangle \subset \langle t_1, t_2 \rangle$ is a solution of the same equation, $y(t_1) = x(t_1)$, then $y(t) = x(t)$ for $t \in \langle t_1, t_3 \rangle$.*

Let D be an open subset of E_n, let the function $f(x, t)$ be defined and continuous for $x \in D$, $t \in \langle -T_1, T_1 \rangle$ and let the function $g(x)$ be defined and continuous for $x \in D$. Suppose, that the values of the functions $f(x, t)$ and $g(x)$ belong to E_n and that

$$\|f(x_1, t) - f(x_2, t)\| \leqq \omega(\|x_2 - x_1\|)\,, \tag{5,4}$$

$$\|g(x_1) - g(x_2)\| \leqq \omega(\|x_2 - x_1\|)\,, \tag{5,5}$$

$$\|f(x, t)\| \leqq K\,, \quad \|g(x)\| \leqq K \tag{5,6}$$

for $x_1, x_2, x \in D$, $t \in \langle -T_1, T_1 \rangle$ ($\omega(\eta)$ has the same meaning as in § 1).

The main result of this section is contained in the following two theorems:

Theorem 5,1. *Let $\varphi_k(t)$ tend to the Dirac function. Let $\chi(\eta)$, $\eta \geqq 0$ be increasing and continuous,* $\chi(0) = 0$, $\int_0^1 \frac{\mathrm{d}\eta}{\chi(\eta)} = \infty$ *and let*

$$\|g(x_1) - g(x_2)\| \leqq \chi(\|x_2 - x_1\|) \quad \text{for} \quad x_1, x_2 \in D\,. \tag{5,7}$$

Let $u(t)$, $t \in \langle -T_0, 0\rangle$ $(0 < T_0 < T_1)$ be a positively unique solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t) \tag{5,8}$$

let the solution $v(t)$ of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x) \tag{5,9}$$

$v\left(-\frac{1}{2}\right) = u(0)$ *be defined for* $t \in \left\langle -\frac{L}{2}, \frac{L}{2}\right\rangle$ *($v(t)$ is unique according to a well-known theorem of Osgood) and let the solution $w(t)$, $t \in \langle 0, T_0\rangle$* of (5,8), $w(0) = v\left(\frac{1}{2}\right)$ *be positively unique. Let $y_k \to u(T_0)$ with $k \to \infty$.*

Then there exists a solution $x_k(t)$, $t \in \langle -T_0, T_0\rangle$ of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t) + g(x)\,\varphi_k(t) \tag{5,10}$$

$x_k(-T_0) = y_k$, *(not necessarily unique) for k sufficiently large and*

$$x_k(t) \to u(t) \;\; \text{if} \;\; -T_0 \leqq t < 0, \;\; x_k(t) \to w(t) \;\; \text{if} \;\; 0 < t \leqq T_0, \;\; k \to \infty\,.$$

For an arbitrary positive ζ the sequence $x_k(t)$ converges uniformly on the intervals $\langle -T_0, -\zeta\rangle$, $\langle \zeta, T_0\rangle$.

Theorem 5,2. *Let $\varphi_k(t)$ tend to the Dirac function positively. Let $u(t)$, $t \in \langle -T_0\ 0\rangle$ $(0 < T_0 < T_1)$ be a positively unique solution of* (5,8). *Let the solution $v(t)$, $t \in \langle -\frac{1}{2}, \frac{1}{2}\rangle$ of* (5,9), $v(-\frac{1}{2}) = u(0)$ *be positively unique and let the solution $w(t)$, $t \in \langle 0, T_0\rangle$ of* (5,8), $w(0) = v(\frac{1}{2})$ *be positively unique. Let $y_k \to u(-T_0)$ with $k \to \infty$.*

Then there exists a solution $x_k(t)$, $t \in \langle -T_0, T_0\rangle$ of (5,10), $x_k(-T_0) = y_k$ *(not necessarily unique) for k sufficiently large and*

$$x_k(t) \to u(t) \;\; \text{if} \;\; -T_0 \leqq t < 0, \;\; x_k(t) \to w(t) \;\; \text{if} \;\; 0 < t \leqq T_0, \;\; k \to \infty\,.$$

For an arbitrary positive ζ the sequence $x_k(t)$ converges uniformly on the intervals $\langle -T_0, -\zeta\rangle$, $\langle \zeta, T_0\rangle$.

In proving Theorem 5,1 the following lemmas will be required:

Lemma 5,1. *Let* (5,7) *be fulfilled. Let $\Psi(t)$, $t \in \langle 0, S\rangle$ be a continuous real-valued* *function of bounded variation and let $y \in D$, $t_0 \in \langle 0, S\rangle$. Then there exists at most one solution $z(\sigma)$ of*

$$z(\sigma) = y + \int_{t_0}^{\sigma} g(z(\tau))\, \mathrm{d}\Psi(\tau) \tag{5,11}$$

$$(= y + \int_{t_0}^{\sigma} \mathrm{D}g(z(\tau))\, \Psi(t), \quad \textit{see Note } 3{,}1).$$

Proof. If Lemma 5,1 is false, then there exist such solutions $z_1(\sigma)$, $z_2(\sigma)$ of (5,11), that $z_1(\sigma_1) = z_2(\sigma_1)$, $z_1(\sigma) \neq z_2(\sigma)$ for $\sigma \in (\sigma_1, \sigma_1 + \zeta\rangle \subset \langle 0, S\rangle$, $\zeta > 0$ (or $\sigma \in \langle \sigma_1 - \zeta, \sigma_1) \subset \langle 0, S\rangle$). According to Lemma 2,1 $z_1(\sigma)$ and $z_2(\sigma)$ are continuous functions of bounded variation. Let $\sigma_1 < \sigma_2 < \sigma_3 \leqq \sigma_1 + \zeta$ and let us put $^*\Psi(t) = \operatorname{var}_{\tau \in \langle 0, t\rangle} \Psi(\tau)$,

$$U(\tau, t) = \frac{\|z_1(t) - z_2(t)\|}{\chi(\|z_1(\tau) - z_2(\tau)\|)}, \quad V(\tau, t) = 2{}^*\Psi(t)\,.$$

As

$$|t - \tau| \frac{\big|\|z_1(t) - z_2(t)\| - \|z_1(\tau) - z_2(\tau)\|\big|}{\chi(\|z_1(\tau) - z_2(\tau)\|)} \leqq$$

$$\leqq \frac{|t - \tau|}{\chi(\|z_1(\tau) - z_2(\tau)\|)} \left\| \int_\tau^t [g(z_1(\sigma)) - g(z_2(\sigma))]\, \mathrm{d}\Psi(\sigma) \right\| \leqq$$

$$\leqq \frac{|t - \tau|}{\chi(\|z_1(\tau) - z_2(\tau)\|)} \left| \int_\tau^t \|g(z_1(\sigma)) - g(z_2(\sigma))\|\, \mathrm{d}^*\Psi(\sigma) \right| \leqq 2(t - \tau)({}^*\Psi(t) - {}^*\Psi(\tau))$$

for $|t - \tau|$ sufficiently small, Lemma 3,1 gives

$$\int_{\|z_1(\sigma_2) - z_2(\sigma_2)\|}^{\|z_1(\sigma_3) - z_2(\sigma_3)\|} \frac{\mathrm{d}\eta}{\chi(\eta)} = \int_{\sigma_2}^{\sigma_3} \frac{\mathrm{d}\|z_1(\sigma) - z_2(\sigma)\|}{\chi(\|z_1(\sigma) - z_2(\sigma)\|)} = \int_{\sigma_2}^{\sigma_3} \mathrm{D}U(\tau, t) \leqq \int_{\sigma_2}^{\sigma_3} \mathrm{D}V(\tau, t) \leqq$$

$$\leqq 2({}^*\Psi(\sigma_3) - {}^*\Psi(\sigma_2))\,.$$

It follows that

$$\int_0^{\|z_1(\sigma_3) - z_2(\sigma_3)\|} \frac{\mathrm{d}\eta}{\chi(\eta)} \leqq 2{}^*\Psi(\sigma_3) - 2{}^*\Psi(\sigma_1) < \infty\,,$$

which contradicts (5,7). Lemma 5,1 is proved.

Lemma 5,2. *Let the assumptions of Theorem 5,1 be fulfilled. Let $\Psi(t)$, $t \in \langle 0, S\rangle$ be a continuous function of bounded variation,*

$$\Psi(0) = -\tfrac{1}{2}\,, \quad \Psi(S) = +\tfrac{1}{2}\,, \quad \operatorname{var}_{t \in \langle 0, S\rangle} \Psi(t) \leqq L\,.$$

Then $z(t) = v(\Psi(t))$ *is the only solution of*

$$z(t) = u(0) + \int_0^t g(z(\tau))\, d\Psi(\tau)\,. \tag{5,12}$$

Proof. Obviously $|\Psi(t)| \leqq \frac{L}{2}$ for $t \in \langle 0, S\rangle$ and $v(\Psi(t))$ is defined. As $v(s) = x(0) + \int_{-\frac{1}{2}}^{s} g(v(\sigma))\, d\sigma$, (5,12) holds and according to Lemma 5,1 $z(t)$ is unique.

We shall show, that Theorem 5,1 follows from Theorem 4,2. According to Theorem 2,2,1 of [1] the classical equations (5,10) are equivalent with

$$\frac{dx}{d\tau} = DF_k(x, t) \tag{5,13}$$

where $F_k(x, t) = \int_{-T_0}^{t} f(x, \tau)\, d\tau + g(x)\, \Phi_k(t)$. Let us put $\Phi(t) = 0$ for $-T_1 \leqq t \leqq 0$; $\Phi(t) = 1$ for $0 < t \leqq T_1$,

$$F_0(x, t) = \int_{-T_0}^{t} f(x, \tau)\, d\tau + (v(\tfrac{1}{2}) - v(-\tfrac{1}{2}))\, \Phi(t)\,,$$

$x(\tau) = u(\tau)$ for $-T_0 \leqq \tau \leqq 0$, $x(\tau) = w(\tau)$ for $0 < \tau \leqq T_0$. $x(\tau)$ is a solution of

$$\frac{dx}{d\tau} = DF_0(x, t) \tag{5,14}$$

which fulfils the unicity condition of Theorem 4,2. Let us put

$$^*\Phi_k(t) = \int_{-T_1}^{t} |\varphi_k(\tau)|\, d\tau,\ h_0(t) = Kt + (K + \|v(\tfrac{1}{2}) - v(-\tfrac{1}{2})\|)\, L\Phi(t)\,,$$

$$h_k(t) = Kt + K^*\Phi_k(t),\ F_0^*(x, t) = F_0(x, t) + [g(x) - v(\tfrac{1}{2}) + v(-\tfrac{1}{2})]\, \Phi(t)\,.$$

Obviously

$$\limsup_{k\to\infty}\, (h_k(t_2) - h_k(t_1)) \leqq h_0(t_2) - h_0(t_1) \text{ for } t_1, t_2 \neq 0,\ t_1 < t_2\,,$$

$$F_k(x, t) \to F_0^*(x, t) \text{ for } t \neq 0,\ x \in D,\ k \to \infty,\ F_k(x, t) \in F(G, h_k, \omega),\ k = 0, 1, 2, \ldots,$$
$$G = D \times (-T_1, T_1)$$

Let us suppose for an instant, that we have already proved, that the sequence $F_k(x, t)$ converges R-emphatically to $F_0(x, t)$ with $k \to \infty$ where the set R consists of the point $(u(0), 0) = (x(0), 0)$ and of all points (x, t), $x \in D$, $t \in (-T_1, T_1)$, $t \neq 0$ and that the condition (Λ) is fulfilled. According to Theorem 4,2 the solutions $x_k(t)$, $t \in \langle -T_0, T_0\rangle$ of (5,13) (and consequently of (5,10)) exist for k sufficiently large and $x_k(t) \to x(t)$ with $k \to \infty$, $t \in \langle -T_0, T_0\rangle$, $t \neq 0$. As $\|x_k(t_2) - x_k(t_1)\| \leqq |h_k(t_2) - h_k(t_1)|$ (see Lemma 2,1), it follows from (5,2) that the sequence $x_k(t)$ converges uniformly on the intervals $\langle -T_0, -\zeta\rangle$, $\langle \zeta, T_0\rangle$ for every positive ζ.

It remains to prove that the conditions (Ω) and (Λ) are fulfilled.

Let us denote by Z the set of values of the function $v(\tau)$, i. e. $Z = E_x\left[x = v(\tau), -\frac{L}{2} \leqq \tau \leqq \frac{L}{2}\right]$. Let $\varrho > 0$ be less than the distance from Z to the complement of D ($\varrho > 0$ arbitrary if $D = E_n$) and let H be the set of all points $x \in D$ at a distance less than ϱ from Z.

Lemma 5,3. *There exists such a $\lambda_0 > 0$ that for every λ, $0 < \lambda < \lambda_0$ there exists a k_1 with the following property:*

if $k > k_1$ and if $y(\tau)$, $\tau \in \langle -\lambda, \xi \rangle \subset \langle -\lambda, \lambda \rangle$ is a solution of (5,10), *$\|y(-\lambda) - u(0)\| < \lambda$, then $y(\tau) \in H$ for $\tau \in \langle -\lambda, \xi \rangle$.*

Note 5,2. It follows from Lemma 5,3, that the condition (Λ) holds.

Proof of lemma 5,3. If Lemma 5,3 is false, then there exists a sequence of such functions $y_j(t)$ that

$$y_j(t) \text{ is defined for } t \in \langle -\lambda_j, \xi_j \rangle \subset \langle -\lambda_j, \lambda_j \rangle,\ \lambda_j \to 0 \textit{ with } j \to \infty, \tag{5,15}$$

$$y_j(-\lambda_j) \to x(0)\,, \quad j \to \infty\,, \tag{5,16}$$

$$y_j(\tau) \text{ is a solution of (5,13) for } k = k_j,\ k_j \to \infty \text{ with } j \to \infty\,, \tag{5,17}$$

$$y_j(\tau) \in H \quad \text{for} \quad \tau \in \langle -\lambda_j, \xi_j)\,, \tag{5,18}$$

$$y_j(\xi_j) \,\tilde{\in}\, H\,, \tag{5,19}$$

$$\int_{-T_1}^{-\lambda_j} |\varphi_{k_j}(t)|\,\mathrm{d}t + \int_{\lambda_j}^{T_1} |\varphi_{k_j}(t)|\,\mathrm{d}t \leqq 2^{-j}\,, \quad j = 1, 2, 3, \ldots \tag{5,20}$$

(first we choose λ_j and then k_j may be found arbitrarily large).

$y_j(t)$ is a solution of (5,10) for $k = k_j$. Let us introduce a new variable

$$\tau = \int_{-\lambda_j}^{t} [\lambda_j^{-\frac{1}{2}} + |\varphi_{k_j}(\sigma)|]\,\mathrm{d}\sigma\,, \quad -\lambda_j \leqq t \leqq \lambda_j\,.$$

Consequently $z_j(\tau) = y_j(t_j(\tau))$ satisfies

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{f(z, t_j(\tau))}{\lambda_j^{-\frac{1}{2}} + |\varphi_{k_j}(t_j(\tau))|} + g(z)\,\frac{\varphi_{k_j}(t_j(\tau))}{\lambda_j^{-\frac{1}{2}} + |\varphi_{k_j}(t_j(\tau))|}\,,$$

$$0 \leqq \tau \leqq T_j = \int_{-\lambda_j}^{\xi_j} [\lambda_j^{-\frac{1}{2}} + |\varphi_{k_j}(\sigma)|]\,\mathrm{d}\sigma \leqq T_j^* = \int_{-\lambda_j}^{\lambda_j} [\lambda_j^{-\frac{1}{2}} + |\varphi_{k_j}(\sigma)|]\,\mathrm{d}\sigma\,,$$

$$z_j(0) = y_j(-\lambda_j)\,,$$

Obviously $\limsup_{j\to\infty} T_j^* = L$ (see (5,2)). Let us put

$$\Psi_j(\tau) = -\frac{1}{2} + \int_0^{\tau} \frac{\varphi_{k_j}(t_j(\sigma))}{\lambda_j^{-\frac{1}{2}} + |\varphi_{k_j}(t_j(\sigma))|}\,\mathrm{d}\sigma\,, \quad \tau \in \langle 0, T_j^* \rangle\,.$$

It follows that

$$\limsup_{j\to\infty} \operatorname*{var}_{t\in\langle 0,T_j^*\rangle} \Psi_j(t) \leqq L\,, \tag{5,21}$$

$$\Psi_j(T_j^*) = -\tfrac{1}{2} + \int_{-\lambda_j}^{\lambda_j} \varphi_{k_j}(t)\,\mathrm{d}t \to \tfrac{1}{2} \tag{5,22}$$

according to (5,2) and (5,3),

$$z_j(\tau) = z_j(0) + \int_0^\tau \frac{f(z_j(\sigma), t_j(\sigma))}{\lambda_j^{-\frac{1}{2}} + |\varphi_{k_j}(t_j(\sigma))|}\,\mathrm{d}\sigma + \int_0^\tau g(z_j(\sigma))\,\mathrm{d}\Psi_j(\sigma)\,, \quad \tau\in\langle 0, t_j\rangle\,. \tag{5,23}$$

As $z_j(\tau)$ and $\Psi_j(\tau)$ fulfil a Lipschitz condition (with a fixed constant), there exists such a subsequence j_l that $T_{j_l}\to T$ $(0\leqq T\leqq L)$, $z_{j_l}(\tau)\to z(\tau)$ uniformly ($z(\tau)$ is defined for $\tau\in\langle 0, T\rangle$) and $\Psi_{j_l}(\tau)\to\Psi(\tau)$ uniformly with $l\to\infty$ ($\Psi(\tau)$ is defined for $\tau\in\langle 0, L\rangle$). Obviously $z(0) = u(0)$, $z(\tau)\in\overline{H}$ for $0\leqq\tau\leqq T$,

$$\lim_{l\to\infty} z_{j_l}(T_{j_l}) = z(T)\in\overline{H} - H\,. \tag{5,24}$$

Passing to a limit in (5,23) for $j = j_l$, $l\to\infty$ we obtain

$$z(\tau) = x(0) + \int_0^\tau g(z(\sigma))\,\mathrm{d}\Psi(\sigma)\,, \quad \tau\in\langle 0, T\rangle\,.$$

As $\Psi(0) = -\frac{1}{2}$, $\Psi(L) = \frac{1}{2}$, $\operatorname*{var}_{t\in\langle 0,L\rangle}\Psi(t)\leqq L$ (see (5,21), (5,22)), according to Lemma 5,2 $z(\tau) = v(\Psi(\tau))$, $\tau\in\langle 0, T\rangle$, specially $z(T) = v(\Psi(T))\in H$, which contradicts (5,24). The proof of Lemma 5,3 is complete.

Let us prove the condition (Ω) from Definition 4,1. If this condition (for equations (5,13) and (5,14) is not fulfilled, there exists a sequence of functions $y_j(t)$ satisfying (5,15)—(5,18), (5,20) with $\xi_j = \lambda_j$ and

$$\liminf_{j\to\infty} \|y_j(\lambda_j) - v(\tfrac{1}{2})\| > 0\,. \tag{5,25}$$

As $\xi_j = \lambda_j$, it follows that $T_j = T_j^*$, $T = L$ and in the same way as before we obtain $v(\frac{1}{2}) = v(\Psi(T)) = z(T) = \lim_{l\to\infty} z_{j_l}(T_{j_l}) = \lim_{l\to\infty} y_{j_l}(\lambda_{j_l})$ which contradicts (5,25). Lemma 5,3 is proved and the proof of Theorem 5,1 is complete.

The proof of Theorem 5,2 is quite similar.

Lemma 5,4. *Let the assumptions of Theorem 5,2 be fulfilled. Let* $\Psi(t)$, $t\in\langle 0, s\rangle$ *be continuous and non-decreasing,* $\Psi(0) = -\frac{1}{2}$, $\Psi(s)\leqq\frac{1}{2}$. *Then* $z(t) = v(\Psi(t))$ *is the only solution of*

$$z(t) = u(0) + \int_0^t g(z(\tau))\,\mathrm{d}\Psi(\tau) \tag{5,26}$$

Proof. Let $z(t)$, $t\in\langle 0, s\rangle$ be a solution of (5,26). As $\Psi(t)$ is non-decreasing, $z(t) = z(t_1)$ for $t\in\langle t_1, t_2\rangle$ if $\Psi(t_1) = \Psi(t_2)$. Consequently there exists such a (continuous) function $q(s)$, $s\in\langle -\frac{1}{2}, \Psi(S)\rangle$, that $z(t) = q(\Psi(t))$. As

$$q(\Psi(t)) = u(0) + \int_0^t g(q(\Psi(\tau)))\, \mathrm{d}\Psi(\tau)\ ,$$

it follows, that

$$\dot{q}(s) = u(0) + \int_{-\frac{1}{2}}^{s} g(q(\sigma))\, \mathrm{d}\sigma$$

and $q(t) = v(t)$ according to the unicity assumption concerning $v(t)$. Lemma 5,4 is proved.

In order to prove Theorem 5,2 we may repeat the proof of Theorem 5,1 with slight changes only.

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Резюме

ОБОБЩЕННЫЕ ОБЫКНОВЕННЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ

ЯРОСЛАВ КУРЦВЕЙЛЬ (Jaroslav Kurzweil), Прага

(Поступило в редакцию 17/II 1958 г.)

Понятие обобщенного дифференциального уравнения было введено в [1], § 2. (Мы пользуемся § 1 и § 2, пп. 1 и 2 работы [1].) Мы докажем существование решения и непрерывную зависимость от параметра для определенного класса обобщенных дифференциальных уравнений, отличного от класса, рассмотренного в [1]. Исследуемый класс обобщенных дифференциальных уравнений имеет решения с ограниченным изменением. Результаты, касающиеся непрерывной зависимости от параметра, применяются к классическим дифференциальным уравнениям, поскольку в их правых частях встречаются функции, близкие к функциям Дирака.

§ 1 носит вспомогательный характер. Доказывается существование интеграла $\int_{\sigma_1}^{\sigma_2} \mathrm{D}F(x(\tau), t)$ при следующих условиях:

Функция $F(x, t)$ определена на открытом подмножестве G декартова произведения $E_n \times \langle 0, T\rangle$ (E_n есть n-мерное евклидово пространство) и удовлетворяет условиям

$$\|F(x, t_2) - F(x, t_1)\| \leqq |h(t_2) - h(t_1)|\,, \tag{0,1}$$

$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \omega(\|x_2 - x_1\|)\,|h(t_2) - h(t_1)|, \tag{0,2}$$

где функция $h(t)$ является для $t \in \langle -T, T\rangle$ возрастающей и непрерывной слева, а функция $\omega(\eta)$ является для $\eta \geqq 0$ возрастающей, непрерывной и $\omega(0) = 0$; функция $x(\tau)$ имеет ограниченное изменение, непрерывна слева и $(x(\tau), \tau) \in G$ для $\tau \in \langle \sigma_1, \sigma_2\rangle$.

В § 2 доказывается теорема существования (теорема 2,1) для обобщенного дифференциального уравнения

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t)\,, \tag{0,3}$$

(если функция $F(x, t)$ удовлетворяет условиям (0,1) и (0,2)).

При этом решение, существование которого доказано, имеет ограниченное изменение и непрерывно слева. От теоремы существования обычного типа теорема 2,1 отличается тем, что для данной точки (x_0, t_0) доказано существование решения $x(\tau)$, $x(t_0) = x_0$ для $t_0 \leqq t \leqq t_0 + \zeta$ $(\zeta > 0)$, однако, решение $x(\tau)$ не обязательно существует для $\tau < t_0$. (Это связано с предположением, что функция $h(t)$ непрерывна слева. Если $h(t)$ непрерывна, то решение $x(\tau)$, $x(t_0) = x_0$ существует в некоторой окрестности точки t_0.)

Поводом к исследованию этого класса обобщенных дифференциальных уравнений было следующее рассуждение: Пусть дана последовательность классических дифференциальных уравнений

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + \varphi_k(t)\,, \quad t \in \langle -T, T\rangle\,, \tag{0,4}$$

где

$$\varphi_k(t) \geqq 0 \quad \text{и} \quad \Phi_k(t) = \int_{-T}^{t} \varphi_k(\tau)\,\mathrm{d}\tau \to 0 \text{ соотв. } 1, \tag{0,5}$$

$$\text{если } -T \leqq t < 0\,, \quad \text{соотв. } 0 < t \leqq T\,, \quad k \to \infty\,.$$

Пусть $x_k(t)$ — решение уравнения (0,4), $x_k(-T) = 1$. Очевидно, $x_k(t) \to x(t)$ для $t \neq 0$, $k \to \infty$ где

$$x(t) = \begin{cases} e^{t+T}\,, & -T \leqq t \leqq 0\,; \\ (e^T + 1)\,e^t\,, & 0 < t \leqq 1\,. \end{cases}$$

Мы задались целью построить такой класс обобщенных дифференциальных уравнений, чтобы функция $x(t)$ была решением подходящего обобщенного уравнения и чтобы сходимость $x_k(t) \to x(t)$ была следствием общей теоремы о непрерывной зависимости от параметра.

Если $f(x, t)$ удовлетворяет условиям Каратеодори

$$\|f(x, t)\| \leqq m(t)\,, \tag{0,6}$$

где $m(t)$ — абсолютно интегрируемая функция и

$$\left.\begin{array}{l} f(x, t) \text{ является при любом фиксированном } t \\ \text{непрерывной функцией переменного } x\,, \end{array}\right\} \tag{0,7}$$

то каждое решение (типа Каратеодори, т. е. абсолютно непрерывное) уравнения

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)\,, \tag{0,8}$$

является также решением уравнения (0,1) и наоборот, если

$$F(x, t) = \int_0^t f(x, \tau)\,\mathrm{d}\tau\,. \tag{0,9}$$

Доказательство вполне аналогично доказательству в § 2 работы [1] для случая, когда $f(x, t)$ непрерывна.

Теорема существования 2,1 содержит в качестве частного случая теорему существования Каратеодори; если положим $h(t) = \int_0^t m(\tau)\,\mathrm{d}\tau$, то (0,2) вытекает из (0,6), (0,7)) и (0,9), но (0,3) не должно иметь места ни для какой функции $\omega(\eta)$. В предположении, что функция $f(x, t)$ определена на множестве $Kx\langle -T, T\rangle$, где K — комтактное множество, и исполняет (0,6) и (0,7), все-таки удается доказать, что всегда существуют такие функции $\omega(\eta)$, $h(t)$ ($h(t)$ отлична от $\int_0^t m(\tau)\mathrm{d}\tau$), что (0,2) и (0,3) исполнены.

В § 3 доказывается теорема об однозначности при условии $\omega(\eta) = K\eta$. (Эта теорема следует из неравенства для расстояния между двумя решениями аналогично тому, как и в классическом случае.)

В § 4 доказывается общая теорема о непрерывной зависимости от параметра (теорема 4,2). Предположения, с которыми мы работаем в § 4, довольно сложны. Главные трудности заключаются в следующих фактах:

1. Классическое уравнение (0,4) равносильно обобщенному уравнению

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[xt + \Phi_k(t)]\,, \tag{0,10}$$

однако ясно, что не существует такой функции $h(t)$, чтобы (0,2) имело место для $F(x, t) = xt + \Phi_k(t)$, $k = 1, 2, 3, \ldots$

2. Сходимость последовательности дифференциальных уравнений (0,4) зависит, очевидно, только от поведения правых частей уравнений в надлежащим образом определенной окрестности разрывного предельного решения $x(t)$.

В § 5 теорема 4,2 используется для исследования предельного поведения решений последовательности уравнений

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t) + g(x)\,\varphi_k(t)\,, \tag{0,11}$$

если $f(x, t)$, $g(x)$ и $\varphi_k(t)$ — непрерывные функции и если последовательность $\varphi_k(t)$ сходится к функции Дирака. Мы говорим, что последовательность $\varphi_k(t)$ сходится к функции Дирака, если

$$\int_{-T}^{T} |\varphi_k(t)| \mathrm{d}t \leqq L < \infty \quad \text{для} \quad k = 1, 2, 3, \dots,$$

$$\lim_{k\to\infty} \left[\int_{-T}^{-\varepsilon} |\varphi_k(t)|\, \mathrm{d}t + \int_{\varepsilon}^{T} |\varphi_k(t)|\, \mathrm{d}t \right] = 0 \quad (\text{для любого } \varepsilon > 0) \tag{0,12}$$

и если

$$\lim_{k\to\infty} \int_{-T}^{t} \varphi_k(\tau)\, \mathrm{d}\tau = 0\,, \quad \text{соотв. } 1 \ (\text{для } -T \leqq t < 0, \text{ соотв. } 0 < t \leqq T)\,.$$

Если, кроме того, $\varphi_k(t) \geqq 0$, то мы говорим, что последовательность $\varphi_k(t)$ сходится к функции Дирака положительно.

Пусть решение $u(t)$, $t \in \langle -T, 0 \rangle$, *уравнения*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t) \tag{0,13}$$

однозначно определяется начальным условием $u(-T) = y_0$. *Пусть решение* $v(t)$, $t \in \langle -\frac{1}{2}, +\frac{1}{2} \rangle$, *уравнения*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x) \tag{0,14}$$

однозначно определяется начальным условием $v(-\frac{1}{2}) = u(0)$ *и пусть, наконец, решение* $w(t)$, $t \in \langle 0, T \rangle$, *уравнения* (0,13) *однозначно определяется начальным условием* $w(0) = v(\frac{1}{2})$. *Пусть* $y_k \to y_0$ *для* $k \to \infty$.

Если последовательность $\varphi_k(t)$ *сходится к функции Дирака положительно, то для достаточно больших* k *существует решение* $x_k(t)$, $t \in \langle -T, T \rangle$ *уравнения* (0,11), $x_k(-T) = y_k$ *и*

$$\lim x_k(t) = u(t)\,, \quad \textit{соотв.} \quad w(t) \quad (\textit{для} \quad -T \leqq t < 0\,, \quad \textit{соотв.} \quad 0 < t \leqq T)\,.$$

Это утверждение справедливо и при более слабом предположении, а именно, что последовательность $\varphi_k(t)$ сходится к функции Дирака, если выполняется условие

$$\|g(x_2) - g(x_1)\| \leqq \chi(\|x_2 - x_1\|)\,, \quad \text{где} \quad \int_0^1 \frac{\mathrm{d}\eta}{\chi(\eta)} = \infty$$

и если решение $v(t)$ уравнения (0,13), $v(-\frac{1}{2}) = u(0)$, определено для

$$t \in \left\langle -\frac{L}{2}, +\frac{L}{2} \right\rangle.$$